Micro I Midterm, February 26, 2019

1. Suppose utility on $\mathbb{R}^3_{++}$ is given by $u(\mathbf{x}) = \frac{1}{4}x_1^2 + x_1x_2x_3 + x_2^2x_3^2$.

   a) List all non-trivial (proper) partitions of $\{1,2,3\}$.

   b) Which non-trivial partitions is u weakly separable on?

   Answer:

   a) The non-trivial partitions are $\Big\{\{1\},\{2\},\{3\}\Big\}$, $\Big\{\{1\},\{2,3\}\Big\}$, $\Big\{\{2\},\{1,3\}\Big\}$, and $\Big\{\{3\},\{1,2\}\Big\}$.

   b) Because u is strictly increasing on $\mathbb{R}^3_{++}$, it induces an order on every singleton group. It follows that u is weakly separable relative to the partition of singletons, $\Big\{\{1\},\{2\},\{3\}\Big\}$.

   We need only check whether it induces an order on each two-element group to determine if it is weakly separable on the other partitions. Now

$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \frac{1}{2}x_1 + x_2x_3 = \frac{1}{2}(x_1 + 2x_2x_3)\\ \frac{\partial u}{\partial x_2} &= x_1x_3 + 2x_2x_3^2 = x_3(x_1 + 2x_2x_3),\\ \frac{\partial u}{\partial x_3} &= x_1x_2 + 2x_2^2x_3 = x_2(x_1 + 2x_2x_3).\end{aligned}$$

   We now calculate the marginal rates of substitution:

$$\text{MRS}_{12} = \frac{1}{2x_3}, \quad \text{MRS}_{13} = \frac{1}{2x_2}, \quad \text{MRS}_{23} = \frac{x_3}{x_2}$$

   Of these, both MRS_{12} and MRS_{13} depend on the third good, indicating that no preference order is induced on either $\{1,2\}$ or $\{1,3\}$. However, MRS_{23} only depends on goods two and three, so a preference order is induced on $\{2,3\}$. Since a preference order is also induced on every singleton, u is weakly separable on the partition $\Big\{\{1\},\{2,3\}\Big\}$. In fact, it is strongly separable there.

2. Consider the expected utility functions $u(x) = x^{1/2}$ and $v(x) = \ln x$.

   a) Which of the two utility functions is more risk-averse?

   b) Suppose a lottery is described by the uniform probability density on $[1,2]$,

$$\rho(x) = \begin{cases} 1 & \text{when } 1 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

   Compute the expected utilities $\mathsf{E}u$ and $\mathsf{E}v$.

Answer:

a) The function ν is more risk averse. An easy way to see that is to notice that $\nu(x) = 2\ln u(x)$. Since ν is a concave transformation of u, it is more risk averse.

Another way to see that ν is more risk averse is to solve part (b), then compute certainty equivalents. They are $c_u = (2/3)^2[2^{3/2} - 1]^2 \approx 1.49$ and $c_\nu = e^{2\ln 2-1} = 4/e \approx 1.47$. The smaller certainty equivalent for ν indicates that ν is more risk averse.

One could also compute the coefficients of risk aversion. For u, the absolute risk aversion is $1/2x$ while for ν it is $1/x$. Since ν has larger absolute risk aversion, it is more risk averse.

b) Now

$$Eu = \int_1^2 x^{1/2}\,dx = (2/3)x^{3/2}\Big|_1^2 = (2/3)(2^{3/2} - 1)$$

and

$$E\nu = \int_1^2 \ln x\,dx = (x\ln x - x)\Big|_1^2 = 2\ln 2 - 1.$$

3. Suppose production is described by the set $Y = \{(y_1, y_2, y_3) : y_1, y_2 \le 0, y_3 \le (y_1y_2)^{1/2}\}$.

a) Show that Y obeys the five conditions defining a production set.

b) For which price vectors $(p_1, p_2, p_3) \gg \mathbf{0}$ can profit be maximized?

Answer:

a) First, $(0, 0, 0) \in Y$ because $y_1 = y_2 = 0 \le 0$ and $y_3 = 0 \le (y_1y_2)^{1/2} = 0$. This establishes that Y is **non-empty** and obeys **inaction**. If $\mathbf{y} \in Y$ and $\mathbf{y} \ge 0$, we must have $y_1 = 0$ and $y_2 = 0$. Then $y_3 \le (y_1y_2)^{1/2} = 0$, so $y_3 = 0$ also. This shows that Y satisifies the **no-free lunch** condition. Now $(y_1, y_2) \to (y_1y_2)^{1/2}$ is continuous and the sets $(y_1, y_2) \le \mathbf{0}$ is closed, so Y is **closed** since weak inqualities are preserved in the limit. Finally, if $\mathbf{y} \in Y$ and $\mathbf{y}' \le \mathbf{y}$, we have $y_1' \le y_1 \le 0$, $y_2' \le y_2 \le 0$, and $y_3' \le y_3 \le (y_1y_2)^{1/2} \le (y_1'y_2')^{1/2}$, showing that Y obeys **free disposal**.

b) The production technology is constant returns to scale, so maximum profit is either zero or infinite. If it is zero, it is attained at $\mathbf{y} = \mathbf{0}$. If it is infinite, profit cannot be maximized.

There are several ways to determine which prices work. E.g., you can use derivatives. Here is another.

If positive profit is possible, the constant returns means we can make positive profit with output one. In other words, we use a vector of the form $(-x, -1/x, 1)$ and profit is $p_3 - xp_1 - p_2/x$. We can use the fact that $0 \leq (\sqrt{xp_1} - \sqrt{p_2/x})^2$ to show that $xp_1 + p_2/x \geq 2\sqrt{p_1p_2}$. Combining with positive profit, we obtain $p_3 > xp_1 + p_2/x \geq 2\sqrt{p_1p_2}$ to see that positive profits can only occur if $p_3 \geq 2\sqrt{p_1p_2}$. Moreover they do occur then, so maximum profit is infinite in this case.

In sum, profit can be maximized if $p_3 \leq 2\sqrt{p_1p_2}$, and cannot be maximized if $p_3 > 2\sqrt{p_1p_2}$.

4. Consider the production function on $\mathbb{R}^3_+$ defined by $f(x_1, x_2, x_3) = (x_1 + x_2)^{1/2} + \min\{x_2, x_3\}$. Show how this production function may be homogenized by adding a fourth factor of production.

Answer: We call the fourth factor x_4. For $x_4 > 0$, we define

$$\begin{aligned} F(x_1, x_2, x_3, x_4) &= x_4 f(x_1/x_4, x_2/x_4, x_3/x_4) \\ &= x_4 \left(\frac{x_1 + x_2}{x_4}\right)^{1/2} + x_4 \min\left\{\frac{x_2}{x_4}, \frac{x_3}{x_4}\right\} \\ &= (x_1x_4 + x_2x_4)^{1/2} + \min\{x_2, x_3\} \end{aligned}$$

Since F is continuous at $x_4 = 0$, the formula for F applies to all of $\mathbb{R}^4_+$.