Homework #2

7.2.2 Let $A = \{(x,y) : x \geq 0, y \leq \sqrt{x}\}$.

a) Show that A is a convex set.

b) Let $\mathbf{b} = (15/4, 3)$. Find the point in A that is closest to $\mathbf{b}$. That is, minimize $\|\mathbf{x}-\mathbf{b}\|^2$ where $\mathbf{x} = (x,y)$ obeys $y \leq \sqrt{x}$ and $x \geq 0$. Then find a $\mathbf{p}$ that separates $\mathbf{b}$ from A

c) Now consider $\mathbf{c} = (-3, -2)$. Find the point in A that is closest to $\mathbf{c}$. Then find a $\mathbf{p}$ that separates $\mathbf{c}$ from A

Answer:

a) The set A is the intersection of two convex sets, $\{(x,y) : x \geq 0\}$ and $\{(x,y) : y \leq \sqrt{x}\}$. The second set is an upper contour set of the concave function $-y + \sqrt{x}$.

b) We note that $\mathbf{b}(15/4, 3) \notin A$ since $3 > 2 > \sqrt{15/4}$. The Lagrangian is $\mathcal{L} = (x - 15/4)^2 + (y-3)^2 - \lambda(\sqrt{x} - y) - \mu x$. The first-order conditions are

$$
\begin{aligned}
0 &= 2(x - 15/4) - \lambda/2\sqrt{x} - \mu \\
0 &= 2(y-3) + \lambda.
\end{aligned}
$$

Clearly $x \neq 0$ does not solve the first-order conditions, so $\mu = 0$ by complementary slackness. It follows that $\lambda = (4x - 15)\sqrt{x}$. The constraint must bind since the other constraint doesn't bind, so $y = \sqrt{x}$. Combining the first-order conditions yields $2\sqrt{x} - 6 + (4x - 15)\sqrt{x} = 0$.

The substitution $z = \sqrt{x}$ yields $4z^3 - 13z - 6 = 0$. One solution is $z = 2$, corresponding to $x = z^2 = 4$ and $y = 2$. Dividing by $(z-2)$ yields $z^2 + 8z + 5 = 0$ or $x + 8\sqrt{x} + 5 = 0$. This cannot be solved by $x \geq 0$, so $(x,y) = (4,2)$ is the only solution. The separating vector is $\mathbf{p} = (15/4, 3) - (4, 2) = (-1/4, 1)$.

c) Because $\mathbf{c}$ is to the left of the negative y-axis, which is the boundary of A, the closest point in A will be directly right of $\mathbf{c} = (-3, -2)$. It is $(0, -2)$. The separating vector is then $\mathbf{c} - (0, -2) = (-3, 0)$.

7.4.1 Let $g(\mathbf{x}) = f(\mathbf{x} - \mathbf{a})$ for some $\mathbf{a} \in \mathbb{R}^L$. Show that $g^*(\mathbf{p}) = \mathbf{p}\cdot\mathbf{a} + f^*(\mathbf{p})$.

Answer: Here

$$\begin{aligned} g^*(\mathbf{p}) &= \inf\{\mathbf{p}\cdot\mathbf{x} - g(\mathbf{x})\} \\ &= \inf_{\mathbf{x}}\{\mathbf{p}\cdot\mathbf{x} - f(\mathbf{x}-\mathbf{a})\} \\ &= \inf_{\mathbf{x}}\{\mathbf{p}\cdot(\mathbf{x}+\mathbf{a}) - f(\mathbf{x})\} \\ &= \mathbf{p}\cdot\mathbf{a} + \inf_{\mathbf{x}}\{\mathbf{p}\cdot\mathbf{x} - f(\mathbf{x})\} \\ &= \mathbf{p}\cdot\mathbf{a} + f^*(\mathbf{p}). \end{aligned}$$

7.5.1 Let $f(x) = \sqrt{x}$ for $x \ge 0$ and $f(x) = -\infty$ for $x < 0$. Use the Legendre transformation to find $f^*(p)$ when possible.

Answer: We compute $df/dx = 1/2\sqrt{x}$ for $x > 0$. For $p > 0$, solve $p = 1/2\sqrt{x}$ for $x(p) = 1/4p^2$. Now $f^*(p) = px(p) - \sqrt{x(p)} = -1/4p$ for $p > 0$. The Legendre transformation does not work for $p \le 0$ since the derivative of f never takes those values. However, it is easy to use the definition of f^* to see that $f^*(p) = -\infty$ for $p \le 0$.

7.5.3 Let $f(x) = -e^{-x}$. Compute f^*.

Answer: We use the Legendre transformation. We solve $df/dx = e^{-x} = p$ for $x(p) = -\ln p$ whenever $p > 0$. Now $f^*(p) = px(p) - f\big(x(p)\big) = -p\ln p + p$. The definition of f^* tells us $f^*(0) = 0$ and $f^*(p) = -\infty$ for $p < 0$. In sum,

$$f^*(p) = \begin{cases} -p\ln p + p & \text{when } p > 0 \\ 0 & \text{when } p = 0 \\ -\infty & \text{when } p < 0. \end{cases}$$

8.1.2 Let $e(\mathbf{p}, \bar{u}) = \bar{u}(p_1 + 3p_2)$. Find the conjugate function $e^*_{\bar{u}}(\mathbf{x})$.

Answer: We must minimize $\mathbf{p}\cdot\mathbf{x} - e(\mathbf{p}, \bar{u}) = p_1(x_1 - \bar{u}) + p_2(x_2 - 3\bar{u})$ with respect to $\mathbf{p}$. Thus

$$e^*_{\bar{u}}(\mathbf{x}) = \begin{cases} 0 & \text{if } x_1 \ge \bar{u} \text{ and } x_2 \ge 3\bar{u} \\ -\infty & \text{otherwise.} \end{cases}$$

Although not part of the problem, it is easy to see that the associated quasiconcave utility function is $u(\mathbf{x}) = \min\{x_1, x_2/3\}$.