Micro I Final, April 23, 2020

1. Suppose u is a constant relative risk aversion utility function. Let $x > 0$ Consider a lottery L_x that pays xG with probability p, $0 < p < 1$, and pays xB with probability $(1-p)$. Here $G > B$. How does the certainty equivalent of this lottery change as x changes?

   Answer: Up to affine transformations, there are two cases. Either $u(c) = c^{1-\sigma}/(1-\sigma)$ or $u(c) = \ln c$. Let $\bar{c}$ be the certainty equivalent of the lottery L_1, defined by $u(\bar{c}) = Eu(L_1)$.

   In the first case, expected utility is

$$
\begin{aligned}
Eu(L_x) &= \frac{p(xG)^{1-\sigma} + (1-p)(xB)^{1-\sigma}}{1-\sigma} \\
&= x^{1-\sigma} Eu(L_1) \\
&= x^{1-\sigma} \frac{\bar{c}^{1-\sigma}}{1-\sigma} \\
&= u(x\bar{c}).
\end{aligned}
$$

   In the second case,

$$
\begin{aligned}
Eu(L_x) &= p \ln xG + (1-p) \ln xB \\
&= \ln x + Eu \\
&= \ln x + \ln \bar{c} \\
&= u(x\bar{c}).
\end{aligned}
$$

   The certainty equivalent of the lottery is $x\bar{c}$, which is linear in x.

2. A two-good, two-person production economy has utility functions $u_i(\mathbf{x}) = \ln x_1 + \ln x_2$, endowments $\boldsymbol{\omega}^1 = (5, 3)$ and $\boldsymbol{\omega}^2 = (2, 2)$, production set $Y = \{(y_1, y_2) : y_2 \le 0, y_1 \le -y_2\}$, and profit shares $\theta^1 = .3$, $\theta^2 = .7$.

   a) Find any Walrasian equilibria $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2, \hat{\mathbf{y}})$ where nothing is produced.

   b) Find any Walrasian equilibria $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2, \hat{\mathbf{y}})$ where something is produced.

   Answer:

   a) We first consider whether there are any equilibria without production. Profit is $p_1y_1 + p_2y_2 \le (p_1 - p_2)(-y_2)$. As $-y_2 \ge 0$, $\mathbf{y} = \mathbf{0}$ will only maximize profit when $p_1 \le p_2$. Demand will be infinite if either price is zero, so we may take good one as numéraire.

Then we can write $\mathbf{p} = (1, p)$. The criterion for no production becomes $p = p_2/p_1 \geq 1$. Income is $m^1 = 5 + 3p$ and $m^2 = 2 + 2p$. Using the fact that preferences are equal-weighted Cobb-Douglas, market demand must be

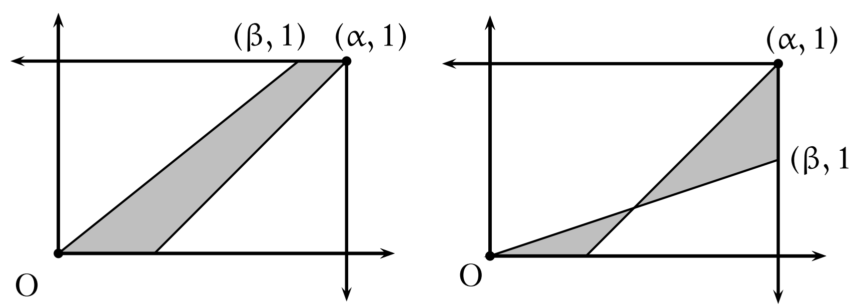
$$\mathbf{x} = \left(\frac{7+5p}{2}, \frac{7+5p}{2p}\right) = \boldsymbol{\omega} = (7, 5).$$

This has solution $p = 7/5$. As required for no production, $p \geq 1$. The resulting Walrasian equilibrium is $\hat{\mathbf{p}} = (1, 7/5)$, $\hat{\mathbf{x}}^1 = (23/5)(1, 5/7)$ and $\hat{\mathbf{x}}^2 = (12/5)(1, 5/7)$. Any price vector of the form $\lambda(1, 7/5)$ where $\lambda > 0$ also yields a Walrasian equilibrium with the same allocation. There are no other equilibria without production.

b) If the firm operates, profit maximization requires that $p_1/p_2 = 1$. We can then normalize prices with $\mathbf{p} = (1, 1)$. Because production is constant returns to scale, there are no profits to consider. It follows that income is $m^1 = 8$ and $m^2 = 4$. The corresponding demands are $\mathbf{x}^1 = (4, 4)$ and $\mathbf{x}^2 = (2, 2)$. By market clearing, $(4, 4) + (2, 2) = \boldsymbol{\omega} + \mathbf{y} = (7, 5) + \mathbf{y}$, so $\mathbf{y} = (-1, +1) \notin Y$. There is no feasible solution, and hence no Walrasian equilibrium with production.

