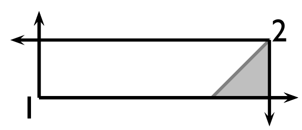
Homework #6

21.2.4 An exchange economy has two consumers with utility $u_1(x^1) = \max\{x_1^1, x_2^1\}$ and $u_2(x^2) = \min\{x_1^2, x_2^2\}$. Their endowments are $\omega^1 = (1,1)$ and $\omega^2 = (3,0)$. Find the core. Be careful, the max in u_1 is not a typo!

Answer: We start by finding the Pareto set. Note that $u_2 \leq \omega_2 = 1$. If $x_1^2 > x_2^2$, we can Pareto improve by giving the excess of good 1 to consumer one who will then have $u_1 = 4 - x_2^2$. Thus $x_1^2 \leq x_2^2$ at any Pareto optimum. Once we have given at least $4 - x_2^2$ of good 1 to consumer one, consumer one's utility cannot be further increased. It follows that the Pareto optima are $\{((4-x, 1-y), (x,y)) : 0 \leq y \leq 1, x \leq y\}$ yielding utility $u_1 = 4 - x$ and $u_2 = x$. (This is the same as problem 11.4.)

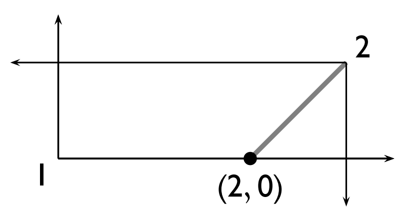
Individual rationality requires $u_1 = 4 - x \geq 1$ and $u_2 \geq 0$, both of which are satisfied at all the Pareto optima.



21.2.7 An exchange economy has two consumers with utility $u_1(x^1) = x_1^1 + 2x_2^1$ and $u_2(x^2) = \min\{x_1^2, x_2^2\}$. Their endowments are $\omega^1 = (1,0)$ and $\omega^2 = (2,1)$. Find the core.

Answer: $MRS_{12}^1 = 1/2$ while MRS_{12}^2 can be interpreted as anything when $x_1^2 = x_2^2$. The interior Pareto optimal allocations run from the upper right corner of the box to $(2,0)$. The two boundary points are included. Note that $(x,0)$ for $x < 2$ is not Pareto optimal as $(2,0)$ is a Pareto improvement (consumer one is better off, consumer two is indifferent). The set of Pareto optima is the diagonal line in the diagram, $\{(x_1, x_2) : x_1 - 2 = x_2, x_1 \geq 2\}$.

Individual rationality requires $u_1(x^1) \geq 1$ and $u_2(x^2) \geq 1$. This leaves the single point $x^1 = (2,0)$ ($x^2 = (1,1)$).



22.4.1 Suppose there are 5 states, $s = 1, \ldots, 5$. Lottery L_1 has probabilities $(1/5, 1/10, 3/10, 1/5, 1/5)$ while lottery L_2 has probabilities $(1/5, 3/10, 1/10, 2/5, 0)$. Suppose $L_3 = (1/5, 1/5, 1/5, 3/10, 1/10)$.

a) Write L_3 as a compound lottery over L_1 and L_2.

b) Suppose u is an expected utility function with $u(L_1) = 1$ and $u(L_2) = 3$. Compute $u(L_3)$.

Answer:

a) Here $L_3 = \frac{1}{2}L_1 \oplus \frac{1}{2}L_2$.

b) Since $L_3 = \frac{1}{2}L_1 \oplus \frac{1}{2}L_2$, $Eu(L_3) = \frac{1}{2}Eu(L_1) + \frac{1}{2}Eu(L_2) = 2$.

22.6.1 Suppose $F(x) = ax$ for $0 \leq x \leq 10$, $F(x) = 0$ for $x \leq 0$ and $F(x) = 10a$ for $x \geq 10$.

a) Find the value of a that makes F a c.d.f.

b) Let $u(x) = x^2$. Using the value of a from part (a), compute $u(F)$.

Answer:

a) Since a c.d.f. must have limit 1 at $+\infty$, a must be $1/10$.

b) Here

$$u(F) = \frac{1}{10}\int_0^{10} x^2 dx = \frac{1}{30}x^3\Big|_0^{10} = \frac{1}{30}(10^3 - 0) = \frac{100}{3}.$$

22.6.4 Suppose the random variable X has distribution F which is described by a probability density function $f(x) = \beta e^{-\alpha x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$ where $\alpha, \beta > 0$.

a) What must β be (in terms of α) for f to be a probability density function?

b) What is the mean of X?

c) Compute the variance of X.

d) Suppose $u(x) = x^2$. Find $Eu(X)$.

Answer:

a) This must obey $\beta\int_0^\infty e^{-\alpha x}\,dx = 1$ to be a probability density function. Evaluating the integral, we obtain β/α, so $\beta = \alpha$.

b) The mean is $\mu = \int_0^\infty \alpha x e^{-\alpha x}\,dx$. This may be integrated by parts to find $\mu = \alpha^{-1}[-ue^{-u} - e^{-u}]_0^{+\infty} = 1/\alpha$.

c) The variance is $\text{var}(X) = E(X^2) - \mu^2 = E(X^2) - \alpha^{-2}$. Now

$$\begin{aligned} E(X^2) &= \alpha\int_0^\infty x^2 e^{-\alpha x}\,dx \\ &= \alpha^{-2}\int_0^\infty u^2 e^{-u}\,du \\ &= \alpha^{-2}\left[-u^2e^{-u} - 2ue^{-u} - 2e^{-u}\right]_0^\infty = 2\alpha^{-2}. \end{aligned}$$

So $\text{var}(X) = E(X^2) - \mu^2 = 1/\alpha^2$.

d) This is just $E(X^2)$ from part (c), which is $Eu(F) = 2/\alpha^2$.