Homework #4

10.4.1 Show that the Laspeyres price and quantity indices are not compatible. **NB:** Although the formula suggests they are not compatible, that is not enough to show it. You must construct an example that shows they are not compatible.

Answer: For the Laspeyres case,

$$P_L\Big(\mathbf{p}^0,\mathbf{p}^1;\mathbf{x}^0\Big)\cdot Q_L\Big(\mathbf{x}^0,\mathbf{x}^1;\mathbf{p}^0\Big) = \frac{\mathbf{p}^1\cdot\mathbf{x}^0}{\mathbf{p}^0\cdot\mathbf{x}^0}\cdot\frac{\mathbf{p}^0\cdot\mathbf{x}^1}{\mathbf{p}^0\cdot\mathbf{x}^0} = \frac{(\mathbf{p}^1\cdot\mathbf{x}^0)(\mathbf{p}^0\cdot\mathbf{x}^1)}{(\mathbf{p}^0\cdot\mathbf{x}^0)^2},$$

which does not appear to be equal to the spending ratio $\mathbf{p}^1\cdot\mathbf{x}^1/\mathbf{p}^0\cdot\mathbf{x}^0$.

Although that formula looks unlikely to hold. We need an example to show that it need not. Suppose utility on $\mathbb{R}^2_+$ has the Cobb-Douglas form $u(\mathbf{x}) = x_1^{1/3}x_2^{2/3}$. Let $m^1 = m^2 = 1$, $\mathbf{p}^0 = (1,1)$ and $\mathbf{p}^1 = (1/3, 2/3)$. We calculate the demands: $\mathbf{x}^0 = (1/3, 2/3)$ and $\mathbf{x}^1 = (1,1)$. Now $\mathbf{p}^0\cdot\mathbf{x}^1 = 2$ and $\mathbf{p}^1\cdot\mathbf{x}^0 = 5/9$.

Then the spending ratio is 1, which is not equal to the product of the Laspeyres indices

$$\frac{(\mathbf{p}^1\cdot\mathbf{x}^0)(\mathbf{p}^0\cdot\mathbf{x}^1)}{(\mathbf{p}^0\cdot\mathbf{x}^0)^2} = \frac{10}{9}.$$

13.1.5

a) Show that conditions (T1)–(T7) are compatible with increasing returns to scale by finding an increasing returns to scale production function with a production set obeying (P1)–(P7).

b) Consider the case of one input and one output with any production function $f(z)$ that obeys $f' > 0$, $f'' > 0$ and $f(0) = 0$. Suppose $f \in \mathcal{C}^2$ and $\mathbf{p} \gg \mathbf{0}$. When is it possible to solve the producer's profit maximization problem?

Answer:

a) We show compatibility by constructing an appropriate example. Consider the increasing returns to scale production function $f(z) = z^2$. Define $Y = \{\mathbf{y} \in \mathbb{R}^2 : y_1 \le 0, y_2 \le f(-y_1)\}$. It is easy to verify that Y satisfies (T1)–(T7), and by construction, exhibits increasing returns to scale. In fact, any continuous increasing returns production function f with $f(\mathbf{0}) = \mathbf{0}$ would do the job.

b) Maximum profit at input z_1 is $p_2 f(z_1) - p_1 z_1$. If $f \in \mathcal{C}^2$, an interior solution for profit maximization will require that the first and second-order necessary conditions hold, that $p_2 f'(z_1) - p_1 = 0$ and that $p_2 f''(z_1) \le 0$. But increasing returns to scale requires $f'' > 0$ and the second-order necessary condition must fail. We are actually minimizing profit! In fact, increasing returns to scale implies that $f(z_1)/z_1$ is increasing, so larger production levels will eventually increase profit without bound. Profit cannot be maximized in such a case.

13.4.1 Suppose there are two inputs and one output with the Leontief production function $f(z_1, z_2) = \min(z_1, 3z_2)$. The output price is $p > 0$ and the input prices are $w_\ell > 0$.

a) Find all profit-maximizing net output vectors.

b) Calculate the profit function.

c) Show directly that the Law of Supply holds.

Answer: We will follow the convention of Example 13.1.1 and write $\mathbf{y} = (-z_1, -z_2, q)$ where q is the output and z_i are inputs. We write price as (w_1, w_2, p) and net output as $\mathbf{y} = (-z_1, -z_2, q)$.

a) Since both inputs are costly, cost is minimized when there is no excess of either input. Thus $z_1 = 3z_2 = q$ and cost is $w_1 q + w_2(q/3)$. Profit is then $(w_1, w_2, p)\cdot(-q, -q/3, q) = (p - w_1 - w_2/3)q$. There is no maximum when $p > w_1 + w_2/3$, and profit is maximized at $q = 0$ when $p < w_1 + w_2/3$. The net output correspondence is:

$$\mathbf{y}(\mathbf{p}) = \begin{cases} \{(-q, -q/3, q) : q \ge 0\} & \text{when } p = w_1 + w_2/3 \\ \{\mathbf{0}\} & \text{when } p < w_1 + w_2/3 \\ \text{undefined} & \text{when } p > w_1 + w_2/3. \end{cases}$$

b) Using part (a), we find the profit function is

$$\pi(p, \mathbf{w}) = \begin{cases} 0 & \text{for } p \le w_1 + w_2/3 \\ +\infty & \text{otherwise.} \end{cases}$$

c) For $p < w_1 + w_2/3$, the net supply vector $\mathbf{y}$ is zero. Then $\Delta\mathbf{p}\cdot\Delta\mathbf{y} = \mathbf{p}'\cdot\mathbf{y}' - \mathbf{p}\cdot\mathbf{y}'$. Now $\mathbf{p}'\cdot\mathbf{y}' = 0$ due to constant returns to scale, and $\mathbf{p}\cdot\mathbf{y}' \le \mathbf{p}\cdot\mathbf{y} = 0$. It follows that $\Delta\mathbf{p}\cdot\Delta\mathbf{y} \ge 0$.

