Micro I Midterm, February 27, 2018

1. Suppose utility on $\mathbb{R}^3_+$ is given by $u(\mathbf{x}) = (x_1+1)x_2(x_3+5)$.

   a) Is there a monotonic transformation that transforms u into an additive separable utility function?

   Answer: Yes. Let $v = \ln u$. Then $v(\mathbf{x}) = \ln(x_1+1) + \ln x_2 + \ln(x_3+5)$, which is in additive separable form.

   One way to find the right transformation is to consider $v(\mathbf{x}) = \phi\big(u(\mathbf{x})\big)$. The second cross partial derivatives must be zero. Now $\partial v/\partial x_1 = \phi' x_2(x_3+5)$ and $\partial^2 u/\partial x_2 \partial x_1 = \phi'(x_3+5) + \phi''(x_1+1)x_2(x_3+5)^2 = 0$. This can be written as $\phi'(x_3+5)+\phi'' u(x_3+5) = 0$. Thus $\phi' + \phi'' u = 0$. Let $\psi = \phi'$ so that $\psi + \psi' u = 0$. In other words, $d\psi/\psi = -du/u$. The solution is $\psi(u) = A/u$ for some constant A which may be of either sign. Now $\phi' = \psi = A/u$. This has general solution $\phi = B + A\ln u$ for some constants A and B. Because ϕ is increasing, $A > 0$. One such function is $\phi(u) = \ln u$. We don't have to worry about the other cross partial derivatives as ϕ converts u into the additive separable form $v(\mathbf{x}) = \ln(x_1+1) + \ln x_2 + \ln(x_3+5)$.

   b) Is u completely separable?

   Answer: Yes. Since it is additive separable it is also completely separable.

   To answer more fully, it induces the same preference order defined by the utility function x_i on $\{i\}$. On $\{1,2\}$, it induces the order defined by the utility function $\ln(x_1+1)+\ln x_2$. On $\{1,3\}$ it induces $\ln(x_1+1)+\ln(x_3+5)$. On $\{2,3\}$ it induces $\ln x_2 + \ln(x_3+5)$. Since it induces an order on every commodity subgroup of $\{1,2,3\}$ it is completely separable.

2. Suppose utility on $\mathbb{R}^2_+$ is given by $u(\mathbf{x}) = \min\{x_1, x_2\}$. In the following, by "revealed preferred" I mean the Samuelson sense, not the Houthakker sense.

   a) Find the Marshallian demand functions for all strictly positive prices $\mathbf{p} \in \mathbb{R}^2_{++}$ and non-negative incomes $m \geq 0$.

   Answer: When $\mathbf{p} \gg \mathbf{0}$, there is a cost, but no gain to consuming an excess amount of either good. That means that $x_1 = x_2$ at the consumer's optimal point. Since Leontief preferences are locally non-satiated, Walras's Law applies. Combining these two facts quickly yields the Marshallian demand

$$\mathbf{x}(\mathbf{p}, m) = \frac{m}{p_1+p_2}\begin{pmatrix}1\\1\end{pmatrix}.$$

b) Is $(5,5)$ revealed preferred to $(10,2)$? If so, find a budget set containing $(10,2)$ and $(5,5)$ as a Marshallian demand point.

Answer: Yes, $(5,5)$ is revealed preferred to $(10,2)$. Setting $\mathbf{p} = (1/2, 1)$ and $m = 7.5$ has $(5,5)$ as Marshallian demand and $\mathbf{p}\cdot(10,2) = 7 < m$, so $(10,2)$ is in the budget set.

c) Can $(6,7)$ be revealed preferred to $(5,5)$? If so, find a budget set that contains $(5,5)$ and has $(6,7)$ as a Marshallian demand point.

Answer: No, because prices are strictly positive, $(6,7)$ cannot be a revealed preferred to $(5,5)$ (or anything else) since it is not a demand point.

