Micro I Final, April 20, 2021

1. A two-good, two-person production economy has utility functions $u_i(\mathbf{x}) = \ln x_1 + \ln x_2$, endowments $\boldsymbol{\omega}^1 = (3,1)$, $\boldsymbol{\omega}^2 = (1,3)$, production set $Y = \{(y_1, y_2) : y_2 \leq 0, y_1 \leq 2(-y_2)^{1/2}\}$, and profit shares $\theta^1 = .3$, $\theta^2 = .7$.

   Find all Walrasian equilibrium prices.

   Answer: We have equal-weighted Cobb-Douglas utility for both consumers. If either price is zero, there will be infinite demand for that good. This can't happen in equilibrium, so both prices must be positive. We take good one as numéraire, so that prices are $\mathbf{p} = (1, p)$. Demand by consumer $i = 1, 2$ is

$$\mathbf{x}^i(p) = \frac{m^i}{2}\begin{pmatrix} 1 \\ 1/p \end{pmatrix}$$

   where $m^i = \mathbf{p}\boldsymbol{\omega}^i + \theta^i \pi(p)$ and π is the firms profit function.

   Now $\pi(p) = \max\{y_1 + py_2 : y_2 \leq 0, y_1 \leq 2(-y_2)^{1/2}\} = \max 2(-y_2)^{1/2} + py_2$. The first order conditions are $-(-y_2)^{-1/2} + p = 0$, so $-1/p^2 = y_2 \leq 0$ and $y_1 = 2/p$. It follows that supply is $\mathbf{y}(p) = (2/p, -1/p^2)$ and profit is $2/p - p/p^2 = 1/p$. Then $m^1 = 3 + p + .3/p$ and $m^2 = 1 + 3p + .7/p$. It follows that market demand is

$$\mathbf{x}^i(p) = (2 + 2p + 1/2p)\begin{pmatrix} 1 \\ 1/p \end{pmatrix}$$

   Market supply is

$$\boldsymbol{\omega} + \mathbf{y}(p) = \begin{pmatrix} 4 \\ 4 \end{pmatrix} + \begin{pmatrix} 2/p \\ -1/p^2 \end{pmatrix}$$

   Market clearing for good one implies $2 + 2p + 1/2p = 4 + 2/p$, so $p = 3/2$ (we have discarded the negative solution).

   It follows that the equilibrium price vectors are any positive scalar multiple of $\hat{\mathbf{p}} = (1, 3/2)$.

2. Suppose there are two goods and two consumers. The consumers have consumption set $\mathbb{R}^2_+$ and utility functions $u_1(\mathbf{x}^1) = \min\{x^1_1, x^1_2\}$ and $u_2(\mathbf{x}^2) = x^2_1 x^2_2$. Endowments are $\boldsymbol{\omega}^1 = (20, 0)$ and $\boldsymbol{\omega}^2 = (0, 4)$. Find all core allocations.

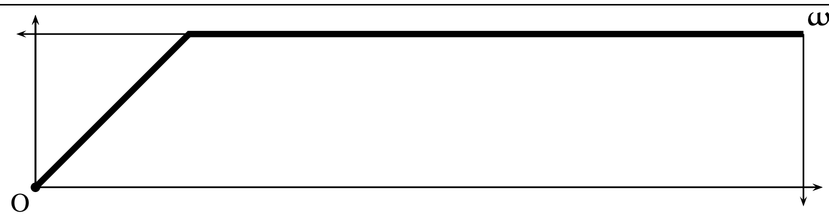
   Answer: Since there are two consumers, the core consists of all individually rational Pareto optima. In this case, $u_1(\boldsymbol{\omega}^1) = 0$ and $u_2(\boldsymbol{\omega}^2) = 0$. As this is the lower bound of utility in both cases, the individual rationality requirement is moot. We need only find the Pareto optima.

   Further, the maximum possible utility for consumer one is $u_1 = 4$. This occurs at any $\mathbf{x}^1 = (x_1, 4)$ with $x_1 \geq 4$. At any of those points, consumer two receives $\mathbf{x}^2 = (20 - x_1, 0)$ yielding utility $u_2(\mathbf{x}^2) = 0$. All of these points are both Pareto optimal and in the core.

   This reduces the problem to thinking about Pareto optima where consumer one has utility less than 4. If $x^1_1 > x^1_2$, we can give the amount $x^1_1 - x^1_2$ of good one to consumer two without reducing consumer one's utility. Since $u_1 = x^1_2 < 4$, this will give consumer two a positive amount of both goods, and more than before, increasing two's utility. It follows that no such allocation can be Pareto optimal. A similar argument shows that $x^1_1 < x^1_2$ is impossible at any Pareto optimum. It follows that the core allocations, which are also the Pareto optima allocations are as illustrated in the diagram.

