Homework #8

25.2.5 Suppose a consumer has discount factor $0 < \delta < 1$ and period utility function $u(c) = \ln c$. The consumer has wealth $W > 0$ and faces prices $p_t = p > 0$ for all times t. Find the optimal consumption path.

Answer: Since the marginal utility of zero consumption is infinite, consumption will always be positive (unless wealth is zero). The first-order conditions are $\delta u'(c_{t+1})/u'(c_t) = p_{t+1}/p_t$. This becomes $\delta c_t/c_{t+1} = p/p = 1$, so $c_{t+1} = \delta c_t$. It follows that $c_t = \delta^t c_0$. The budget constraint is $W = \sum_t pc_t = \sum_t p\delta^t c_0 = pc_0/(1 - \delta)$. Thus $c_0 = (1 - \delta)W/p$ and $c_t = (1 - \delta)\delta^t W/p$.

If you don't recall how to sum the infinite series, let $S = \sum_{t=0}^{\infty} \delta^t$. Then $1 + \delta S = \delta^0 + \sum_{t=1}^{\infty} \delta^t = S$. It follows that $S = (1 - \delta)^{-1}$. This requires $|\delta| < 1$ for the summation to converge.

25.2.8 A consumer has period utility $u(c) = \ln c$ and discount factor $\delta = 0.8$. Overall wealth is W. Suppose the budget constraint is $W \geq \sum p_t c_t$ where $p_t = (1.1)^{-t}$. Find the consumption path that maximizes utility over the budget set.

Answer: The first-order conditions are $u'(c_t)/\delta u'(c_{t+1}) = (1.1)^{-t}/(1.1)^{-t-1} = 1.1$. Thus $c_{t+1} = .88c_t$, implying $c_t = (.88)^t c_0$. Applying the budget constraint, we obtain $W = \sum_{t=0}^{\infty}(1.1)^{-t}(.88)^t c_0 = \sum(.8)^t c_0 = c_0/(1 - 0.8) = 5c_0$. It follows that $c_0 = W/5$ and $c_t = (.88)^t W/5$.

25.3.3 Consider the one-sector production technology described by a production function $f(k) = k^\gamma$ for $0 < \gamma < 1$. Suppose prices are $p_t = p(1 + r)^{-t}$ where $r > 0$ is the interest rate. Find k_t for every t.

Answer: The firm seeks to maximize $p_{t+1}f(k_t) - p_t k_t = p(1 + r)^{-t-1}[k_t^\gamma - (1 + r)k_t]$. The solution obeys the first-order condition $\gamma k_t^{\gamma-1} = 1 + r$. It follows that $k_t = [(1 + r)/\gamma)]^{1/(\gamma-1)}$ for all t.

25.5.1 Consider the following Ramsey problem. Suppose a consumer has utility $\sum_{t=0}^{\infty} \delta^t u(c_t)$ where $0 < \delta < 1$ and the felicity function is $u(c) = -1/c$. The production function is $f(a) = \beta a$ where $\beta > 1$. Suppose that there is an optimal path with $c_t > 0$ for every t.

a) Does consumption grow? If so, what is the growth factor.

b) Is the (consumption) transversality condition satisfied?

Answer:

a) The Euler equations are

$$\delta f'(a_t)u'(c_{t+1}) = u'(c_t)$$

yielding

$$\delta\beta/c_{t+1}^2 = 1/c_t^2.$$

It follows that $c_{t+1} = (\delta\beta)^{1/2}c_t$, implying that $c_t = (\delta\beta)^{t/2}c_0$.

Consumption grows by the growth factor $(\delta\beta)^{1/2}$ when $\delta\beta > 1$, is constant if $\delta\beta = 1$, and shrinks if $\delta\beta < 1$, all of which are possible.

b) The consumption transversality condition is that $p_t c_t \to 0$ where $p_t = \delta^t u'(c_t) = \delta^t/c_t^2$. Thus $p_t c_t = \delta^t/c_t$. Now $c_t = (\delta\beta)^{t/2}c_0$, so $p_t c_t = (\delta/\beta)^{t/2}/c_0$. Since $\delta < 1$ and $\beta > 1$, $\delta/\beta < 1$, implying that the transversality condition is satisfied.

27.2.1 Consider a contingent goods economy with two consumers and two states with two goods in each state. There is one firm with production set $Y = \{y \in \mathbb{R}^4 : y_{1,s} \le -2y_{2,s}, y_{2,s} \le 0$, for $s = 1, 2\}$, as in Example 27.2.2. Shares of the firm are $\theta_1^i = 1/2$.

