Micro I Midterm, March 3, 2020

1. The production function is $f(z_1, z_2) = z_1 + z_2^{1/2}$.

   a) Use the production function to define a production set in $\mathbb{R}^3$ in the usual way.

   b) For what price vectors (p_1, p_2, p_3) can profit be maximized?

   c) Find the profit function.

   Answer:

   a) The production set is $Y = \{(y_1, y_2, y_3) : y_1, y_2 \leq 0, \text{ and } y_3 \leq -y_1 + (-y_2)^{1/2}\}$

   b) Profit is $\mathbf{p}{\cdot}\mathbf{y} \leq p_1y_1 + p_2y_2 - p_3y_1 + p_3(-y_2)^{1/2} = (p_1 - p_3)y_1 + p_2y_2 + p_3(-y_2)^{1/2}$. Note that maximum profit is zero if $p_3 = 0$ because $y_1, y_2 \leq 0$. For the remainder of part (b), we assume $p_3 > 0$. Due to free disposal, we need only consider the cases with $\mathbf{p} \geq \mathbf{0}$.

   We first consider the use of good one as a input. Since $y_1 \leq 0$, we must have $p_1 \leq p_3$ to be able to maximize profit. The maximum occurs at $y_1 = 0$ if $p_1 < p_3$, and at any $y_1 \leq 0$ if $p_1 = p_3$.

   Next we use good two as an input. The first order condition is $p_2 = p_3(-y_2)^{-1/2}/2$, so $y_2 = -(p_3/2p_2)^2$. It follows that the restrictions on prices that avoid infinite profit are $p_1 \leq p_3$ and $\mathbf{p} \geq \mathbf{0}$ and $p_2 > 0$.

   c) When profit can be maximized, $(p_1 - p_3)y_1 = 0$, so profit is $p_2y_2 + p_3(-y_2)^{1/2} = -p_3^2/4p_2 + p_3^2/2p_2 = p_3^2/4p_2$. Thus

$$\pi(\mathbf{p}) = \begin{cases} 0 & \text{when } p_3 = 0,\ p_2 \geq 0, \text{ and } p_1 \leq p_3 \geq 0 \\ p_3^2/4p_2 & \text{when } \mathbf{p} \gg \mathbf{0},\ p_1 \leq p_3, \text{ and } p_2 > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

2. The utility function is

$$u(\mathbf{x}) = x_1 + \sqrt{x_1x_2} + \sqrt{x_1x_3} + \sqrt{x_2x_3}$$

   on $\mathbb{R}^3_{++}$. What can you say about the separability, or lack thereof, of u.

   Answer: Because u is increasing in each argument, it is weakly separable with respect to the partition of singletons, $\mathcal{P}_s = \big\{\{1\}, \{2\}, \{3\}\big\}$. To see if there is any other separability, we consider the marginal rates of substitution.

The marginal utilities are:

$$MU_1 = \frac{\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}}{2\sqrt{x_1}}.$$

$$MU_2 = \frac{\sqrt{x_1}+\sqrt{x_3}}{2\sqrt{x_2}}, \text{ and } MU_3 = \frac{\sqrt{x_1}+\sqrt{x_2}}{2\sqrt{x_3}}$$

This means the marginal rates of substitution are:

$$\begin{aligned} MRS_{12} &= \frac{\sqrt{x_2}}{\sqrt{x_1}} \cdot \frac{\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}}{\sqrt{x_1}+\sqrt{x_3}} \\ MRS_{13} &= \frac{\sqrt{x_3}}{\sqrt{x_1}} \cdot \frac{\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}}{\sqrt{x_1}+\sqrt{x_2}} \\ MRS_{23} &= \frac{\sqrt{x_3}}{\sqrt{x_2}} \cdot \frac{\sqrt{x_1}+\sqrt{x_3}}{\sqrt{x_1}+\sqrt{x_2}}. \end{aligned}$$

All three marginal rates of substitution are affected by all three variables through the second term of the products. This means that the functions is neither weakly nor strongly separable with respect to any other proper partition.

3. Consider a two-good, two-person exchange economy where $\mathfrak{X}_i = \mathbb{R}^2_+$, $\boldsymbol{\omega}^1 = (1,2)$, $\boldsymbol{\omega}^2 = (3,1)$, and utility is $u_1(\mathbf{x}) = x_1 + x_2$ and $u_2(\mathbf{x}) = -1/x_1 - 1/x_2$.

   Find all equilibrium prices and allocations.

Answer: The total endowment is $\boldsymbol{\omega} = (4,3)$. Consumer one considers the goods to be perfect substitutes, with $MRS_{12} = 1$. This consumer will only consume both goods if $p_1/p_2 = 1$, otherwise they consume the cheaper good.

We can use good one as numéraire, setting $p_2 = p$. Consumer one has income $1 + 2p$, and will spend it all on the cheaper good. If $p = 1$, then consumer one can choose any point on the budget line.

Consumer two's income is then $3+p$. Consumer two's first order conditions are $1/x_1^2 = \lambda$ and $1/x_2^2 = \lambda p$, which implies $x_2^2/x_1^2 = 1/p$. It follows that $x_2 = x_1/\sqrt{p}$. Using the budget constraint, $3 + p = x_1 + px_2 = x_1 + x_1\sqrt{p}$, so $x_1 = (3+p)/(1+\sqrt{p})$. Thus

$$\mathbf{x}^2(p) = \begin{pmatrix} \frac{3+p}{1+\sqrt{p}} \\ \frac{3+p}{p+\sqrt{p}} \end{pmatrix}$$

We first look for solutions where consumer one consumes both goods. That requires $p = 1$. Then $\hat{\mathbf{x}}^2 = (2,2)$. Setting supply equal to demand we obtain $\mathbf{x}^1 = (3,1)$. This will work regardless of the price level, so there are equilibria with prices (p, p) for $p > 0$ and $\mathbf{x}^1 = (2, 1)$, $\mathbf{x}^2 = (2, 2)$.

We now consider the case where $p < 1$. Good two is cheaper, so consumer one only consumes good two. Market clearing for good one requires $(3 + p)/(1 + \sqrt{p}) = 4$. That is, $p - 4\sqrt{p} - 1 = 0$. Solving the quadratic for $\sqrt{p}$, which must be non-negative (otherwise consumption of one good is negative), we find $\sqrt{p} = 2 + \sqrt{5}$, implying $p > 1$. This contradicts $p < 1$ and cannot be an equilibrium.

If $p > 1$, consumer one does not consume good two. It is only consumed by consumer two. By market clearing, $3 = (3 + p)/(p + \sqrt{p})$. Then $0 = 2p + 3\sqrt{p} - 3$. It follows that $\sqrt{p} = (-3 + \sqrt{33})/4$, which implies $p < 1$. This contradiction implies it is not an equilibrium.

The only equilibrium is $\mathbf{p} = (p, p)$, $\mathbf{x}^1 = (2, 1)$, and $\mathbf{x}^2 = (2, 2)$.

4. Homogenize the production function $f(z_1, z_2) = z_1 + z_1^{1/3} z_2^{1/2}$ by adding an extra factor of production.

Answer: Define $F(z_1, z_2, z_3) = z_3 f(z_1/z_3, z_2/z_3) = z_1 + z_3 z_1^{1/3} z_3^{-1/3} z_3^{1/2} z_3^{-1/2} = z_1 + z_1^{1/3} z_2^{1/2} z_3^{1/6}$ for $z_3 > 0$. Since the formula is continuous at $z_3 = 0$, we can use the same formula to define F in that case, where $F(z_1, z_2, 0) = z_1$.