Homework #3

9.1.2 When possible, compute the income and price elasticities for:

a) Cobb-Douglas utility.

b) Leontief utility.

c) Linear utility.

Answer:

a) Let utility be $u(\mathbf{x}) = \prod_\ell x_\ell^{\gamma_\ell}$ with $\gamma_\ell > 0$ and $\sum_\ell \gamma_\ell = 1$. Demand is then $x_\ell(\mathbf{p}, m) = \gamma_\ell m / p_\ell$. The elasticities are $\varepsilon_{k\ell} = 0$ for $k \neq \ell$, $\varepsilon_{\ell\ell} = -1$, $\eta_\ell = 1$.

b) Let utility be $u(\mathbf{x}) = \min\{\alpha_\ell x_\ell\}$ where $\alpha_\ell > 0$. Demand is then

$$x_\ell(\mathbf{p}, m) = \frac{m}{\alpha_\ell}\left(\sum_k \frac{p_k}{\alpha_k}\right)^{-1}.$$

The elasticities are then $\eta_\ell = 1$ and

$$\varepsilon_{k\ell} = -\frac{p_\ell}{\alpha_\ell}\left(\sum_j \frac{p_j}{\alpha_j}\right)^{-1} = -s_\ell$$

where s_ℓ is the budget share of good ℓ.

c) Here $u(\mathbf{x}) = \sum_\ell \alpha_\ell x_\ell$. These elasticities can only be computed when the derivatives of demand exist. This requires that prices be such that there be a unique lowest price-alpha ratio (call it p_k/α_k). Demand is zero for the other other goods, and so all their elasticities are zero. Demand for k is m/p_k For k, the cross-price elasticity is zero, $\varepsilon_{kk} = -1$, and $\eta_k = 1$.

9.1.7 We know that Hicksian demand obeys the symmetry condition $\partial h_k/\partial p_\ell = \partial h_\ell/\partial p_k$. Suppose Marshallian demand obeys the similar symmetry condition $\partial x_k/\partial p_\ell = \partial x_\ell/\partial p_k$.

Show $\varepsilon_{k\ell} = \varepsilon_{\ell k}$ if and only if k and ℓ have equal income shares.

Answer: Consider $\varepsilon_{k\ell} = p_\ell/x_k \partial x_k/\partial p_\ell$. Using the symmetry condition, we find $\varepsilon_{k\ell} = p_\ell/x_k \partial x_\ell/\partial p_k$. Now this is $\varepsilon_{\ell k} = (p_k/x_\ell)\partial x_\ell/\partial p_k$ if and only if $p_\ell/x_k = p_k/x_\ell$, that is, if and only if $p_\ell x_\ell = p_k x_k$. Spending on both goods (and hence their income shares) must be equal.

9.1.8 Demand functions of the form $\ln x_\ell = a_\ell + \eta_\ell \ln m + \sum_k \varepsilon_{\ell k} \ln p_k$ are sometimes used to estimate demand.

a) Show that η_ℓ is the income elasticity of demand and $\varepsilon_{\ell k}$ are the cross-price elasticities of demand.

b) Show that Walras' Law requires $\eta_\ell = 1$ for all ℓ.

c) If $\eta_\ell = 1$, show that $\varepsilon_{\ell k} = 0$ for $\ell \neq k$ and $\varepsilon_{kk} = -1$ for all k.

Answer:

a) Direct calculation shows that the income elasticity of demand is $\frac{\partial \ln x_\ell}{\partial \ln m} = \eta_\ell$ and the cross-price elasticities are $\frac{\partial \ln x_\ell}{\partial \ln p_k} = \varepsilon_{\ell k}$.

b) Now $p_\ell x_\ell = p_\ell e^{a_\ell} \prod_k p_k^{\varepsilon_{\ell k}} m^{\eta_\ell}$ By Walras' Law, $\mathbf{p}\cdot\mathbf{x} = m$, so $m = \sum_\ell a_\ell(\mathbf{p}) m^{\eta_\ell}$. Because the various powers of m are linearly independent, the coefficients on the m^{η_ℓ} terms must sum to zero for $\eta_\ell \neq 1$. With this form, that can only happen if there are no such terms. In that case, the sum would be zero, which is impossible for $m > 0$. Thus all terms have $\eta_\ell = 1$.