Utility is $U_i(x) = \ln x_{1,1} + \ln x_{2,1} + 2\ln x_{1,2} + \ln x_{2,2}$ and the endowments are $\omega^i = \big((0,3),(0,3)\big)$.

Find the contingent commodity equilibrium.

Answer: As in Example 27.2.2 production is constant returns to scale. Since there is no endowment of good one in either state, and its price cannot be zero due to the Cobb-Douglas utility, the firm must produce good one in both states. This implies that $p_{2,s} = 2p_{1,s}$.

Letting $m^i = \mathbf{p}\cdot\omega^i$, we find that market demand is

$$x(\mathbf{p}) = \frac{m}{4}\left(\frac{1}{p_{1,1}}, \frac{1}{p_{2,1}}, \frac{2}{p_{1,2}}, \frac{1}{p_{2,2}}\right) = \frac{m}{4}\left(\frac{1}{p_{1,1}}, \frac{1}{2p_{1,1}}, \frac{2}{p_{1,2}}, \frac{1}{2p_{2,2}}\right)$$

where $m = m^1 + m^2$. It follows that $x_{1,1} = 2x_{2,1}$ and $x_{1,2} = 4x_{2,2}$.

Market clearing, combined with the fact that $y_{1,s} = -2y_{2,s}$ tells us that $-2y_{2,1} = 2(6 + y_{2,1})$ and $-2y_{2,2} = 4(6 + y_{2,2})$. It follows that $y_{2,1} = -3$ and $y_{2,2} = -4$. Production and consumption are then

$$y = \big((6,-3),(8,-4)\big) \quad \text{and} \quad x = \big((6,3),(8,2)\big).$$

We normalize prices so that $p_{1,1} = 1$. This yields equilibrium price vector

$$\mathbf{p} = \big((1,2),(3/2,3)\big).$$

Incomes are $m^i = 15$, and and $m = 30$. The individual consumption vectors are

$$x^1 = x^2 = \big((3, 3/2), (4, 1)\big),$$

completing the equilibrium.

27.3.4 Suppose a contingent goods exchange economy has two goods, two states, and two consumers. Each consumer has utility function $U_i(x) = \frac{1}{2}\ln(x_{11} + x_{21}) + \ln(x_{12} + x_{22})$. The endowments are $\big((1, 2), (2, 1)\big)$ and $\big((2, 1), (1, 2)\big)$.

a) Find all Arrow-Debreu equilibria.

b) Show that there is an Arrow-Debreu equilibrium with $\hat{x}^1 = \omega^1$ and $\hat{x}^2 = \omega^2$.

c) Why does neither consumer care that they are not fully insured?

Answer:

a) The marginal rate of substitution between goods one and two is 1 in either state. It follows that $p_{1,s} = p_{2,s}$ must be the same in equilibrium. We can then write $\mathbf{p} = \big((p_1, p_1), (p_2, p_2)\big)$.

Since each consumer's utility depends only on the total amount consumed in each state, not how consumption is distributed between the goods. Let $y_s^i = x_{1s}^i + x_{2s}^i$ denote the total consumed by consumer i in state s. We must maximize $v(\mathbf{y}^i) = \frac{1}{2}\ln y_1^i + \ln y_2^i$ under the budget constraint $p_1 y_1^i + p_2 y_2^i = 3(p_1 + p_2)$. Here $y_1^i = (p_1 + p_2)/p_1$ and $y_2^i = 2(p_1 + p_2)/p_2$. Market clearing requires $y_s^1 + y_s^2 = 6$.

Thus $\hat{\mathbf{p}} = \big((1, 1), (2, 2)\big)$, or a positive scalar multiple thereof. The consumption goods must be distributed so that $y_s^i = 3$. It follows that any $\mathbf{x}^i$ with $x_{11}^1 + x_{21}^1 = 3 = x_{12}^1 + x_{22}^1$, $0 \le x_{\ell s}^1 \le 3$ and $x_{\ell s}^2 = 3 - x_{\ell s}^1$ will be an equilibrium allocation.

b) Consumption by each consumer in each state is feasible (adds up to 3) , so by part (a), it is an Arrow-Debreu equilibrium for $\hat{\mathbf{p}} = (1, 1, 1, 1)$.

c) Although the consumers are not fully insured in the sense that the consumption vector is certain, they are fully insured in the weaker sense that the utility is the same in each state. This happens even though the consumers are risk averse (logarithmic utility).