# Homework #1

3.1.5 Let $u$ and $v$ be equivalent utility functions on $\mathbb{R}^L_+$.

*a*) Suppose $u$ and $v$ are both homogeneous of degree one. Show that $v = Cu$ for some $C > 0$.

*b*) Suppose $u$ and $v$ are homogeneous of degree $\beta$ and $\gamma$, respectively. Show that $u = Cv^{(\beta/\gamma)}$ for some $C > 0$.

**Answer:**

*a*) **Method 1:** Let $t > 0$ and $\mathbf{x} \in \mathbb{R}^L_+$ be arbitrary. Since the utility functions are equivalent, there is an increasing function $\varphi$ with $v(\mathbf{x}) = \varphi\big(u(\mathbf{x})\big)$. Now

$$\begin{aligned}\varphi\big(u(\mathbf{x})\big) &= tv(\mathbf{x})\\ &= v(t\mathbf{x})\\ &= \varphi\big(tu(\mathbf{x})\big)\\ &= \varphi\big(u(t\mathbf{x})\big).\end{aligned}$$

This implies that $\varphi$ itself is homogeneous of degree 1. Since $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(z) = Cz$ for some $C$. Moreover, since $\varphi$ is increasing, $C > 0$.

**Method 2:** When the functions are differentiable, an alternative method is to use Euler's formula. Since the utilities are equivalent, there is an increasing function $\varphi$ with $v(\mathbf{x}) = \varphi\big(u(\mathbf{x})\big)$. Differentiate to obtain $dv = \varphi' du$. Take the dot product with $\mathbf{x}$ and apply Euler's formula. This yields $v(\mathbf{x}) = dv{\cdot}\mathbf{x} = \varphi' du{\cdot}\mathbf{x} = \varphi'\big(u(\mathbf{x})\big)u(\mathbf{x})$. Since $u$ and $v$ are homogeneous of degree one in $\mathbf{x}$, we can conclude that $\varphi'$ is homogeneous of degree zero in $u$. This implies $\varphi'$ is some constant $C > 0$ and that $u(\mathbf{x}) = Cv(\mathbf{x})$.

*b*) Consider $\psi(\mathbf{x}) = [u(\mathbf{x})]^{1/\beta}$ and $\phi(\mathbf{x}) = [v(\mathbf{x})]^{1/\gamma}$. Apply part (a) to find $\zeta$ so that $\psi = \zeta\phi$. Then raise to the $\beta$ power to get $u = \zeta^\beta v^{(\beta/\gamma)}$. Set $C = \zeta^\beta$ to obtain the result. There is also an alternative method as in part (a).

3.2.1 List all 15 partitions of $\{1, 2, 3, 4\}$.

**Answer:** The 14 non-trivial partitions are

$$\begin{array}{lcl} \Big\{\{1\},\{2,3,4\}\Big\}, & & \Big\{\{1\},\{2\},\{3\},\{4\}\Big\}, \\ \Big\{\{2\},\{1,3,4\}\Big\}, & & \Big\{\{1,2\},\{3,4\}\Big\}, \\ \Big\{\{3\},\{1,2,4\}\Big\}, & & \Big\{\{1,3\},\{2,4\}\Big\}, \\ \Big\{\{4\},\{1,2,3\}\Big\}, & & \Big\{\{1,4\},\{2,3\}\Big\}, \\ \Big\{\{1\},\{2\},\{3,4\}\Big\}, & & \Big\{\{2\},\{3\},\{1,4\}\Big\}, \\ \Big\{\{1\},\{3\},\{2,4\}\Big\}, & & \Big\{\{2\},\{4\},\{1,3\}\Big\}, \\ \Big\{\{1\},\{4\},\{2,2\}\Big\}, & \text{and} & \Big\{\{3\},\{4\},\{1,2\}\Big\}. \end{array}$$

There is also the trivial partition $\Big\{\{1,2,3\}\Big\}$, making 15 in all.

