Homework #3

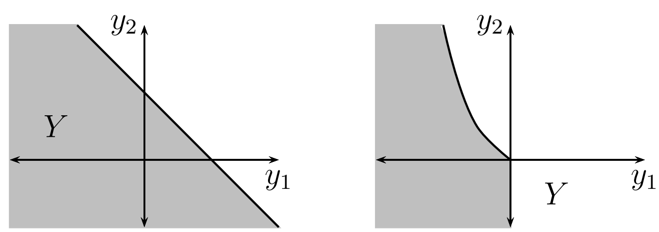
13.1.3 Find sets obeying the following conditions:

a) The set is non-empty, closed, and obeys inaction and free disposal, but fails the no free lunch condition.

b) The set is a production set that is additive, but not divisible.

c) The set is a production set that is not convex.

Answer: Many such examples are possible. We will use the two in Figure .



(a) **(b, c)**

F: igures for Problem 13.1.3.

a) The set $Y = \{\mathbf{y} \in \mathbb{R}^2 : y_1 + y_2 \leq 2\}$ is a set that fails the no free lunch condition (T3) because $(1, 1) \in Y$. It does obey the other conditions for a production set: non-emptiness, closure, inaction, and free disposal.

b) Consider the production function $f(z) = z^2$. This increasing returns to scale production function yields a production set that fails convexity. Let $Y = \{(-z, q) : q \leq z^2, z \leq 0\}$. Then $(-1, 1) \in Y$, but $\frac{1}{2}(-1, 1) + \frac{1}{2}(0, 0) = (-\frac{1}{2}, \frac{1}{2}) \notin Y$ because $\frac{1}{2} \not\leq (\frac{1}{2})^2 = \frac{1}{4}$. Since f is continuous and $f(0) = 0$, we showed in Example 13.1.1 that Y is a production set.

Now $1 \leq (-1)^2$, so $(-1, 1) \in Y$. But $1/2 \not\leq (-1/2)^2 = 1/4$, so $(-1/2, 1/2) \notin Y$. Since $(-1, 1) \in Y$, but $(-1/2, 1/2) \notin Y$, this set is not divisible.

It is additive because if $(-z_1, q_1) \in Y$ and $(-z_2, q_2) \in Y$, $q_1 + q_2 \leq z_1^2 + z_2^2 \leq (z_1 + z_2)^2$. From this it follows that $(q_1 + q_2, z_1 + z_2) \in Y$.

c) Using the production set from (b), we find that $(0, 0), (-1, 1) \in Y$, but $(-1/2, 1/2) \notin Y$, so Y is a production set that is not convex.

These are illustrated in the figures below.

13.1.5

a) Show that conditions (T1)–(T7) are compatible with increasing returns to scale by finding an increasing returns to scale production function with a production set obeying (T1)–(T7).

b) Consider the case of one input and one output with any production function $f(z)$ that obeys $f' > 0$, $f'' > 0$ and $f(0) = 0$. Suppose $f \in \mathcal{C}^2$ and $\mathbf{p} \gg \mathbf{0}$. When is it possible to solve the producer's profit maximization problem, $\max_x[pf(x) - wx]$?

Answer:

a) We show compatibility by constructing an appropriate example. Consider the increasing returns to scale production function $f(z) = z^2$. Define $Y = \{\mathbf{y} \in \mathbb{R}^2 : y_1 \leq 0, y_2 \leq f(-y_1)\}$. It is easy to verify that Y satisfies (T1)–(T7), and by construction, exhibits increasing returns to scale. In fact, any continuous increasing returns production function f with $f(\mathbf{0}) = \mathbf{0}$ would do the job.

b) Maximum profit at input z_1 is $p_2 f(z_1) - p_1 z_1$. If $f \in \mathcal{C}^2$, an interior solution for profit maximization will require that the first and second-order necessary conditions hold, that $p_2 f'(z_1) - p_1 = 0$ and that $p_2 f''(z_1) \leq 0$. But increasing returns to scale requires $f'' > 0$ and the second-order necessary condition must fail. We are actually minimizing profit! In fact, increasing returns to scale implies that $f(z_1)/z_1$ is increasing, so larger production levels will eventually increase profit without bound. Profit cannot be maximized in such a case.

13.5.1 Suppose production is described by a concave production function $f\colon \mathbb{R}^L_+ \to \mathbb{R}_+$ that is $\mathcal{C}^1$ with $df \gg \mathbf{0}$ and obeys $f(\mathbf{0}) = 0$. Use f to describe the efficient net output vectors for this technology.

Answer: We can describe the production set as $Y = \{(q, -\mathbf{x}) : q \leq f(\mathbf{x}), \mathbf{x} \geq \mathbf{0}\} \subset \mathbb{R}^{L+1}$ If $q < f(\mathbf{x})$, $(q, -\mathbf{x})$ is not efficient since $(q + \epsilon, -\mathbf{x}) \in Y$ for small $\epsilon > 0$. Thus $q = f(\mathbf{x})$ at all efficient points.

We next show that every point in Y that obeys $q = f(\mathbf{x})$ is efficient.

Suppose by way of contradiction that there is $(q, -\mathbf{x}) \in Y$ with $q = f(\mathbf{x})$ that is not efficient. Then there is $(q', -\mathbf{x}') \in Y$ with $q' \geq q$, $-\mathbf{x}' \geq -\mathbf{x}$ and $(q, \mathbf{x}) \neq (q', \mathbf{x}')$. Either $-\mathbf{x}' > \mathbf{x}$ or $q' > q$.

If $-\mathbf{x}' > -\mathbf{x}$, $\mathbf{x}' < \mathbf{x}$. Then since $df \gg \mathbf{0}$, it follows that $q' \leq f(\mathbf{x}') < f(\mathbf{x}) = q$, contradicting $q' \geq q$. So it must be that $\mathbf{x} = \mathbf{x}'$. But then $q' \leq f(\mathbf{x}') = f(\mathbf{x}) = q$, so $q' = q$. This contradicts that $(q, -\mathbf{x})$ is not efficient. We can conclude that all points $(q, -\mathbf{x}) \in Y$ with $q = f(\mathbf{x})$ are efficient.

In other words, the efficient points in Y are precisely the points $(q, -\mathbf{x})$ with $q = f(\mathbf{x})$.

14.2.2 The production set is $Y = \{(y_1, y_2) : y_1 \le 0, y_2 \le -(2y_1 + 1/y_1)\}$. Find all prices (p_1, p_2) where profit maximization is possible.

Answer: The production set allows free disposal, so profit-maximizing prices must be non-negative. Profit is then $p_1y_1+p_2y_2$. To maximize profit, we must set $y_2 = -2y_1-1/y_1$, yielding profit $p_1y_1 - 2p_2y_1 - p_2/y_1$. The first derivative is $(p_1 - 2p_2) + p_2/y_1^2$ and the second derivative is negative. If the first derivative is always positive, profit cannot be maximized. Thus $p_1 < 2p_2$ for profit maximization. If $p_1 < 2p_2$, profit is maximized at $y_1 = \big(p_2/(2p_2 - p_1)\big)^{1/2}$.

It follows that profit may be maximized whenever $\mathbf{p} \ge \mathbf{0}$ and $p_1 < 2p_2$.