Notice however, that if we merely require $\mathbf{p} > 0$,the pointe $(6,7)$ *is* revealed preferred to $(5,5)$. To see this, set $\mathbf{p} = (1,0)$ and $m = 6$. Then $(6,7)$ is in the demand correspondence and $(5,5)$ is in the budget set.

3. Suppose utility on $\mathbb{R}^2_+$ is defined by $u(\mathbf{x}) = \sqrt{x_1} + \sqrt{x_2}$.

a) Find the Marshallian demand functions.

Answer: The utility function is concave, so we need only check the first order conditions. Moreover, the fact that marginal utility is ∞ when either $x_1 = 0$ or $x_2 = 0$ means we need only consider interior solutions.

The Lagrangian is $\mathcal{L} = \sqrt{x_1} + \sqrt{x_2} + \lambda(\mu - p_1x_1 + p_2x_2)$. The first order conditions are $1/2\sqrt{x_i} = \lambda p_i$. It follows that $p_1/p_2 = \sqrt{x_2/x_1}$. Squaring and rearranging, we find that $x_x = p_1^2 x_1/p_2^2$. Using the budget constraint, we find the Marshallian demands:

$$\mathbf{x}(\mathbf{p}, m) = \frac{m}{p_1 + p_2}\begin{pmatrix} p_2/p_1 \\ p_1/p_2 \end{pmatrix}.$$

b) Find the indirect utility function.

Answer: We compute $v(\mathbf{p}, m) = u\Big(\mathbf{x}(\mathbf{p}, m)\Big)$.

$$v(\mathbf{p}, m) = \sqrt{\frac{m}{p_1 + p_2}}\sqrt{\frac{p_1^2 + p_2^2}{p_1 p_2}}.$$

c) Find the expenditure function.

Answer: We use the fact that $v\Big(\mathbf{p}, e(\mathbf{p}, \bar{u})\Big) = \bar{u}$. Solving, we obtain

$$e(\mathbf{p}, \bar{u}) = \bar{u}^2 \frac{p_1 p_2 (p_1 + p_2)}{p_1^2 + p_2^2}$$

4. Suppose $e(\mathbf{p}, \bar{u}) = p_1 + 2p_2 + 2\bar{u}\sqrt{p_1 p_2}$ is a consumer's expenditure function for $\bar{u} \geq 0$.

a) Find the Hicksian (compensated) demand functions.

Answer: The Shepard-McKenzie Lemma tells us that $\mathbf{h}(\mathbf{p}, \bar{u}) = d_{\mathbf{p}} e(\mathbf{p}, \bar{u})$. Thus

$$\begin{pmatrix} h_1(\mathbf{p}, \bar{u}) \\ h_2(\mathbf{p}, \bar{u}) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \bar{u} \begin{pmatrix} \sqrt{p_2/p_1} \\ \sqrt{p_1/p_2} \end{pmatrix}.$$

b) What restrictions are there on the Hicksian demands due to the fact that $\bar{u} \geq 0$.

Answer: The restrictions are $h_1 \geq 1$ and $h_2 \geq 2$.

c) Find the direct utility function $u(\mathbf{x})$ and state any restrictions on x_1 and x_2.

Answer: We use the Hicksian demands and eliminate p_1/p_2. To do this, use the fact that $h_1 - 1 = \bar{u}\sqrt{p_2/p_1}$ and $h_2 - 2 = \bar{u}\sqrt{p_1/p_2}$ and multiply. We obtain $(h_1 - 1)(h_2 - 2) = \bar{u}^2$, implying that the utility has the Stone-Geary form $u(\mathbf{x}) = \sqrt{(x_1 - 1)(x_2 - 2)}$ for $x_1 \geq 1$ and $x_2 \geq 2$.

This utility function will only give us a real value when $x_1 \geq 1$ and $x_2 \geq 2$. Thus $\mathfrak{X} = \{\mathbf{x} \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 2\}$.