3.3.7 Let $u(\mathbf{x}) = x_1^2+2x_1x_2x_3+x_2^2x_3^2$. Is $u$ separable on $\mathbb{R}^3_{++}$ relative to any partition? If so, is $u$ strongly separable relative to that partition? Does $u$ have a quasi-linear representation?

**Answer:** There are 4 non-trivial partitions to consider: $\Big\{\{1\},\{2\},\{3\}\Big\}$, $\Big\{\{1,2\},\{3\}\Big\}$, $\Big\{\{1,3\},\{2\}\Big\}$, and $\Big\{\{1\},\{2,3\}\Big\}$.

Since $u$ is increasing, it is separable relative to the partition $\Big\{\{1\},\{2\},\{3\}\Big\}$. For the rest, it will be helpful to calculate the marginal rates of substitution. They are $\text{MRS}_{12} = 1/x_3$, $\text{MRS}_{13} = 1/x_2$ and $\text{MRS}_{23} = x_3/x_2$. Since $\text{MRS}_{12}$ depends on $x_3$, which is not in $\{1\}\cup\{2\}$, $u$ is not strongly separable relative to $\Big\{\{1\},\{2\},\{3\}\Big\}$.

The same marginal rate of substitution also tells us that $u$ is not separable relative to $\Big\{\{1,2\},\{3\}\Big\}$. It is also not separable relative to $\Big\{\{1,3\},\{2\}\Big\}$ because $\text{MRS}_{13} = 1/x_2$. The lack of separability in the last two cases implies there are no quasi-linear representations relative to $x_2$ or $x_3$

That leaves $\Big\{\{1\},\{2,3\}\Big\}$, which passes the marginal rate of substitution test. In fact, $u(\mathbf{x}) = (x_1+x_2x_3)^2$, and is equivalent to $v(\mathbf{x}) = \sqrt{u(\mathbf{x})} = x_1+x_2x_3$. Regardless of the value of $x_1$, the ranking of $(x_1,x_2,x_3)$ and $(x_1,y_2,y_3)$ only depends on whether $x_2x_3$ is bigger than $y_2y_3$. These preferences are both separable and strongly separable relative to $\Big\{\{1\},\{2,3\}\Big\}$. Moreover, we have quasi-linear representation relative to $x_1$, $v(\mathbf{x}) = x_1+x_2x_3$.

3.3.12 Let $u$ be a continuous utility function and $\mathcal{P}$ be a partition of goods. Suppose there is a continuous function $v$ that is increasing in each argument and continuous subutility functions $u_P$ defined on each $\mathbf{x}_P$ with $u(\mathbf{x}) = v\big[\big(u_P(\mathbf{x}_P)\big)_{P\in\mathcal{P}}\big]$. Show that $u$ is weakly separable relative to $\mathcal{P}$.

**Answer:** We must show that $u$ induces an order on each commodity group $P \in \mathcal{P}$. Let $\mathcal{P} = \{P_i\}_{i=1}^K$. Now suppose $\mathbf{x} = (\mathbf{x}_{P_1}, \mathbf{x}_{\sim P_1}) \succsim \mathbf{x} = (\mathbf{y}_{P_1}, \mathbf{x}_{\sim P_1})$. Let $u_i = u_{P_i}(\mathbf{x}_{P_i})$. We can then write this preference in utility terms as

$$v(u_1, \ldots, u_K) \geq v(u_{P_1}(\mathbf{y}_{P_1}), u_2, \ldots, u_K)$$

Since $v$ is increasing in argument 1, this is equivalent to $u_1 = u_{P_1}(\mathbf{x}_{P_1}) \geq u_{P_1}(\mathbf{y}_{P_1})$. The actual values of $u_2, \ldots, u_K$ don't matter here, so $v$ induces an order on $P_1$. In fact, $\mathbf{x} \succsim_{P_1} \mathbf{y}$ if and only if $u_1 = u_{P_1}(\mathbf{x}_{P_1}) \geq u_{P_1}(\mathbf{y}_{P_1})$.

By repeating for every $P_i$, we find that $v$ induces an order on each $P_i$. In other words, it is weakly separable relative to $\mathcal{P} = \{P_i\}$.