Micro II Final, April 27, 2017

1. Suppose a consumer has utility $\sum_{t=0}^{\infty} \delta^t u(c_t)$ where $0 < \delta < 1$ and the felicity function is $u(c) = \ln c$. Prices are $p_0 = p$ and $p_t = (1+r)^{-t}$ for $t = 1, 2, 3, \dots$ where $r > 0$ is the interest rate. The budget constraint is $\sum_{t=0}^{\infty} p_t c_t = W$. Suppose the rate of interest and rate of impatience are equal ($r = \delta^{-1} - 1$).

   a) Solve the intertemporal consumer's problem.

   b) Find $\partial c_t/\partial p$ and $\partial c_t/\partial W$ for every $t = 0, 1, \dots$.

   Answer:

   a) The first-order conditions are

   $$\frac{\delta c_t}{c_{t+1}} = \frac{\delta u'(c_{t+1})}{u'(c_t)} = \frac{p_{t+1}}{p_t}.$$

   Since $r = \delta^{-1} - 1$, $\delta(1+r) = 1$. Using the values for p_t in the first-order conditions, we find $c_t = c_{t+1}$ for $t = 1, 2, \dots$ and $c_1 = p(1+r)\delta c_0 = pc_0$. Thus $c_t = pc_0$ for $t = 1, 2, \dots$.

   The budget constraint is then $W = p_0 c_0 + pc_0 \sum_{t=1}^{\infty}(1+r)^{-t} = pc_0(1 + 1/r)$, so $c_0 = rW/p(1+r)$ and $c_t = rW/(1+r)$ for $t = 1, 2, \dots$.

   b) We now compute $\partial c_0/\partial p = -rW/p^2(1+r)$ and $\partial c_t/\partial p = 0$ for $t = 1, 2, \dots$. Further, $\partial c_0/\partial = r/p(1+r)$ and $\partial c_t/\partial W = r/(1+r)$ for $t = 1, 2, \dots$.

2. Consider a production economy with 2 goods and 2 consumers. There is one firm with technology set $Y = \{(y_1, y_2) : y_1 \le 0, 2y_2 \le -y_1\}$. Both consumers have the same utility function, $u(\mathbf{x}) = x_1 + x_2$. Endowments are $\boldsymbol{\omega}^1 = (1, 2)$ and $\boldsymbol{\omega}^2 = (0, 3)$. Find all competitive equilibria $(\mathbf{x}^i, \mathbf{y}, \mathbf{p})$.

   Answer: There are two cases to consider, depending on whether the firm operates or not. The technology is constant returns to scale, so maximum profit is zero. If the firm produces good 2, prices must be a multiple of $(1, 2)$. In that case, good 1 is cheaper and demand by each conusmer is $(m^i, 0)$ where $m^1 = 5$ and $m^2 = 6$ are consumer incomes. Then market demand is $(11, 0)$. The total endowment is $(1, 5)$, so this is not feasible (requires production of good 1, not good 2).

   That leaves the case where the firm does not produce. Then $\mathbf{y} = (0, 0)$ and $\mathbf{x}^1 + \mathbf{x}^2 = \boldsymbol{\omega} = (1, 5)$. In this case prices must be a multiple of $(1, 1)$ because both goods are consumed. Then profit is $y_1 + y_2$. To maximize profit, we set $y_2 = -y_1/2$, yielding profit $(1/2)y_1$. As $y_1 \le 0$, this is maximized when $y_1 = 0$. The second equilibrium has $\mathbf{p} = (1, 1)$, $\mathbf{y} = (0, 0)$, and any $\mathbf{x}^1, \mathbf{x}^2$ with $x_1^1 + x_2^1 = 3$, $x_1^2 + x_2^2 = 3$, $x_1^1 + x_1^2 = 1$ and $x_2^1 + x_2^2 = 5$.

   Summing up, the equilibria are $\mathbf{p} = (1, 1)$ (or any positive multiple), $\mathbf{y} = (0, 0)$ and $\mathbf{x}^1 = (x, 3-x)$ and $\mathbf{x}^2 = (1-x, 2+x)$ where x obeys $0 \le x \le 1$.

