Homework #7

23.1.4 Suppose F is uniformly distributed over $[1, a]$ for $a > 1$. Calculate the risk premium for the following utility functions.

a) $u(x) = x^3$.

b) $u(x) = x^{1/2}$.

c) $u(x) = \ln x$.

Answer: Note that the probability density for all three parts is $1/(a-1)$, and that this density has expected value of $(1+a)/2$.

a) The expected utility is $(\int_1^a x^3\,dx)/(a-1) = (a^4-1)/4(a-1)$. This has certainty equivalent $[(a^4-1)/4(a-1)]^{1/3}$, so the risk premium is $R(u,F) = (1+a)/2 - [(a^4-1)/4(a-1)]^{1/3}$.

b) The expected utility is $(\int_1^a x^{1/2}\,dx)/(a-1) = 2(a^{3/2}-1)/3(a-1)$. This has certainty equivalent $[2(a^{3/2}-1)/3(a-1)]^2$, so the risk premium is $R(u,F) = (1+a)/2 - [2(a^{3/2} - 1)/3(a-1)]^2$.

c) The expected utility is $(\int_1^a \ln x\,dx)/(a-1) = (a\ln a - a + 1)/(a-1)$. This has certainty equivalent $\exp[(a\ln a - a + 1)/(a-1)$, so the risk premium is $R(u,F) = (1+a)/2 - \exp\left[(a\ln a - a + 1)/(a-1)\right]$.

23.3.4 A firm with cost function $C(q) = 2q$ faces an uncertain price p. The firm chooses the production level q before the price is revealed.

a) Write expected profit in terms of expected price Ep and the chosen quantity of output q.

b) Suppose the firm maximizes expected profit. At what expected prices can the firm maximize expected profit?

c) When expected profit can be maximized, what quantity q maximizes expected profit?

d) Suppose the owner of the firm maximizes *expected utility* of income where $u(m) = \ln m$. The owner has \$10 of other income in addition to profit income. There is 50% chance that the price is \$1 and 50% chance that the price is \$3. Find the profit maximizing quantity q.

Answer:

a) Profit is $\pi(q) = pq - C(q) = pq - 2q$. Expected profit is $E\big(\pi(q)\big) = E(pq-2q) = q(Ep-2)$.

b) If $Ep > 2$, expected profit cannot be maximized because $\lim_{q\to\infty} E\pi = +\infty$. If $Ep = 2$, profit is also zero, which is the maximum. If $Ep < 2$, profit is negative for $q > 0$, and maximum profit is zero (at $q = 0$).

c) When $\mathrm{E}p = 2$, any $q \geq 0$ maximizes profit. When $\mathrm{E}p < 2$, $q = 0$ maximizes profit.

d) The owner's income is $10 + (p - 2)q$. The expected utility of income is

$$\mathrm{E}u = \frac{1}{2}\ln(10 + q) + \frac{1}{2}\ln(10 - q).$$

We differentiate with respect to q to find the first-order conditions.

$$\frac{1}{10 + q} - \frac{1}{10 - q} = 0.$$

It follows that expected profit is maximized when $q = 0$.

25.2.5 Suppose a consumer has discount factor $0 < \delta < 1$ and period utility function $u(c) = \ln c$. The consumer has wealth $W > 0$ and faces prices $p_t = p > 0$ for all times t. Find the optimal consumption path.

Answer: Since the marginal utility of zero consumption is infinite, consumption will always be positive (unless wealth is zero). The first-order conditions are $\delta u'(c_{t+1})/u'(c_t) = p_{t+1}/p_t$. This becomes $\delta c_t/c_{t+1} = p/p = 1$, so $c_{t+1} = \delta c_t$. It follows that $c_t = \delta^t c_0$. The budget constraint is $W = \sum_t pc_t = \sum_t p\delta^t c_0 = pc_0/(1 - \delta)$. Thus $c_0 = (1 - \delta)W/p$ and $c_t = (1 - \delta)\delta^t W/p$.

If you don't recall how to sum the infinite series, let $S = \sum_{t=0}^{\infty} \delta^t$. Then $1 + \delta S = \delta^0 + \sum_{t=1}^{\infty} \delta^t = S$. It follows that $S = (1 - \delta)^{-1}$. This requires $|\delta| < 1$ for the summation to converge.

25.2.8 A consumer has period utility $u(c) = \ln c$ and discount factor $\delta = 0.8$. Overall wealth is W. Suppose the budget constraint is $W \geq \sum p_t c_t$ where $p_t = (1.1)^{-t}$. Find the consumption path that maximizes utility over the budget set.

Answer: The first-order conditions are $u'(c_t)/\delta u'(c_{t+1}) = (1.1)^{-t}/(1.1)^{-t-1} = 1.1$. Thus $c_{t+1} = .88c_t$, implying $c_t = (.88)^t c_0$. Applying the budget constraint, we obtain $W = \sum_{t=0}^{\infty}(1.1)^{-t}(.88)^t c_0 = \sum(.8)^t c_0 = c_0/(1 - 0.8) = 5c_0$. It follows that $c_0 = W/5$ and $c_t = (.88)^t W/5$.

25.3.1 Show that the storage technology in Example 25.3.1 obeys all seven conditions for an input-output technology set.

Answer: Here $T = \{(a, b) \in \mathbb{R}^{2L}_+ : b \leq a\}$. This is clearly a closed and convex set (1) with $(0, 0) \in T$ (inaction). If $(0, b) \in T$, $0 \leq b \leq 0$, so $b = 0$ (no free lunch).

Now suppose $a' \geq a$ and $0 \leq b'b$ with $(a, b) \in T$. Then $b \leq a \leq a'$, so $b' \leq a'$, showing that $(a', b') \in T$ (free disposal).

Finally, let $a = (1, \ldots, 1)$ Then $(a, a) \in T$, showing that productivity is satisfied.

25.5.1 Consider the following Ramsey problem. Suppose a consumer has utility $\sum_{t=0}^{\infty} \delta^t u(c_t)$ where $0 < \delta < 1$ and the felicity function is $u(c) = -1/c$. The production function is $f(a) = \beta a$ where $\beta > 1$. Suppose that there is an optimal path with $c_t > 0$ for every t.

a) Does consumption grow? If so, what is the growth factor.

b) Is the (consumption) transversality condition satisfied?

Answer:

a) The Euler equations are

$$\delta f'(a_t)u'(c_{t+1}) = u'(c_t)$$

yielding

$$\delta\beta/c_{t+1}^2 = 1/c_t^2.$$

It follows that $c_{t+1} = (\delta\beta)^{1/2}c_t$, implying that $c_t = (\delta\beta)^{t/2}c_0$.

Consumption grows by the growth factor $(\delta\beta)^{1/2}$ when $\delta\beta > 1$, is constant if $\delta\beta = 1$, and shrinks if $\delta\beta < 1$, all of which are possible.

b) The consumption transversality condition is that $p_t c_t \to 0$ where $p_t = \delta^t u'(c_t) = \delta^t/c_t^2$. Thus $p_t c_t = \delta^t/c_t$. Now $c_t = (\delta\beta)^{t/2}c_0$, so $p_t c_t = (\delta/\beta)^{t/2}/c_0$. Since $\delta < 1$ and $\beta > 1$, $\delta/\beta < 1$, implying that the transversality condition is satisfied.