Homework #6

20.2.3 Suppose there are two goods and two consumers, both with consumption set $\mathfrak{X} = \mathbb{R}^2_+$. The aggregate endowment is $\boldsymbol{\omega} = (100, 200)$. Consumer one has utility function $u_1(x_1, x_2) = (x_1x_2)^{1/3}$ and consumer two has utility $u_2(x_1, x_2) = x_1 + x_2$. Find the utility possibility set.

Answer: We first look for interior Pareto optima. These have $x_1^1/x_2^1 = \text{MRS}_{12}^1 = \text{MRS}_{12}^2 = 1$, so $x_1^1 = x_2^1$. Since $x_1^1 \leq 100$, these optima require $x_2^1 \leq 100$. We set $z = x_2^1$. The interior Pareto optima have $0 \leq z \leq 100$ with $u_1 = z^{2/3}$ and $u_2 = 300 - 2z$. Thus $u_2 = 300 - 2u_1^{3/2}$ for $0 \leq u_1 \leq 10^{4/3}$ or $u_1 = (150 - u_2/2)^{2/3}$ for $200 \leq u_2 \leq 300$.

If $100 < z \leq 200$, we are no longer in an interior solution, and consumer one will get all of good one. Then $u_1 = (100z)^{1/3}$ and $u_2 = 200 - z$, so $u_2 = 200 - (u_1)^3/100$ or $u_1 = \big(100(200 - u_2)\big)^{1/3}$ for $0 \leq u_2 \leq 100$.

Summing up, the Pareto frontier is given by

$$u_1 = \begin{cases} (20000 - 100u_2)^{1/3} & \text{when } 0 \leq u_2 \leq 100 \\ \left(\frac{300-u_2}{2}\right)^{2/3} & \text{when } 100 \leq u_2 \leq 300 \end{cases}$$

The utility possibility set is all points to the left and below the Pareto frontier.

20.2.5 Suppose there are two consumers and two goods in an exchange economy $\mathcal{E}$. Both consumers have identical utility $u(\mathbf{x}) = \sqrt{x_1x_2}$ with consumption set $\mathbb{R}^2_+$. Consumer one has endowment $\boldsymbol{\omega}^1 = (1, 3)$ and consumer two has endowment $\boldsymbol{\omega}^2 = (1, 5)$. The social welfare function is $W(\mathbf{u}) = u_1^\alpha u_2^{1-\alpha}$ for some α, $0 < \alpha < 1$.

a) Find all social welfare maxima.

b) For each social welfare maximum, find prices and transfers that make it a quasi-equilibrium with taxes and transfers. Is it a Walrasian equilibrium with taxes and transfers?

c) Is there an α where the transfers are zero?

Answer:

a) The aggregate endowment is $\boldsymbol{\omega} = (2, 8)$. By Example 19.2.5, the Pareto set is $\{(u_1, u_2) \in \mathbb{R}^2_+ : u_1 + u_2 = \sqrt{16} = 4\}$. We must maximize W over the Pareto set. Setting $\mathcal{L} = u_1^\alpha u_2^{1-\alpha} + \lambda(\sqrt{18} - u_1 - u_2) + \mu_1 u_1 + \mu_2 u_2$, we obtain the first-order conditions $\alpha u_1^{\alpha-1}u_2^{1-\alpha} + \mu_1 = \lambda$ and $(1-\alpha)u_1^\alpha u_2^{-1\alpha} + \mu_2 = \lambda$. If $u_1, u_2 > 0$, this yields $\alpha u_2 = (1-\alpha)u_1$ so $\alpha(4 - u_1) = (1-\alpha)u_1$. Thus $u_1 = 4\alpha$ and $u_2 = 4 - 4\alpha$. The corner solutions are the cases $\alpha = 0$ and $\alpha = 1$. By Example 19.2.5, the corresponding allocation of goods is $\mathbf{x}^1 = \alpha\boldsymbol{\omega}$ and $\mathbf{x}^2 = (1-\alpha)\boldsymbol{\omega}$.

b) By Example 19.2.5, the common $\text{MRS}_{12} = x_2/x_1 = \omega_2/\omega_1 = 8/2 = 4$. It follows that the equilibrium price ratio is $p_1/p_2 = \text{MRS}_{12} = 4$. Choosing good two as the numéraire we obtain $\mathbf{p} = (4,1)$. Aggregate wealth is $\mathbf{p}\cdot\boldsymbol{\omega} = 16$. The corresponding wealth levels are $m^1 = 16\alpha$ and $u_2 = 16(1-\alpha)$. As long as $0 < \alpha < 1$, both consumers will satisfy the cheaper point condition. By Corollary 20.3.3, this is a Walrasian equilibrium with taxes and transfers. Further, since one of the consumers will consume nothing when $\alpha = 0$ or $\alpha = 1$, those cases are easily seen to be price equilibria with taxes and transfers.

c) The transfer will be zero if consumer i's income from their endowment is m^i. For consumer one, that means $7 = 16\alpha$ and for consumer two, $9 = 16 - 16\alpha$. This only happens if $\alpha = 7/16$.

