Micro I Final, April 28, 2022

1. Suppose there are two goods and two consumers. The consumers have consumption set $\mathbb{R}^2_+$ and utility functions $u_1(\mathbf{x}^1) = x^1_1 + x^1_2$ and $u_2(\mathbf{x}^2) = \min\{x^2_1, x^2_2\}$. Endowments are $\boldsymbol{\omega}^1 = (3, 0)$ and $\boldsymbol{\omega}^2 = (0, 2)$.

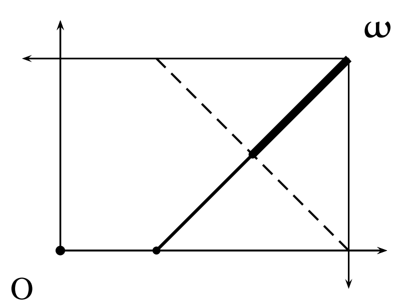
   a) Find all Pareto optima.

   b) Find all core allocations

   Answer:

   a) Consumer one has $\text{MRS}^1 = 1$. Since u_2 is concave but not always differentiable, it is enough if 1 is a supergradient of u_2. The supergradient is 0 on the horizontal portion of the indifference curve, and $+\infty$ on the vertical portion. Neither match. That leaves the points where $x^2_1 = x^2_2$ and the supergradient is $\mathbb{R}_+$. Since $1 \in \mathbb{R}_+$, we have a mutual tangency there. All feasible allocations with $x^2_1 = x^2_2$ are Pareto optimal.

   When we are on the lower side of the Edgeworth box, $x^2_2 = 2$, and consumer two's utility cannot be further increased. While consumer two is indifferent between allocations with $x^2_1 = 2$ and $x^2_1 \geq 2$, consumer one is not indifferent. Then $\mathbf{x}^1 = (1, 0)$ Pareto improves on anything of the form $\mathbf{x}^1 = (x^1_1, 0)$ for $x^1_1 \leq 1$. It follows that the diagonal line in the Edgeworth box below is the set of Pareto optimal allocations.



Pareto Optima and Core: The diagonal line shows the set of Pareto optima while the heavier portion shows the core.

   b) With two consumers, core allocations must be both Pareto optimal and individually rational. That is, for consumer one, $u_1(\mathbf{x}^1) = x^1_1 + x^1_2 \geq u_1(3, 0) = 3$ and for consumer two, $u_2(\mathbf{x}^2) = \min\{x^2_1, x^2_2\} \geq 0$. The second condition does not rule out any Pareto optima. For the first, we must have $x^1_2 + (1 + x^1_2) \geq 3$, or $x^1_2 \geq 1$. The heavy portion of the diagonal line shows the core, which lies on and above the dashed line where $u_1(\mathbf{x}^1) \geq 3$.

2. A consumer has Cobb-Douglas utility on $\mathbb{R}^2_+$ given by $u(x_1, x_2) = x_1 x_2$.
   a) Find the expenditure function $e(\mathbf{p}, \bar{u})$.
   b) Suppose prices rise from $(1, 3)$ to $(2, 4)$. Use the Konüs true cost of living index to compute the increase in the price level.

Answer:

a) As this is equal weighted Cobb-Douglas, half of expenditure will be on each good. So if e is the expenditure function, $x_1 = e/2p_1$ and $x_2 = e/2p_2$. It follows that

$$\bar{u} = u(x_1, x_1) = \frac{e^2}{4p_1p_2}$$

so

$$e(\mathbf{p}, \bar{u}) = 2\sqrt{p_1p_2\bar{u}}.$$

b) We calculate

$$\frac{e\big((2,4), \bar{u}\big)}{e\big((1,3), \bar{u}\big)} = \frac{\sqrt{8\bar{u}}}{\sqrt{3\bar{u}}} = \sqrt{\frac{8}{3}},$$

indicating the true cost of living index has risen by about 63%.

3. A consumer has period utility $u(c) = c^{1/2}$ for $c \in \mathbb{R}_+$ and discount factor δ, $0 < \delta < 1$. Overall wealth is $W > 0$. Suppose the budget constraint is $W \geq \sum p_t c_t$ where $p_t = (1+r)^{-2t}$.
   a) Find the consumption path that maximizes utility over the budget set. Don't forget to state any conditions needed to make sense of your calculations.
   b) Is the consumption transversality condition satisfied?
   c) For what values of r and δ does consumption grow over time?