3. Suppose $u_1(\mathbf{x}^1) = \min\{x_1^1, \beta x_2^1\}$, $u_2(\mathbf{x}^2) = \min\{x_1^2, x_2^2\}$, and $\boldsymbol{\omega} = (\alpha, 1)$ where $\alpha > 1$. Find all Pareto optimal allocations.

Answer: There are two cases to consider depending on whether or not the lines $x_1^1 = \beta x_2^1$ and $x_1^2 = x_2^2$ intersect. Drawing in some indifference curves shows that in either case, the set of Pareto optima is the area between the two lines. The case $\beta < \alpha$ is on the left, and $\beta > \alpha$ on the right. If $\beta = \alpha$, the diagonal is the set of Pareto optima (not illustrated).



4. Let $\mathcal{E}$ be a contingent commodities exchange economy with $S = 10$ states and $L = 1$ good in each state. There are $I = 10$ consumers. Consumer i has endowment $\boldsymbol{\omega}^i = 10\mathbf{e}^i$ where $\mathbf{e}^i$ is the i^{th} basis vector and utility $u_i(\mathbf{x}^i) = \sum_{s=1}^{10} \pi_s u(x_s^i)$, where u and the π_s are the same for every consumer. The subutility $u \in \mathcal{C}^2$ obeys $u' > 0$ and $u'' < 0$. Here $\sum_s \pi_s = 1$ with each π_s obeying $0 < \pi_s < 1$.

There are 10 assets with return matrix $\mathbf{R}$. The matrix $\mathbf{R}$ is invertible. Asset one is a safe asset paying one unit of good one in every state. The expected payoff from each asset is 1.

a) Is there a complete set of assets? Why?

b) If we treat this as an Arrow-Debreu economy rather than a Radner economy, is there full insurance in equilibrium? Describe the Arrow-Debreu equilibrium.

c) Based on part (b), give a formula for the corresponding equilibrium asset price vectors $\hat{q}$ under full insurance.

d) Consider Example 26.3.6. Find an equivalent Arrovian securities equilibrium for this example.

e) Continue with the model of part (d). Show that both assets defined by the return matrix

$$\mathbf{R} = \begin{pmatrix} 1 & 3/2 \\ 1 & 3/4 \end{pmatrix}$$

have expected payoff 1 using the common probabilities $\boldsymbol{\pi} = (1/3, 2/3)$. Then use arbitrage pricing to find the prices of both of the Radner assets.

f) Consider Example 26.3.5. Find an equivalent Arrovian securities equilibrium for this example.

g) Continue with the model of part (f). Show that both assets defined by the return matrix

$$\mathbf{R} = \begin{pmatrix} 1 & 4/3 \\ 1 & 3/4 \end{pmatrix}$$

have expected payoff 1 using the market probabilities. Then use arbitrage pricing to find the prices of both of the Radner assets.

h) What, if any, conjectures can you make about when a risky asset will have a lower price than a safe asset with the same expected return. Consider the similarities and differences between your results for parts (d)-(e) and (f)-(g).

Answer:

a) The 10×10 payoff matrix $\mathbf{R}$ is invertible. It follows that the set of assets is complete.

b) We have an aggregate endowment of $(10, 10, \dots, 10)$, which is certain. There is no aggregate uncertainty. The Full Insurance Theorem applies, yielding equilibrium prices $\hat{\mathbf{p}} = \bar{p}(\pi_s)$ where the π_s are the common probabilities in the utility functions. We may take $\bar{p} = 1$. Income of consumer i is then π_i and $\hat{\mathbf{x}}^i = \pi_i(10, 10, \dots, 10)$.

c) By Theorem 27.3.6, the corresponding asset prices are $\hat{\mathbf{q}} = \mathbf{R}^{\mathsf{T}}\boldsymbol{\pi}$. It follows that the price of each asset is its expected value, so $\hat{\mathbf{q}} = (1, \dots, 1)$.

d) In Example 26.3.6 endowments are $\boldsymbol{\omega}^1 = (3, 0)$ and $\boldsymbol{\omega}^2 = (0, 2)$. Utility is $u_i(\mathbf{x}^1) = \frac{1}{3}\ln x_1^i + \frac{2}{3}\ln x_2^i$ for each of the two consumers $i = 1, 2$.

