Homework #5

Problems 16.3.3, 16.7.2, 19.2.5, 19.2.6, and 20.4.4 are due on Tuesday, March 28.

16.3.3 Suppose an exchange economy has 2 consumers and 2 goods. Consumer one has endowment $\omega^1 = (1, 0)$. Utility is $u_1(x^1) = \sqrt{x_1^1}$. Consumer two has endowment $\omega^2 = (0, 1)$. Utility is $u_2(x^2) = \sqrt{x_1^2} + \sqrt{x_2^2}$. The consumption sets are $\mathfrak{X}_i = \mathbb{R}^2_+$.

a) Show that there is no competitive equilibrium.

b) Which hypotheses of the Equilibrium Existence Theorem are violated. Make sure you list all of them.

Answer:

a) We first calculate demands. Since consumer two's utility is strictly increasing in each good, prices must be strictly positive in equilibrium.

Since consumer one only values good one, $x^1(p) = (1, 0)$.

Consumer two has income p_2. Setting the marginal rate of substitution equal to the relative price, we find $p_1/p_2 = \sqrt{x_2^2}/\sqrt{x_1^2}$. It follows that

$$p_1 x_1^2 + p_2 x_2^2 = \frac{p_1 + p_2}{p_2} p_1 x_1^2.$$

Setting this equal to income, we find $x_1^2 = p_2^2/p_1(p_1 + p_2)$ and $x_2^2 = p_1/(p_1 + p_2)$.

In equilibrium, we must have $x_2^2 = 1$, implying $p_2 = 0$, which is impossible. There cannot be an equilibrium.

b) Two hypotheses are violated. (1) Consumer one's utility function is not strictly concave, although consumer two's utility function is strictly concave. (2) The endowments of both consumers are not strictly positive. Since we still have a demand function for consumer one, the former violation is unimportant. The latter is responsible for the non-existence of equilibrium.

16.7.2 Consumers 1 and 2 both have utility $u_i(x_1^i, x_2^i) = x_1^i x_2^i$. Their endowments are $\omega^1 = (1, 0)$ and $\omega^2 = (0, 1)$ while the firm shares are $\theta^1 = 1/3$, $\theta^2 = 2/3$. There is one firm with technology set $Y = \{(y_1, y_2) : y_1 \le 0, y_2 \le \sqrt{-y_1}\}$. Using good one as numéraire, solve the firm's problem to find net output $\mathbf{y}(\mathbf{p})$ and the profit function $\pi(\mathbf{p})$. Then solve the consumer's problem to obtain $\mathbf{x}(\mathbf{p})$. Finally, set excess demand to zero and solve for the equilibrium price $\mathbf{p}$.

Answer: Let p be the price of good two. Profit is then $py_2 + y_1$ and will be maximized when

$y_2 = \sqrt{-y_1}$. To find y_1, maximize $p\sqrt{-y_1} + y_1$. The first-order condition is $p/2\sqrt{-y_1} = 1$, yielding $-p^2/4 = y_1$ and $y_2 = p/2$. Maximized profit is $p^2/4$ and $\mathbf{y}(p) = (-p^2/4, p/2)$.

Consumer incomes are $m_1 = 1 + p^2/12$ and $m_2 = p + p^2/6$. Total income is $1 + p + p^2/4$. With equal weighted Cobb-Douglas preferences, we find that consumer demand is

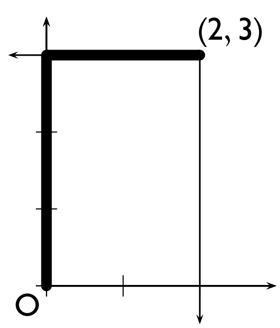
$$x(p) = \frac{1 + p + p^2/4}{2}(1, 1/p).$$

Now supply is $\omega + \mathbf{y}(p) = (1 - p^2/4, 1 + p/2)$. We equate supply and demand and solve for p. By Walras' Law, we only need clear one market so we focus on good one. Then $1 - p^2/4 = 1/2 + p/2 + p^2/8$. The solutions are $p = 2/3$ and $p = -2$. The latter makes no economic sense, so $p = 2/3$ is the correct solution. Thus $\mathbf{p} = (1, 2/3)$.