3. A consumer has period utility $u(c) = \ln c$ for $c \in \mathbb{R}_+$ and discount factor δ, $0 < \delta < 1$. Overall wealth is $W > 0$. Suppose the budget constraint is $W \geq \sum p_t c_t$ where $p_t = (1 + r)^{-t}$.

   a) Find the consumption path that maximizes utility over the budget set. (20)



Core: The heavy line shows the core for problem 2.

b) Show that the consumption transversality condition is satisfied. (5)

c) For what values of r and δ does consumption grow over time? (5)

Answer:

a) The first-order conditions are $u'(c_t)/\delta u'(c_{t+1}) = (1+r)^{-t}/(1+r)^{-t-1} = 1+r$. Thus $c_{t+1} = \delta(1+r)c_t$, implying $c_t = [\delta(1+r)]^t c_0$. Applying the budget constraint, we obtain

$$W = \sum_{t=0}^{\infty}(1+r)^{-t}\delta^t c_0 = \sum \delta^t c_0 = c_0/(1-\delta).$$

It follows that $c_0 = (1-\delta)W$ and $c_t = [\delta(1+r)]^t(1-\delta)W$.

b) Now $p_t c_t = \delta^t(1-\delta)W \to 0$ as $t \to \infty$ because $0 < \delta < 1$. This establishes the consumption transversality condition.

c) Consumption grows over time if and only if $\delta(1+r) > 1$.

4. Consider a contingent commodity economy with 1 good in each of 3 states. The consumers have identical utility functions, $u(\mathbf{x}) =\leq x_1+\ln x_2+\ln x_3$ and endowments $\boldsymbol{\omega}^1 = (3,0,0)$, $\boldsymbol{\omega}^2 = (0,2,1)$, and $\boldsymbol{\omega}^3 = (0,1,2)$. The return matrix is

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Find a Radner equilibrium.

Answer: We start by finding the Arrow-Debreu equilibrium for this economy. The total endowment is $\boldsymbol{\omega} = (3,3,3)$. The identical Cobb-Douglas consumers will each consume a share of this. To calculate the shares, we need only consider their incomes. The marginal rates of substitution are identical, so the prices must be $\hat{\mathbf{p}} = (1,1,1)$ (or a positive multiple thereof). The incomes are identical, $m^i = 3$, so consumption is identical, $\hat{\mathbf{x}}^i = (1,1,1)$. We need to translate this into a Radner equilibrium.

To do that, we consider an Arrovian securities equilibrium. We can set the price of the Arrovian security in state s equal to 1. Let $\mathbf{q} = (1,1,1)$ be the Arrovian securities prices.

We now find the Radner asset prices using the relation $\hat{\mathbf{q}} = \mathbf{R}\mathbf{q} = (1,1,1)$. The Radner portfolios $\hat{\mathbf{z}}^i$ must satisfy $\hat{\mathbf{x}}^i = \boldsymbol{\omega}^i + \mathbf{R}\hat{\mathbf{z}}^i$. Thus $\hat{\mathbf{z}}^i = \mathbf{R}^{-1}(\hat{\mathbf{x}}^i - \boldsymbol{\omega}^i)$. Now

$$\mathbf{R}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Multiplying, we find $\hat{\mathbf{x}}^1 - \boldsymbol{\omega}^1 = (-2,1,1)$, $\hat{\mathbf{x}}^2 - \boldsymbol{\omega}^2 = (1,-1,0)$ and $\hat{\mathbf{x}}^3 - \boldsymbol{\omega}^3 = (1,0,-1)$. It follows that $\hat{\mathbf{z}}^1 = (-1/2,-1/2,1)$, $\hat{\mathbf{z}}^2 = (0,1/2,-1/2)$, and $\hat{\mathbf{z}}^3 = (1/2,0,-1/2)$.

5. Consider the function

$$F(x) = \begin{cases} 0 & \text{when } x < 0 \\ x/20 & \text{when } 0 \leq x < 10 \\ 1 & \text{when } 10 \leq x \end{cases}$$

a) Compute the expected value of F. (10)

b) Suppose $u(x) = \sqrt{x}$. Compute its expected value using F. (10)

c) Find the certainty equivalent of the lottery F under u. (10)

Answer:

a) The expectation of F is

$$\begin{aligned} \mathsf{E}F &= \int_{-\infty}^{+\infty} x \, dF(x) \\ &= \frac{1}{20}\int_0 10x \, dx + \frac{1}{2}(10) \\ &= 2.5 + 5 = 7.5 \end{aligned}$$

b) The expectation of u is

$$\begin{aligned} \mathsf{E}F &= \int_{-\infty}^{+\infty} x^{1/2} \, dF(x) \\ &= \frac{1}{20}\int_0 10x^{1/2} \, dx + \frac{1}{2}10^{1/2} \\ &= \frac{2}{3(20)}10^{3/2} + \frac{1}{2}10^{1/2} \\ &= \frac{5}{6}\sqrt{10} \end{aligned}$$

c) Let $\bar{c}$ be the certainty equivalent. It is defined by $u(\bar{c}) = \bar{c}^{1/2} = (5/6)\sqrt{10}$, so $\bar{c} = 250/36 = 125/18$.