Homework #4

14.3.2 Let $f\colon \mathbb{R}^L_+ \to \mathbb{R}_+$ be defined by $f(z) = a{\cdot}z$ where $a \gg 0$. Let F be the corresponding homogenized production function. Determine the formula for F.

Answer: When $z_{L+1} > 0$ we have $F(z, z_{L+1}) = z_{L+1}a{\cdot}(z/z_{L+1}) = a{\cdot}z = f(z)$. We take the limit as $z_{L+1} \to 0$ to find $F(z, 0)$, so $F(z, 0) = a{\cdot}z = f(z)$.

14.3.7 Suppose a convex production set Y is also a constant returns production set. Find the augmented production set $\hat{Y}$. What price do you expect the entrepreneurial factor to have? Why?

Answer: The augmented production set is given by $\hat{Y} = \text{cl}\{(zy, -z) : y \in Y, z > 0\}$. Note that if $y \in zY$ for any $z > 0$, then $y \in z'Y$ for all $z' > 0$. Then $\hat{Y} = \overline{(Y \times (-\mathbb{R}_{++})} = Y \times (-\mathbb{R}_+)$.

Since the entrepreneurial factor is not needed to allow a firm using $\hat{Y}$ to operate at any scale, it will not be demanded at any positive price, and we expect the price of the entrepreneurial factor to be zero.

15.3.1 Consider a two-person, two-good exchange economy. Endowments are $\omega^1 = (1, 2)$ and $\omega^2 = (1, 3)$ and utility is $u_1(x^1) = x^1_1 + 2x^1_2$ and $u_2(x^2) = \sqrt{x^2_1 x^2_2}$. Find all Walrasian equilibrium prices and allocations.

Answer: Since consumer two has Cobb-Douglas utility, both prices will have to be positive in equilibrium. We normalize so $(p_1, p_2) = (1, p)$. Then consumer incomes are $m^1 = 1 + 2p$ and $m^2 = 1 + 3p$. Consumer two has equally weighted Cobb-Douglas utility, so $x^2(p) = (1 + 3p)(1/2, 1/2p)$. Consumer one has linear utility, and will be at a corner unless $p = 2$. Consumer one's demand is:

$$x^1(p) = \begin{cases} (1 + 2p, 0) & \text{if } p > 2 \\ \{(x, (5 - x)/2) : 0 \le x \le 5\} & \text{if } p = 2 \\ (0, (1 + 2p)/p) & \text{if } p < 2. \end{cases}$$

where we have used the fact that $m^1 = 5$ when $p = 2$.

The aggregate endowment is $\omega = (2, 5)$. If $p > 2$, only consumer two demands good 2 and market clearing requires $(1 + 3p)/2p = 5$. Then $p = 1/7$, contradicting the fact that $p > 2$. If $p < 2$, only consumer one demands good 1 and market clearing requires $(1 + 3p)/2 = 2$. Then $p = 1$ is an equilibrium. The corresponding allocation is $x^1 = (0, 3)$ and $x^2 = (2, 2)$. Finally, if $p = 2$, $x^2 = (7/2, 7/4)$, meaning there is excess demand for good 1. This is not an equilibrium.

15.4.6 An economy has two consumers. Consumer one has Cobb-Douglas utility $u_1(x) = \sqrt{x_1 x_2}$. Consumer two has lexicographic preferences. That is $x \succsim_2 x'$ if and only if either $x_1 > x_1'$ or $x_1 = x_1'$ and $x_2 \geq x_2'$. Good one can be used to produce good two. This is described by the production set $Y = \{(y_1, y_2) : y_1 \leq 0 \text{ and } y_2 \leq -y_1\}$. Endowments of goods are $\omega^1 = (2, 0)$ and $\omega^2 = (3, 0)$. Consumer one owns 100% of the firm, while consumer two owns none of the firm. Find the equilibrium prices $\hat{p}$ and all equilibrium allocations $(\hat{x}^1, \hat{x}^2, \hat{y})$.

Answer: If the price of either good is zero, consumer one will not be able to maximize utility, so both prices must be positive. Moreover, there will be excess demand for good two unless it is produced. This means that good two must be produced in equilibrium. As the marginal rate of transformation is 1, $p_1 = p_2$. We can normalize prices so that $\hat{p} = (1, 1)$.

The income earned by the firm will be zero, so the only income is from the endowments. Using $\hat{p} = (1, 1)$, we calculate incomes $m_1 = 2$ and $m_3 = 3$. Consumer one has Cobb-Douglas utility, so demand is $x^1 = (m_1/2)(1/p_1, 1/p_2) = (1, 1)$.

Consumer two, with lexicographic utility, will not waste income on good two. Everything is spent on good 1 and demand is $x^2 = (m_2/p_1, 0) = (3, 0)$.

Thus $\hat{x}^1 = (1, 1)$ and $\hat{x}^2 = (3, 0)$. Market clearing requires $(5, 0) + \hat{y} = \omega + \hat{y} = \hat{x}^1 + \hat{x}^2 = (4, 1)$. It follows that $\hat{y} = (-1, 1)$, which is in Y.

To sum up, in equilibrium, $\hat{p} = (1, 1)$ or any positive multiple thereof, $\hat{x}^1 = (1, 1)$, $\hat{x}^2 = (3, 0)$ and $\hat{y} = (-1, 1)$. This is the only equilibrium allocation.