3. Suppose an economy has three states, $s = 1, 2, 3$. There are two consumers with identical utility functions, $u_i(\mathbf{x}^i) = \ln x_1^i + \ln x_2^i + \ln x_3^i$. Endowments are $\boldsymbol{\omega}^1 = (1, 0, 3)$ and $\boldsymbol{\omega}^2 = (3, 4, 1)$. There are two Arrovian securities, for states 1 and 2. There is no Arrovian security for state 3. (That means this is technically a Radner equilibrium due to the missing Arrovian security.)

   a) Find the Radner equilibrium.

   b) Is the equilibrium Pareto optimal.

   Answer:

   a) Of course, we can normalize the spot prices so that $p_s = 1$ in every state s. Now indirect utility is $v_1(\mathbf{z}^1) = \ln(1 + z_1^1) + \ln z_2^1 + \ln 3$ and $v_2(\mathbf{z}^2) = \ln(3 + z_1^2) + \ln(4 + z_2^2)$. Both asset prices must be positive as there would otherwise be infinite demand for them. We can normalize prices so that $q_1 = 1$ and $q_2 = q$. For each consumer, we maximize indirect utility using the budget constraint $q_1 z_1^i + q_2 z_2^i = z_1^i + q z_2^i = 0$. The first-order conditions are

$$\frac{1}{3 + z_1^1} = \lambda, \quad \frac{1}{z_2^1} = \lambda q$$

   and

$$\frac{1}{1 + z_1^2} = \mu, \quad \frac{1}{4 + z_2^2} = \mu q.$$

   It follows that $\lambda(3 + z_1^1 + q z_2^1) = 2$ and $\mu(1 + 4q + z_1^2 + q z_2^2) = 2$. Using the budget constraints $z_1^i + q z_2^i = 0$, we find $3\lambda = 2$ and $\mu(1 + 4q) = 2$. It follows that $\lambda = 2/3$ and $\mu = 2/(1 + 4q)$.

   Asset demand by consumer one is then $z_1^1 = -3/2$ and $z_2^1 = 3/2q$, while asset demand by consumer two is $z_1^2 = -1/2 + 2q$ and $z_2^2 = -2 + 1/2q$. Market clearing for good 1 requires $-3/2 - 1/2 + 2q = 0$, so $q = 1$.

   Equilibrium asset prices are $\mathbf{q} = (1, 1)$ (or any positive multiple) and asset demands are $\mathbf{z}^1 = (-3/2, 3/2)$ and $\mathbf{z}^2 = (3/2, -3/2)$. The corresponding goods allocation is $\mathbf{x}^1 = (3/2, 3/2, 3)$ and $\mathbf{x}^2 = (5/2, 5/2, 1)$. Finally, spot prices can be any positive numbers (we used $\mathbf{p} = (1, 1, 1)$).

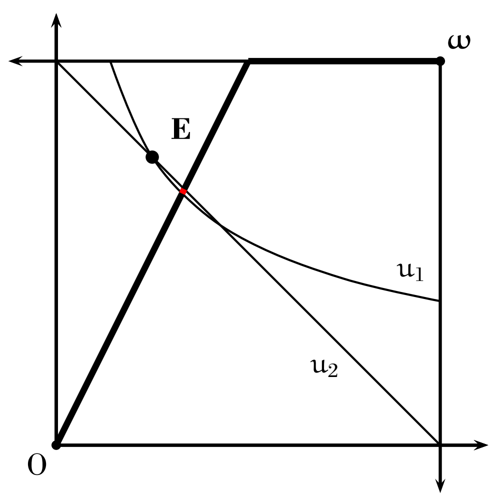
   b) Since the allocation is interior, the marginal rates of substitution must obey $\mathrm{MRS}_{13}^1 = \mathrm{MRS}_{13}^2$ to be Pareto optimal where $\mathrm{MRS}_{13}^i = x_3^i / x_1^i$. But $\mathrm{MRS}_{13}^1 = 2$ and $\mathrm{MRS}_{13}^2 = 2/5$. As these are not equal, the equilibrium allocation is not Pareto optimal.