21.1.2 An exchange economy has two consumers with utility $u_1(\mathbf{x}^1) = (x_1^1 x_2^1 x_3^1)^{1/3}$ and $u_2(\mathbf{x}^2) = (x_1^2 x_2^2 x_3^2)^{1/3}$. Their endowments are $\boldsymbol{\omega}^1 = (2,1,1)$ and $\boldsymbol{\omega}^2 = (3,1,2)$. Find the core.

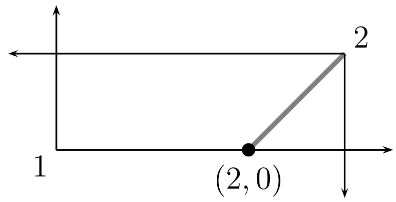
Answer: Here we have identical Cobb-Douglas preferences. The aggregate endowment is $\boldsymbol{\omega} = (5,2,3)$. Pareto optimal allocations will have the form $\mathbf{x}^1 = \alpha\boldsymbol{\omega}$ and $\mathbf{x}^2 = (1-\alpha)\boldsymbol{\omega}$ where $0 \le \alpha \le 1$.

We must also satisfy the requirement of individual rationality. This requires $u_1(\mathbf{x}^1) \ge u_1(2,1,1) = 2^{1/3}$ and $u_2(\mathbf{x}^2) \ge u_2(3,1,2) = 6^{1/3}$. Since $u_1(\alpha\boldsymbol{\omega}) = \alpha 30^{1/3}$ and $u_2\big((1-\alpha)\boldsymbol{\omega}\big) = (1-\alpha)30^{1/3}$, the core allocations are the Pareto optima that obey $(1/15)^{-1/3} \le \alpha \le 1-(1/5)^{-1/3}$. The range of α is fairly narrow, about 0.4055–0.4152.

21.1.7 An exchange economy has two consumers with utility $u_1(\mathbf{x}^1) = x_1^1 + 2x_2^1$ and $u_2(\mathbf{x}^2) = \min\{x_1^2, x_2^2\}$. Their endowments are $\boldsymbol{\omega}^1 = (1,0)$ and $\boldsymbol{\omega}^2 = (2,1)$. Find the core.

Answer: $\text{MRS}_{12}^1 = 1/2$ while MRS_{12}^2 can be interpreted as anything when $x_1^2 = x_2^2$. The interior Pareto optimal allocations run from the upper right corner of the box to $(2,0)$. The two boundary points are included. Note that $(x,0)$ for $x < 2$ is not Pareto optimal as $(2,0)$ is a Pareto improvement (consumer one is better off, consumer two is indifferent). The set of Pareto optima is the diagonal line in the diagram, $\{(x_1,x_2) : x_1 - 2 = x_2, x_1 \ge 2\}$.

Individual rationality requires $u_1(\mathbf{x}^1) \ge 1$ and $u_2(\mathbf{x}^2) \ge 1$. This leaves the single point $\mathbf{x}^1 = (2,0)$ $(\mathbf{x}^2 = (1,1))$.



22.3.1 Suppose there are 5 states, $s = 1, \ldots, 5$. Lottery L_1 has probabilities $(1/5, 1/10, 3/10, 1/5, 1/5)$ while lottery L_2 has probabilities $(1/5, 3/10, 1/10, 2/5, 0)$. Suppose $L_3 = (1/5, 1/5, 1/5, 3/10, 1/10)$.

a) Write L_3 as a compound lottery over L_1 and L_2.

b) Suppose u is an expected utility function with $u(L_1) = 1$ and $u(L_2) = 3$. Compute $u(L_3)$.

Answer:

a) Here $L_3 = \frac{1}{2}L_1 \oplus \frac{1}{2}L_2$.

b) Since $L_3 = \frac{1}{2}L_1 \oplus \frac{1}{2}L_2$, $Eu(L_3) = \frac{1}{2}Eu(L_1) + \frac{1}{2}Eu(L_2) = 2$.

22.3.3 Suppose $u(w) = w + \sqrt{w}$.

a) Find the expected utility of the lottery that pays 1 with probability p and 0 with probability $(1 - p)$.

b) Find the expected utility of the lottery that pays 3 with probability p and 1 with probability $(1 - p)$.

Answer:

a) Expected utility is $Eu(L) = pu(1) + (1 - p)u(0) = 2p$.

b) Expected utility is $Eu(L) = pu(3)+(1-p)u(1) = p(3+\sqrt{3})+2(1-p) = 2+p(1+\sqrt{3})$.