Homework #7

22.4.2 Suppose $F(x) = 0$ for $x \le 0$, $F(x) = 1/2$ for $0 \le x \le 1$, $F(x) = 3/4$ for $1 \le x < 2$, and $F(x) = 1$ for $x \ge 2$. Let $u(x) = x^2$. Compute $u(F)$.

Answer: The c.d.f. has a jump of 1/2 at 0, a jump of 1/4 at 1, and a jump of 1/4 at 2. This means that $u(F) = (1/2)u(0) + (1/4)u(1) + (1/4)u(2) = 1/4 + (1/4)4 = 5/4$.

22.4.3 Suppose a lottery has probability density $f(x) = 3x^2$ for $0 \le x \le 1$ and zero otherwise. Let $u(x) = x^{1/4}$. Compute the expected utility.

Answer: The expected utility is

$$\begin{aligned} Eu &= \int_0^1 u(x)f(x)\,dx = \int_0^1 x^{1/4}\,(3x^2)\,dF(x) \\ &= \int_0^1 3x^{9/4}\,dx = \left[\frac{12}{13}x^{13/4}\right]_0^1 \\ &= 12/13. \end{aligned}$$

23.1.2 Suppose F is uniformly distributed over $[1, 10]$. Calculate the expected utility $u(F)$ and the certainty equivalent $c(u, F)$ for the following utility functions.

a) $u(x) = 15x$.

b) $u(x) = \ln x$.

c) $u(x) = 20x - x^2$.

d) $u(x) = x^2$.

Answer: The distribution function is

$$F(x) = \begin{cases} 0 & \text{when } x \le 1 \\ (x-1)/9 & \text{when } 1 \le x \le 10 \\ 1 & \text{when } x \ge 10. \end{cases}$$

It follows that $dF = dx/9$ and that $EF = (1/9)\int_1^{10} x\,dx = (1/18)x^2\big|_1^{10} = 11/2$. We are now ready to tackle the four utility functions.

a) Here $Eu = (15/9)\int_1^{10} x\,dx = (5/6)x^2\big|_1^{10} = 165/2$. The certainty equivalent is found by solving $165/2 = u(c) = 15c$, so $c = 11/2$. The risk neutrality of linear utility means the certainty equivalent is the expected value.

b) Here $\mathbb{E}u = (1/9)\int_1^{10} \ln x\,dx = (1/9)\big[x\ln x - x\big]_1^{10} = (10/9)\ln 10 - 1$. The certainty equivalent is found by solving $u(c) = \ln c = (10/9)\ln 10 - 1$. So $c = \exp[(10/9)\ln 10 - 1] = 10^{10/9}/e$. Here $c \approx 4.75 < 11/2$. This consumer is risk averse.

c) Here $\mathbb{E}u = (1/9)\int_1^{10} 20x - x^2\,dx = (1/9)\big[10x^2 - x^3/3\big]_1^{10} = \frac{1}{9}[1000 - 1000/3 - (10 - 1/3)] = \frac{1}{9}[2001/3 - 10] = 73$. The certainty equivalent is found by solving $u(c) = 20c - c^2 = 73$. The solutions are $10 \pm 3\sqrt{3}$. Oddly enough, this utility function has two certainty equivalents, one where utility is increasing, the other where utility is decreasing. You will recall that some of the theorems regarding various notions of risk aversion required $u' > 0$. This problem shows what can happen otherwise.

d) Here $\mathbb{E}u = (1/9)\int_1^{10} x^2\,dx = (1/9)\big[x^3/3\big]_1^{10} = \frac{1}{27}[1000 - 1] = \frac{1}{27}[999] = 111/3 = 37$. The certainty equivalent is found by solving $u(c) = c^2 = 37$ so $c = \sqrt{37}$.

23.1.6 Suppose the distribution F is described by a probability density function $f(x) = (2\pi)^{-1/2}e^{-(x-\mu)^2/2}$ and utility is $u(x) = -e^{-\alpha x}$.

a) Calculate the risk premium for F.

b) How do changes in α affect the risk premium?