Similarly, if $p' < w_1' + w_2'/3$, $\mathbf{y}' = \mathbf{0}$ and $\Delta\mathbf{p}\cdot\Delta\mathbf{y} = -\mathbf{p}'\cdot\mathbf{y} \ge 0$.

Now suppose $p = w_1 + w_2/3$ and $p' = w_1' + w_2'/3$. Then $\Delta\mathbf{p} = (w_1' - w_1, w_2' - w_2, p' - p)$. Net outputs are $\mathbf{y} = (-q, -q/3, q)$ and $\mathbf{y}' = (-q', -q'/3, q')$, so $\Delta\mathbf{y} = \Delta q(-1, -1/3, 1)$. Now $\Delta\mathbf{p}\cdot\Delta\mathbf{y} = (\Delta p - \Delta w_1 - \Delta w_2/3)\Delta q$. Then $\Delta p - \Delta w_1 - \Delta w_2/3 = 0$, so $\Delta\mathbf{p}\cdot\Delta\mathbf{y} = 0$.

Either way, the Law of Supply holds.

13.5.1 Suppose production is described by a concave production function $f\colon \mathbb{R}^L_+ \to \mathbb{R}_+$ that is $\mathcal{C}^1$ with $df \gg \mathbf{0}$ and obeys $f(\mathbf{0}) = 0$. Use f to describe the efficient net output vectors for this technology.

Answer: We can describe the production set as $Y = \{(q, -\mathbf{x}) : q \le f(\mathbf{x}), \mathbf{x} \ge \mathbf{0}\} \subset \mathbb{R}^{L+1}$ If $q < f(\mathbf{x})$, $(q, -\mathbf{x})$ is not efficient since $(q + \epsilon, -\mathbf{x}) \in Y$ for small $\epsilon > 0$. Thus $q = f(\mathbf{x})$ at all efficient points.

We next show that every point in Y that obeys $q = f(\mathbf{x})$ is efficient.

Suppose by way of contradiction that there is $(q, -\mathbf{x}) \in Y$ with $q = f(\mathbf{x})$ that is not efficient. Then there is $(q', -\mathbf{x}') \in Y$ with $q' \ge q$, $-\mathbf{x}' \ge -\mathbf{x}$ and $(q, \mathbf{x}) \ne (q', \mathbf{x}')$. Either $-\mathbf{x}' > \mathbf{x}$ or $q' > q$.

If $-\mathbf{x}' > -\mathbf{x}$, $\mathbf{x}' < \mathbf{x}$. Then since $df \gg \mathbf{0}$, it follows that $q' \le f(\mathbf{x}') < f(\mathbf{x}) = q$, contradicting $q' \ge q$. So it must be that $\mathbf{x} = \mathbf{x}'$. But then $q' \le f(\mathbf{x}') = f(\mathbf{x}) = q$, so $q' = q$. This contradicts that $(q, -\mathbf{x})$ is not efficient. We can conclude that all points $(q, -\mathbf{x}) \in Y$ with $q = f(\mathbf{x})$ are efficient.

In other words, the efficient points in Y are precisely the points $(q, -\mathbf{x})$ with $q = f(\mathbf{x})$.

14.2.1 Consider the production function from Example 14.2.1: $f(z) = 1 + z - 1/(1 + z)$ with associated production set $Y = \{(q, -z) : q \le f(z), z \ge 0\}$. Suppose the price vector is $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2_+$. For which values of $\mathbf{p}$ can profit be maximized? For which values is $\pi(\mathbf{p}) = +\infty$?

Answer: Profit is $h(z) = p_1\bigl(1 + z - 1/(1 + z)\bigr) - p_2 z$. The derivative is $h'(z) = p_1\bigl(1 + 1/(1 + z)^2\bigr) - p_2$.

If $1 < p_2/p_1 < 2$, we can solve $h'(z) = 0$ and profit is maximized when $z = -1 + \sqrt{p_1/(p_2 - p_1)}$.

The case $p_2/p_1 = 1$ was covered in Example 14.2.1. Here $\pi(\mathbf{p}) = p_1 = p_2$, but profit cannot be maximized.

If $p_2/p_1 < 1$, the derivative is bounded above zero. It follows that profit becomes arbitrarily large as $z \to \infty$, so $\pi(\mathbf{p}) = +\infty$.

Finally, if $p_2/p_1 \ge 2$, $h'/p_1 = 1 + 1/(1 + z)^2 - p_2/p_1 \le -1 + 1/(1 + z)^2 < 0$ for $z > 0$. In this case, profit decreases as z increases and the maximum profit is at $z = 0$. Putting the cases together, we find profit can be maximized if and only if $p_2/p_1 > 1$ while the profit function is finite for $p_2/p_1 \ge 1$.