For the record, the equilibrium is $x^1 = \frac{14}{27}(1, 3/2)$, $x^2 = \frac{10}{27}(1, 3/2)$, $\mathbf{y} = (-1/9, 1/3)$ and $\mathbf{p} = (1, 2/3)$.

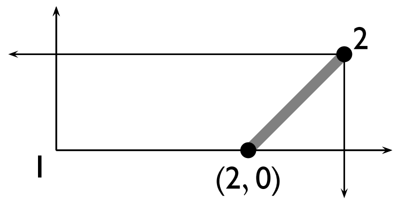
19.2.5 Suppose $u_1(x^1) = x_1^1 + 2x_2^1$, $u_2(x^2) = x_1^2 + x_2^2$, and $\omega = (2, 3)$. Find all Pareto optimal allocations.

Answer: Here $\text{MRS}_{12}^1 = 1/2 \neq \text{MRS}_{12}^2 = 1$. Since the marginal rates of substitution can never be equal, there are no interior Pareto optima. All of the Pareto optima lie on the boundary of the Edgeworth box. Since $\text{MRS}_{12}^1 < \text{MRS}_{12}^2$, good 1 preferentially goes to consumer 2 and good 2 goes to consumer 1 first. Thus the Pareto set is $\{((0, y), (2, 3 - y)) : 0 \leq y \leq 3\} \cup \{((x, 3), (2 - x, 0)) : 0 \leq x \leq 2\}$. In other words, the Pareto set is the left edge and top of the Edgeworth box as in the diagram below.



19.2.6 Suppose $u_1(x^1) = x_1^1 + 2x_2^1$ and $u_2(x^2) = \min\{x_1^2, x_2^2\}$, with $\omega = (3, 1)$. Find all Pareto optimal allocations. Be careful on the boundary.

Answer: Here $\text{MRS}_{12}^1 = 1/2$ while MRS_{12}^2 can be interpreted as anything when $x_1^2 = x_2^2$. The interior Pareto optimal allocations run from the upper right corner of the box to $(2, 0)$. Only two boundary points are included. Note that $(x, 0)$ for $x < 2$ is not Pareto optimal as $(2, 0)$ is a Pareto improvement (consumer 1 is better off, consumer 2 is indifferent). The set of Pareto optima is the heavy line in the diagram, $\{(x_1, x_2) : x_1 - 2 = x_2, x_1 \geq 2\}$.



20.4.4 Suppose there are two goods, aggregate endowment is $\omega = (2000, 0)$, and good 1 is used in production, with production function $f(z) = 75z^{1/3}$. Utility is $u_i(x^i) = x_1^i + 4x_2^i$. The social welfare function is $W(u) = u_1u_2$. Find the social welfare maximizing allocation.

Answer: The solution must be a Pareto optimum and good two must be produced. The marginal rate of substitution for both consumers is $MRS_{12} = 1/4$. The marginal rate of transformation is $25z^{-2/3} = 1/4$. It follows that $z = 1000$ and $y = (-1000, 750)$. This means that $y + \omega = (1000, 750)$ is available for consumption.

There is a 1-1 utility trade-off as we move goods from one consumer to another, so $u_1 + u_2 = u(1000, 750) = 4000$ with $u_1, u_2 \geq 0$. We must maximize u_1u_2 under this constraint. This is the same as maximizing Cobb-Douglas utility over a budget set with equal prices, thus $u_2/u_1 = 1$. It follows that $u_1 = u_2 = 2000$ at the optimum.

There are a number of ways to allocate goods to obtain the optimal utility levels. They all have the form $x^1 = (x_1^1, 500 - x_1^1/4)$, $x^2 = (1000 - x_1^1, 250 + x_1^1/4)$ with $0 \leq x_1^1 \leq 1000$.