Homework #1

3.1.1 Consider the constant elasticity of substitution (CES) utility function $u(x_1, x_2) = [\delta x_1^{-\rho} + (1 - \delta)x_2^{-\rho}]^{-\nu/\rho}$ where $0 < \delta < 1$, $\nu > 0$, $\rho > -1$ and $\rho \neq 0$.

a) Is this utility function homothetic?

b) Compute the limit of utility as $\rho \to -1$.

c) When $\rho \to 0$, show u converges to a Cobb-Douglas utility function. [Hint: consider $\ln u$.]

d) When $\rho \to \infty$ and $\nu = 1$, show u converges to a Leontief utility function. Then consider $\nu \neq 1$.

Answer:

a) Yes, in fact it is homogeneous of degree ν. This can be seen by computing $u(tx_1, tx_2) = [\delta(tx_1)^{-\rho} + (1 - \delta)(tx_2)^{-\rho}]^{-\nu/\rho} = [t^{-\rho}(\delta x_1^{-\rho} + (1 - \delta)x_2^{-\rho})]^{-\nu/\rho} = t^{\nu}u(x_1, x_2)$.

b) As $\rho \to -1$, the limit is $[\delta x_1 + (1 - \delta)x_2]^{\nu}$, which is equivalent to linear utility.

c) We consider $\ln u = -(\nu/\rho)\ln[\delta x_1^{-\rho} + (1 - \delta)x_2^{-\rho}]$. Both the numerator and denominator approach zero, so we can use l'Hôpital's rule. This yields

$$\begin{aligned}\lim_{\rho\to 0} \ln u &= -\nu \lim \frac{-\delta \ln x_1 x_1^{-\rho} - (1 - \delta)\ln x_2 x_2^{-\rho}}{\delta x_1^{-\rho} + (1 - \delta)x_2^{-\rho}} \\ &= \nu[\delta \ln x_1 + (1 - \delta)\ln x_2].\end{aligned}$$

It follows that u converges to $(x_1^{\delta}x_2^{1-\delta})^{\nu}$, which is equivalent to Cobb-Douglas utility.

d) If $x = 0$, $u(x) = 0$ and we're done. Otherwise, let $x = \max\{x_1, x_2\} > 0$. Then $u(x_1, x_2) = x[\delta(x_1/x)^{-\rho} + (1 - \delta)(x_2/x)^{-\rho}]^{-1/\rho}$. If $x_1 = x_2 = x$, $u(x) \to x = \max\{x_1, x_2\}$. If $x_2 < x$, $u(x) \to x \lim[\delta]^{-1/\rho} = x = \max\{x_1, x_2\}$. The case $x_1 < x$ is similar.

We get the same limit for all $\nu > 0$.

3.2.3 Suppose u is additive separable on $\mathbb{R}^L_+$.

a) True or False. For each good ℓ, marginal utility $MU_\ell = \partial u/\partial x_\ell$ is diminishing in x_ℓ.

b) If you answered True to (a), prove it. If you answered False, provide a counter-example.

Answer:

a) **False**.

b) Consider $u(x) = \sum_{\ell=1}^{L} x_\ell^2$. This is additive because $\partial^2 u/\partial x_k \partial x_\ell = 0$ for all $k \neq \ell$. The marginal utility of good ℓ is $MU_\ell = 2x_\ell$, which is increasing in x_ℓ.

Even if we require we also require convexity of u, so that $\partial^2 u/\partial x_k \partial x_\ell \leq 0$, there is still the borderline case $u(x) = \sum_{\ell=1}^{L} \alpha_\ell x_\ell$ where $\text{MU}_\ell = \alpha_\ell$ is constant.

3.3.1 In Example 3.3.5 we considered the utility function $u(x) = (x_1 + x_2)(x_2 + x_3)$. Show that this utility function is not additive separable.

Answer: The marginal utilities are $MU_1 = (x_2 + x_3)$, $MU_2 = (x_2 + x_3) + (x_1 + x_3)$, and $MU_3 = (x_1 + x_2)$. This yields marginal rates of substitutions

$$
\begin{aligned}
\text{MRS}_{21} &= 1 + \frac{x_1 + x_3}{x_2 + x_3} \\
\text{MRS}_{13} &= \frac{x_2 + x_3}{x_1 + x_2} \\
\text{MRS}_{23} &= 1 + \frac{x_2 + x_3}{x_1 + x_3}.
\end{aligned}
$$

We consider MRS_{21} instead of its inverse MRS_{12} due to its simpler form. It is evident that each of this marginal rates of substitution depend on consumption levels of all three goods. This means the utility is not additive separable.

3.4.2 Let $\succsim_{lex}$ be the lexicographic preference order on $\mathbb{R}^L_+$.

a) Does $\succsim_{lex}$ induce an order on every singleton?

b) Is $\succsim_{lex}$ completely separable?

c) We know that $\succsim_{lex}$ does not have an additive separable utility representation. Explain why this does not contradict Corollary 3.4.7 for $L \geq 3$.

Answer:

a) Yes, lexicographic preferences are weakly monotonic, so Proposition 3.3.4 shows that $\succsim_{lex}$ induces an order on every singleton.

b) If $L = 1$ or $L = 2$, $\succsim_{lex}$ is completely separable. When $L = 1$, we need only consider the commodity group $\{1\}$, where $\succsim_{lex}$ induces an order by definition. When $L = 2$, we need only consider the commodity groups $\{1\}$, $\{2\}$, and $\{1, 2\}$. The last group includes all commodities, so the induced order is $\succsim_{lex}$ itself. For the other two groups, part (a) shows that $\succsim_{lex}$ induces an order.

Now consider the case $L > 2$. If P is a commodity group, $(\mathbf{x}_P, \mathbf{x}_{\sim P}) \succsim_{lex} (\mathbf{y}_P, \mathbf{x}_{\sim P})$ if and only if $\mathbf{x}_P$ is lexicographically weakly preferred to $\mathbf{y}_P$ on $\mathbb{R}^P_+$ because all of the other coordinates remain fixed.

To make this clear, let $L = 3$, and consider the commodity group $P = \{1, 3\}$. Then $(\mathbf{x}_P, \mathbf{x}_{\sim P}) \succsim_{lex} (\mathbf{y}_P, \mathbf{x}_{\sim P})$ if and only if $(x_1, x_2, x_3) \succsim_{lex} (y_1, x_2, y_3)$, which happens if and only if either $x_1 > y_1$ or $x_1 = y_1$ and $x_3 \geq y_3$. But that is equivalent to $(x_1, z_2, x_3) \succsim_{lex} (y_1, z_2, y_3)$, or in other words, $(\mathbf{x}_P, \mathbf{z}_{\sim P}) \succsim_{lex} (\mathbf{y}_P, \mathbf{z}_{\sim P})$.

c) Since $L \geq 3$, we have at least three essential goods. Corollary 3.4.7 tells us that a completely separable preference order (such as $\succsim_{lex}$) has a continuous additive separable utility representation if and only if it is continuous. But $\succsim_{lex}$ is not continuous, and so the corollary does not apply. Indeed, we found in Example 2.1.3 that $\succsim_{lex}$ has no utility representation of any kind.