Answer:

a) The first-order conditions are

$$\frac{u'(c_t)}{\delta u'(c_{t+1})} = \frac{(1+r)^{-2t}}{(1+r)^{-2t-2}} = (1+r)^2.$$

This can be rewritten

$$\frac{c_{t+1}^{1/2}}{\delta c_t^{1/2}} = (1+r)^2 \quad \text{or} \quad c_{t+1} = [\delta(1+r)^2]^2 c_t.$$

Thus $c_{t+1} = \delta^2(1+r)^4 c_t$, implying $c_t = [\delta^2(1+r)^4]^t c_0$.

Applying the budget constraint, we obtain

$$\begin{aligned} W &= \sum_{t=0}^{\infty} \delta^{2t}(1+r)^{4t}(1+r)^{-2t}c_0 \\ &= \sum_{t=0}^{\infty} [\delta^2(1+r)^2]^t c_0 \\ &= \frac{c_0}{1-\delta^2(1+r)^2} \end{aligned}$$

provided that $\delta(1+r) < 1$.

It follows that $c_0 = (1-\delta^2(1+r)^2)W$ and $c_t = \left[\delta(1+r)^2\right]^{2t}(1-\delta^2(1+r)^2)W$.

b) Now $p_t c_t = \left[\delta(1+r)\right]^{2t}(1-\delta^2(1+r)^2)W \to 0$ as $t \to \infty$ under the assumption that $0 < \delta(1+r) < 1$. This establishes the consumption transversality condition when $\delta(1+r) < 1$. It fails if $\delta(1+r) \geq 1$.

c) Consumption grows over time if and only if $\delta(1+r)^2 > 1$.

4. Suppose utility has constant absolute risk with $u(x) = -e^{-\alpha x}$ for $\alpha > 0$. Let $x > 0$ and consider the lottery L_x which pays $(G+x)$ with probability p and $(B+x)$ with probability $(1-p)$ where $0 < p < 1$. Here $G > B$.

How does the certainty equivalent c_x of lottery L_x change as x changes?

Answer: The lottery has expected utility

$$\begin{aligned} -pe^{-\alpha(x+G)} - (1-p)e^{-\alpha(x+B)} &= e^{-\alpha x}\left[-pe^{-\alpha G} - (1-p)e^{-\alpha B}\right] \\ &= e^{-\alpha x} Eu(L_0). \end{aligned}$$

Let c_0 be the certainty equivalent of L_0. The certainty equivalent of L_x, c_x is defined by

$$u(c_x) = Eu(L_x) = e^{-\alpha x} Eu(L_0) = e^{-\alpha x} u(c_0).$$

In other words,

$$-e^{-\alpha c_x} = -e^{-\alpha x} e^{-\alpha c_0}$$

so

$$\alpha c_x = \alpha x + \alpha c_0 \quad \text{or} \quad c_x = x + c_0$$

showing that c_x is linear in x.

5. Consider a contingent commodity economy with 1 good in each of 2 states. Consumer one has utility $u_1(\mathbf{x}^1) = \frac{1}{4}\ln x_1^1 + \frac{3}{4}\ln x_2^1$ and consumer two has utility $u_2(\mathbf{x}) = \frac{1}{3}\ln x_1^2 + \frac{2}{3}\ln x_2^2$. Their endowments are identical $\boldsymbol{\omega}^1 = \boldsymbol{\omega}^2 = (1,1)$.

   a) Find all Arrow-Debreu equilibrium price vectors.

   b) Is there full insurance in equilibrium?

Answer:

a) Since these are Cobb-Douglas utility functions, both contingent goods must have positive prices. We use the consumption good in state one as numéraire and write $\mathbf{p} = (1, p)$.

Income is $m^1 = 1 + p = m^2$. Demands given the above Cobb-Douglas utility functions are

$$\mathbf{x}^1(p) = (1+p)\begin{pmatrix} 1/4 \\ 3/4p \end{pmatrix} \quad \text{and} \quad \mathbf{x}^2(p) = (1+p)\begin{pmatrix} 1/3 \\ 2/3p \end{pmatrix}$$

for a market demand of

$$\mathbf{x}(p) = (1+p)\begin{pmatrix} 7/12 \\ 35/12p \end{pmatrix}.$$

Setting demand equal to supply in market one, we obtain

$$2 = (1+p)(7/12)$$

Then the normalized equilibrium price vector is

$$\hat{\mathbf{p}} = (1, 17/7).$$

The other equilibrium price vectors are positive scalar multiples of $\hat{\mathbf{p}}$.

b) In equilibrium, $\mathbf{x}^1 = (24/7)(1/4, 21/68) = (6/7, 18/7)$. Consumption differs in the two states, showing that consumer one is not fully insured.

Since the endowment is the same in both states, consumer two is not fully insured either.