4. An exchange economy has two consumers with utility functions $u_1(\mathbf{x}^1) = (x_1^1)^{1/3}(x_2^1)^{2/3}$ and $u_2(\mathbf{x}^2) = x_1^2 + x_2^2$. Endowments are $\boldsymbol{\omega}^1 = (1, 3)$ and $\boldsymbol{\omega}^2 = (3, 1)$. Find the core.

   Answer: Since there are two consumers, the core consists of allocations that are both Pareto optimal and individually rational. Interior Pareto optima must have the same marginal rate of substitution for both consumers. Thus $\mathrm{MRS}^1_{12} = (1/2)x^1_2/x^1_1 = \mathrm{MRS}^2_{12} = 1$, or $2x^1_1 = x^1_2$. Note that $\omega = (4,4)$. Thus the line $\mathbf{x}^1 = (x, 2x)$, $\mathbf{x}^2 = (4-x, 4-2x)$ for $0 \le x \le 2$ consists of Pareto optimal allocations. Moreover, it is easily seen that any points with $x^1_1 \ge 2$, $x^2_2 = 4$, $x^2_1 \le 2$ and $x^2_2 = 0$ are also Pareto optimal.

Individual rationality requires that $(x^1_1)^{1/3}(x^1_2)^{2/3} \ge 3^{2/3}$ and $x^2_1 + x^2_2 \ge 4$. For the interior optima, this translates to $2^{2/3}x \ge 3^{2/3}$ and $8 - 3x \ge 4$, i.e. $3x \le 4$. Note that the boundary Pareto optima have $u_2(\mathbf{x}^2) \le 2$, and are not individually rational. Summing up, the core consists of the Pareto optimal allocations with $4/3 \ge x \ge (3/2)^{2/3} \approx 1.31$, as illustrated on the diagram.

In the diagram, the endowment point is E and the core is shown in red. Indifference curves through the endowment are marked u_1 and u_2. As you can see, the core is very small.



5. Consider the production function $f(\mathbf{z}) = (z_1 + z_2)^{1/2}$. Let $Y = \{(q, -\mathbf{z}) : q \le f(\mathbf{z})\}$. Find the profit function and optimal net output vector for all price vectors $(p, w_1, w_2) \gg \mathbf{0}$.

Answer: Profit is $p(z_1 + z_2)^{1/2} - w_1 z_1 - w_2 z_2$. Notice that the two inputs are perfect substitutes in production. It follows that if $w_1 \ne w_2$, the same output can be produced more cheaply by only using the cheaper input. Thus profit is $p(z_1 + z_2)^{1/2} - \min\{w_1, w_2\}(z_1 + z_2)$. Setting $z_1 + z_2 = z$, we obtain a simpler expression for profit, $pz^{1/2} - \min\{w_1, w_2\}z$.

The first-order conditions are $(p/2)z^{-1/2} = \min\{w_1, w_2\}$. It follows that $z = p^2/4\min\{w_1, w_2\}^2$. The optimal net output vector is then

$$\mathbf{y}(p, w_1, w_2) = \begin{cases} \left(\frac{p}{2w_1}, -\frac{p^2}{4w_1^2}, 0\right) & \text{when } w_1 < w_2 \\ \left(\frac{p}{2w_2}, 0, -\frac{p^2}{4w_2^2}\right) & \text{when } w_1 > w_2 \\ \left\{\left(\frac{p}{2w_1}, -z_1, -z_2\right) : z_1 + z_2 = \frac{p^2}{4w_1^2}, z_1, z_2 \ge 0\right\} & \text{when } w_1 = w_2 \end{cases}.$$

This yields profit function

$$\pi(p, w_1, w_2) = \frac{p^2}{4 \min\{w_1, w_2\}^2}.$$