c) Walras' Law now yields

$$\sum_\ell p_\ell e^{a_\ell} \prod_k p_k^{\varepsilon_{\ell k}} = 1.$$

Since the right-hand side does not depend on any prices, and since we are taking positive linear combinations, the left-hand side cannot depend on any of the prices either. That implies $\varepsilon_{kk} = -1$ for all k and $\varepsilon_{k\ell} = 0$ for $k \neq \ell$.

9.3.4 Suppose utility on $\mathbb{R}^L_+$ is defined by $u(\mathbf{x}) = \big(\min\{x_\ell/\alpha_\ell\}\big)^\gamma$ where $\gamma > 0$ and each $\alpha_\ell > 0$. Compute the Konüs true cost-of-living index for utility level $\bar{u} > 0$ and $\mathbf{p}^0, \mathbf{p}^1 > \mathbf{0}$.

Answer: Expenditure is minimized by consuming the minimum required to attain utility $\bar{u}$, so $x_\ell = \bar{u}^{1/\gamma}\alpha_\ell$. It follows that $e(\mathbf{p}, \bar{u}) = \bar{u}^{1/\gamma}\mathbf{p}\cdot\boldsymbol{\alpha}$. The true cost-of-living index is then $\mathbf{p}^1\cdot\boldsymbol{\alpha}/\mathbf{p}^0\cdot\boldsymbol{\alpha}$.

10.2.1 Suppose that a utility function u obeys $v = \varphi \circ u$ where φ is an increasing function and v is homogeneous of degree one.

a) Use equation (10.2.2) to write the equivalent variation in terms of utility changes and the old price.

b) Use equation (10.2.3) to write the equivalent variation in terms of the new utility and both prices.

c) Repeat parts (a) and (b) for the compensating variation.

d) How do the compensating and equivalent variations compare when utility is homothetic?

Answer:

a) $EV(\mathbf{p}^0, \mathbf{p}^1; m) = [\varphi(u^1) - \varphi(u^0)]\bar{e}(\mathbf{p}^0)$.

b) $EV(\mathbf{p}^0, \mathbf{p}^1; m) = \varphi(u^1)[\bar{e}(\mathbf{p}^0) - \bar{e}(\mathbf{p}^1)]$.

c) For the compensating variation, we have $CV(\mathbf{p}^0, \mathbf{p}^1; m) = [\varphi(u^1) - \varphi(u^0)]\bar{e}(\mathbf{p}^1)$ and $CV(\mathbf{p}^0, \mathbf{p}^1; m) = \varphi(u^0)[\bar{e}(\mathbf{p}^0) - \bar{e}(\mathbf{p}^1)]$.

d) We can use the forms $EV = [\varphi(u^1) - \varphi(u^0)]\bar{e}(\mathbf{p}^0)$ and $CV = [\varphi(u^1) - \varphi(u^0)]\bar{e}(\mathbf{p}^1)$. Then $EV, CV > 0$ if and only if $\varphi(u^1) - \varphi(u^0) > 0$. Using the other forms, we obtain $EV, CV > 0$ if and only if $\bar{e}(\mathbf{p}^0) > \bar{e}(\mathbf{p}^1)$. From this, we see that either both EV and CV are positive, or both are negative.

Thus if $EV, CV > 0$, $\bar{e}(\mathbf{p}^0) > \bar{e}(\mathbf{p}^1)$. We can then use the first form (as in part a) to see that $EV > CV > 0$. When $EV, CV < 0$, $\bar{e}(\mathbf{p}^0) < \bar{e}(\mathbf{p}^1)$. But then $\varphi(u^1) - \varphi(u^0) < 0$, so $0 > EV > CV$.