Micro I Midterm, March 2, 2021

1. Consider an economy where there are two goods, two consumers and one firm. The consumers have Cobb-Douglas utility $u_i(\mathbf{x}^i) = (x_1^i)^{1/2}(x_2^i)^{1/2}$ and endowments $\boldsymbol{\omega}^1 = (3,1)$ and $\boldsymbol{\omega}^2 = (1,3)$, so the aggregate endowment is $\boldsymbol{\omega} = (4,4)$. Good two can be produced from good one. The production set is $Y = \{(y_1, y_2) : y_2 \le -y_1/2, y_1 \le 0\}$.

   Find all equilibrium prices and allocations.

   Answer: Profits will be maximized when $y_2 = -y_1/2$, so we must maximize

$$p_1 y_1 + p_2\left(-\frac{1}{2}y_1\right) = \left(p_1 - \frac{1}{2}p_2\right)y_1$$

   under the constraint $y_1 \le 0$. There is no maximum if $p_1 < p_2/2$ and so no equilibrium. If $p_1 > p_2/2$, the maximum is only at $y_1 = y_2 = 0$. If nothing is produced, the consumers have only the aggregate endowment to consume. It becomes an exchange economy. If we take good one as numéraire, the equilibrium is unique, $\hat{\mathbf{p}} = (1,1)$. The corresponding allocation is $\hat{\mathbf{x}}^1 = \hat{\mathbf{x}}^2 = (2,2)$. Since $1 = p_1 > p_2/2 = 1/2$, production must be $\hat{\mathbf{y}} = (0,0)$. We have found an equilibrium, but it doesn't use the technology at all.

   Are there any equilibria that do use the technology? For the technology to be productive, we must have $p_2 = 2p_1$. Again taking good one as numéraire, $\mathbf{p} = (1,2)$. Incomes are $m^1 = 5$ and $m^2 = 7$. Each consumer spends half of their income on each good, so demands are $\mathbf{x}^1 = (5/2, 5/4)$ and $\mathbf{x}^2 = (7/2, 7/4)$. Market demand is $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2 = (6,3)$. By market clearing,

$$\begin{pmatrix}6\\3\end{pmatrix} = \boldsymbol{\omega} + \mathbf{y} = \begin{pmatrix}4\\4\end{pmatrix} + \mathbf{y},$$

   so $\mathbf{y} = (2,-1)$. But $\mathbf{y} \notin Y$ as $y_1 > 0$. It follows that the only equilibrium has $\hat{\mathbf{p}} = (1,1)$, $\hat{\mathbf{x}}^1 = \hat{\mathbf{x}}^2 = (2,2)$, and $\hat{\mathbf{y}} = \mathbf{0}$. It does not use the production technology at all!

2. Suppose the production function is $f(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x}$ for some $\mathbf{a} \in \mathbb{R}^L$ with $\mathbf{a} \gg \mathbf{0}$. Find the true cost of production index for output $q > 0$ and factor prices $\mathbf{w} \gg \mathbf{0}$.

   Answer: Since the inputs are perfect subsitutes, the only inputs that will be used will be those with the highest value of a_ℓ/w_ℓ. Let ℓ be such a good. Then we minimize cost by using only good ℓ, with $x_\ell = q/a_\ell$. The cost of production is the $c(\mathbf{w}, q) = q(w_\ell/a_\ell) = q\min\{w_\ell/a_\ell\}$. It follows that the cost of production index is $\min\{w_\ell^1/a_\ell\}/\min\{w_\ell^0/a_\ell\}$.

3. A consumer has consumption set $\mathbb{R}_+^L$ and utility function is $u(\mathbf{x}) = \mathbf{b}\cdot\mathbf{x}$ for $\mathbf{b} \gg \mathbf{0}$.

   a) Find the distance function $\partial(\mathbf{x}, \bar{u})$ for any $\bar{u} \gg 0$.

b) Compute the normalized inverse Hicksian demands $\mathbf{a}_{\bar{u}}(\mathbf{x})$.

c) Compute the Antonelli matrix.

Answer: I overlooked the fact that the linear of u would cause problems. Sorry.

a) Using Corollary 8.3.6, we find $\mathfrak{d}(\mathbf{x}, \bar{u}) = u(\mathbf{x})/\bar{u} = \frac{1}{\bar{u}}\mathbf{b}\cdot\mathbf{x}$.

b) After Theorem 8.3.12, we found that $\mathfrak{d}(\mathbf{x}, \bar{u}) = \mathbf{a}_{\bar{u}}(\mathbf{x})\cdot\mathbf{x}$. However, due to linearity, this has multiple solutions. One is $\mathbf{a}_{\bar{u}}(\mathbf{x}) = \frac{1}{\bar{u}}\mathbf{b}$, another is $\frac{\mathbf{b}\cdot\mathbf{x}}{2\bar{u}}(1/x_2, 1/x_1)$. Convex combinations are also solutions.

c) From (b), $\mathbf{A} = \mathbf{0}$. Alternatively, use the fact that $\mathbf{A} = D^2\mathfrak{d}_{\bar{u}}$ to find $\mathbf{A} = \mathbf{0}$. However, this fact relies on a unique value of $\mathbf{a}_{\bar{u}}(\mathbf{x})$. If your solution had a $\frac{\mathbf{b}\cdot\mathbf{x}}{2\bar{u}}(1/x_2, 1/x_1)$ component, you would get a different answer.

4. The transformation function is

$$T(y_1, y_2) = \begin{cases} (y_1 - 1)(y_2 - 1) - 1 & \text{for } y_1, y_2, \le 1. \\ +\infty & \text{otherwise.} \end{cases}$$

Solve the profit maximization problem for $\mathbf{p} \gg \mathbf{0}$.

Answer: Form the Lagrangian

$$\mathcal{L} = p_1 y_1 + p_2 y_2 - \lambda T(y_1, y_2).$$

The first-order conditions are

$$\begin{aligned} p_1 &= \lambda(y_2 - 1) \\ p_2 &= \lambda(y_1 - 1). \end{aligned}$$

We combine them to obtain $p_1/p_2 = (y_2 - 1)/(y_1 - 1)$ or $p_1(y_1 - 1) = p_2(y_2 - 1)$. Then

$$y_2 = 1 + \frac{p_1}{p_2}(y_1 - 1) \le 1.$$

Profit maximization will occur on the boundary of the production set, so $T(y_1, y_2) = 0$, meaning that $y_2 = 1 + 1/(y_1 - 1)$. It follows that

$$\frac{p_1}{p_2}(y_1 - 1) = \frac{1}{y_1 - 1} \quad \text{or} \quad (y_1 - 1)^2 = \frac{p_2}{p_1}.$$

Then

$$y_1 = 1 - \sqrt{\frac{p_2}{p_1}}$$

as the positive square root would violate the condition $y_1 \leq 1$. Also,

$$y_2 = 1 - \sqrt{\frac{p_1}{p_2}}.$$

Notice that if $p2 > p_1$, good one is the input and good two the output, while if $p_1 < p_2$, good one is the output and good two is the input.

Finally, maximum profit is

$$p_1 y_1 + p_2 y_2 = p_1 + p_2 - 2\sqrt{p_1 p_2} = (\sqrt{p_1} - \sqrt{p_2})^2 \geq 0.$$

Profit is only zero when $p_1 = p_2$ implying the optimal net output is zero.