As shown in Example 26.3.6, the Arrow-Debreu equilibrium is $\mathbf{p} = (1, 3)$ with equilibrium allocation $\hat{\mathbf{x}}^1 = (1, 2/3)$ and $\hat{\mathbf{x}}^2 = (2, 4/3)$. The Arrovian Equivalence Theorem applies with Arrovian securities prices $\hat{\mathbf{q}} = (1, 3)$. The resulting asset demands are $\hat{\mathbf{z}}^1 = (-2, +2/3)$, $\hat{\mathbf{z}}^2 = (+2, -2/3)$.

Now that we have derived the Arrovian securities prices, we have the option of renormalizing the spot market prices to $\hat{\mathbf{p}} = (1, 1)$. This renormalization is *not* allowed in Arrow-Debreu model, and so cannot be done before using the Arrovian Equivalence Theorem, only afterward.

e) The common probabilities are 1/3 and 2/3, so asset one has expected payoff $(1/3) \times 1 + (2/3) \times 1 = 1$ and asset two has expected payoff $(1/3) \times (3/2) + (2/3) \times (3/4) = 1$.

Arbitrage pricing tells us that the price of asset one is $q_1 = 1 + 3 = 4$ and that $q_2 = (3/2) \times 1 + (3/4) \times 3 = 15/4 = 3\ 3/4$. The safe asset is more expensive than the risky asset.

f) In Example 26.3.5 endowments are $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (0, 2)$. Utility is $u_1(\mathbf{x}^1) = \frac{1}{3}\ln x_1^1 + \frac{2}{3}\ln x_2^1$ for consumer one, but consumer two's utility is $u_2(\mathbf{x}^2) = \frac{1}{2}\ln x_1^2 + \frac{1}{2}\ln x_2^2$.

As shown in Example 26.3.5, the Arrow-Debreu equilibrium is $\mathbf{p} = (1, 4/3)$ with equilibrium allocation $\hat{\mathbf{x}}^1 = (2/3, 1)$ and $\hat{\mathbf{x}}^2 = (4/3, 1)$. The Arrovian Equivalence Theorem applies with Arrovian securities prices $\hat{\mathbf{q}} = (1, 4/3)$. The resulting asset demands are $\hat{\mathbf{z}}^1 = (-4/3, +1)$ and $\hat{\mathbf{z}}^2 = (+4/3, -1)$.

As in (d), we may renormalize the spot prices to $\hat{\mathbf{p}} = (1, 1)$.

g) The market probabilities from Example 26.3.5 are $\pi_1 = 3/7$ and $\pi_2 = 4/7$. Then asset one has expected payoff $\pi_1 + \pi_2 = 1$ and asset two has expected payoff $(4/3)\pi_1 + (3/4)\pi_2 = 4/7 + 3/7 = 1$.

Arbitrage pricing tells us that the price of asset one is $q_1 = 1 + 4/3 = 7/3$ and the price of asset two is $q_2 = (4/3) + (3/4)(4/3) = 7/3$. Both the safe and risky asset have the same price.

h) The usual intuition is that the safe asset should have a higher price than the risky asset. This failed in Example 27.1.4 (full insurance) and in parts (f)-(g). In both cases, assets with equal expected returns had equal prices, regardless of the risk.

However, in part (d)-(e), assets with equal expected returns had different prices. What is causing the difference?

The common probabilities in parts (d)-(e), $(1/3, 2/3)$, are not the market probabilities $(1/4, 3/4)$. Using the market probabilities, asset two in (d)-(e) has market expected return 15/16, not 1, so it is not surprising it is cheaper.

Consider a case where the risky asset has a market expected return of 1. E.g.,

$$\mathbf{R} = \begin{pmatrix} 1 & 2 \\ 1 & 2/3 \end{pmatrix}.$$

Then the risky and safe assets have the same price ($q_i = 4$).

In general, if there is a complete set of assets in the Radner model, then any asset the same expected return will have the same price. To see this, recall that we can normalize prices so that $\hat{\mathbf{q}} = \mathbf{R}^{\mathsf{T}}\boldsymbol{\pi}$ where π is the vector of market probabilities. Then each asset's price is its expected return: $\hat{q}_k = (\mathbf{r}^k)^{\mathsf{T}}\boldsymbol{\pi} = \sum_s \pi_s r^k_s = \mathsf{E}\mathbf{r}^k$. So when asset markets are complete, equal expected returns imply equal price, regardless of the risk or lack thereof. We conjecture that risk premia can only arise when markets are incomplete.