Answer:

a) We use the results of Problem 22.6.5 (included below) to find $\mathbb{E}F = \mu$ and $\mathbb{E}u = -e^{\alpha^2/2-\alpha\mu}$. We find the certainty equivalent by setting $-e^{-\alpha c} = -e^{\alpha^2/2-\alpha\mu}$, so $-\alpha c = \alpha^2/2 - \alpha\mu$. This yields the certainty equivalent $c(\alpha) = \mu - \alpha/2$. The risk premium is then $\mathbb{E}F - c(\alpha) = \alpha/2$.

b) Increases in α increase the risk premium.

22.6.5 Suppose the distribution F is described by the probability density function $f(x) = (2\pi)^{-1/2}e^{-(x-\mu)^2/2}$ defined on all of $\mathbb{R}$.

a) Compute the mean of F. You may use the fact that $\int_{\mathbb{R}} f(x)\,dx = 1$.

b) Compute the variance of F.

c) Let utility be $u(x) = -e^{-ax}$. Compute $\mathbb{E}u(F)$.

Answer:

a) We start with the mean $\mathbb{E}F$.

$$
\begin{aligned}
\mathbb{E}F &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} xe^{-(x-\mu)^2/2}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (x-\mu)e^{-(x-\mu)^2/2}\,dx + \frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-(x-\mu)^2/2}\,dx
\end{aligned}
$$

Since the last term is just the integral of f multiplied by μ, it evaluates to μ. The substitution $u = x - \mu$ converts the first term to $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} ue^{u^2/2}\,du$. As the integral of an odd function ($f(-u) = -f(u)$), it must be zero. Thus $EF = \mu$.

b) We turn our attention the variance, $\text{var}(F) = E\big((X - \mu)^2\big)$. Thus

$$\begin{aligned}
\text{var}(F) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x-\mu)^2 e^{-(x-\mu)^2/2}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z^2 e^{-z^2/2}\,dz \\
&= \frac{1}{\sqrt{2\pi}} \Big[-ze^{-z^2/2}\Big]_{-\infty}^{+\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-z^2/2}\,dz \\
&= 1.
\end{aligned}$$

Where we have integrated by parts using $v = z$ and $du = ze^{-z^2/2}\,dz$, so that $u = -e^{-z^2/2}$ and $dv = dz$.

c) We compute

$$\begin{aligned}
Eu(F) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha x} e^{-(x-\mu)^2/2}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(x^2 - 2(\mu-\alpha)x + \mu^2)/2}\,dx \\
&= \left(e^{\alpha^2/2 - \alpha\mu}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(x-(\mu-\alpha))^2/2}\,dx \\
&= e^{\alpha^2/2 - \alpha\mu}.
\end{aligned}$$

23.2.2 Suppose a risk-neutral firm is uncertain about its costs but not its price p. Specifically, suppose costs are $\alpha c(q)$ where α is a positive random variable and c is a $\mathcal{C}^2$ cost function obeying $c', c'' > 0$. Set up and solve the firm's problem. How does the solution with this type of uncertainty compare to the case where α is replaced by $E\alpha$, and thus known with certainty.

Answer: Suppose the firm is a price-taker with output price p. Expected profit from output q is $\int [pq - \alpha c(q)]\,dF(\alpha) = pq - (E\alpha)c(q)$. Since the firm is risk neutral, it maximizes expected profit (not the expected utility from profit). The first-order conditions are $p = (E\alpha)c'(q)$, which is precisely what a firm with certain cost of $(E\alpha)c(q)$ would do.

23.3.1 Let $u(x) = -(x+1)^{-2}$.

a) Compute the absolute risk aversion $r_A(x, u)$.

b) Compute the relative risk aversion $r_R(x, u)$.

c) Is absolute risk aversion increasing or decreasing in x? What about relative risk aversion?

Answer:

a) Here $u' = 2(x + 1)^{-3}$ and $u'' = -6(x + 1)^{-4}$, so $r_A = -u''/u' = 3/(x + 1)$.

b) Now $r_R = xr_A = 3x/(x + 1)$.

c) Absolute risk aversion is decreasing in x. We also find $d(r_R)/dx = 1/(x + 1)^2 > 0$, so relative risk aversion is increasing in x.