Micro I Midterm, March 8, 2022

1. Consider an exchange economy where there are two goods, two consumers and one firm. Consumer one has utility $u_1(\mathbf{x}) = \ln x_1 + \ln x_2$ and endowment $\boldsymbol{\omega}^1 = (3, 2)$. Consumer two has utility $u_2(\mathbf{x}) = 2\ln x_1 + \ln x_2$ and endowment $(2, 3)$.

   Find all equilibrium prices and allocations.

   Answer: If either good had price zero, demand would be infinite, so both prices must be positive in equilibrium. We can use either good as numéraire. Set $\mathbf{p} = (1, p)$. Then consumer one has income $m^1 = \mathbf{p}\cdot\boldsymbol{\omega}^1 = 3 + 2p$ and consumer two has income $m^2 = \mathbf{p}\cdot\boldsymbol{\omega}^2 = 2 + 3p$.

   Consumer one has equal weighted Cobb-Douglas utility, so consumer one's demand is

$$\mathbf{x}^1(p) = m^1\left(\frac{1}{2}, \frac{1}{2p}\right).$$

   For consumer two, the weights are $\frac{2}{3}$, $\frac{1}{3}$, so demand is

$$\mathbf{x}^2(p) = m^2\left(\frac{2}{3}, \frac{1}{3p}\right).$$

   It is enough to clear the market for good one. Demand is $m^1/2 + 2m^2/3$. Setting this equal to the supply (endowment) of good one, we find $p = 13/18$.

   It follows that $m^1 = 3 + 2p = 40/9$ and $m^2 = 2 + 3p = 25/6$. The equilibrium price vector is $(1, 13/18)$, or any positive scalar multiple and the equilibrium allocation is

$$\mathbf{x}^1 = (20/9, 40/13), \mathbf{x}^2 = (25/9, 25/13)$$

2. Consider the utility function $u(x, y) = (1 + x)(1 + y) + y^{1/2}$. Is it equivalent to an additive separable utility function on $\mathbb{R}^2_+$?

   Answer: Method 1: Compute the marginal rate of substitution

$$\text{MRS} = \frac{1 + y}{1 + x + \frac{1}{2}y^{-1/2}}$$

   and take its logarithm,

$$\ln \text{MRS} = \ln(1 + y) - \ln(1 + x - \frac{1}{2}y^{-1/2}).$$

The cross-partial deriviative is not zero , so it is not additive separable, and neither is u (Proposition 3.2.6).

Method 2: Suppose there is a φ so that $v = \varphi \circ u$ is additive separable. We now compute $\partial v/\partial x = (1+y)\varphi(u)$ and $\partial^2 v\big/\partial x\partial y = \varphi'(u) + (1+y)[1+x+\frac{1}{2}y^{-1/2}]\varphi''(u) = \varphi'(u) + [u - \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}]\varphi''(u)$. For v to be additive separable, we must have $0 = \varphi'(u) + [u - \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}]\varphi''(u)$. However, the presence of y indicates that φ is not solely a function of u. Thus it is impossible to find a monotonic function of u that yields the required condition $(\partial^2 v\big/\partial x\partial y)$.

3. The production function is $f(z_1, z_2) = z_1^{1/2} + z_2^{1/2}$.
   a) Use the production function to define a production set in $\mathbb{R}^3$ in the usual way, with good three as the output.
   b) For what price vectors (p_1, p_2, p_3) can profit be maximized?
   c) Find the profit function.

Answer:

a) The production set is

$$Y = \left\{(y_1, y_2, y_3) : y_1, y_2 \le 0, \text{ and } y_3 \le (-y_1)^{1/2} + (-y_2)^{1/2}\right\}$$

b) Profit is $\mathbf{p}\cdot\mathbf{y} \le p_1 y_1 + p_2 y_2 + p_3(-y_1)^{1/2} + p_3(-y_2)^{1/2}$. When $p_3 = 0$ the maximum profit is zero because $y_1, y_2 \le 0$. It is infinite if either $p_1 \le 0$ or $p_2 \le 0$. For the remainder of part (b), we assume $\mathbf{p} > \mathbf{0}$. Due to free disposal, we need only consider the cases with $\mathbf{p} \ge \mathbf{0}$.

The first order conditions are

$$p_1 = \frac{p_3}{2}(-y_1)^{-1/2} \quad \text{and} \quad p_2 = \frac{p_3}{2}(-y_2)^{-1/2}$$

It follows that $y_1 = -(p_3/2p_1)^2$ and $y_2 = -(p_3/2p_2)^2$. Then $y_3 = p_3/2p_1 + p_3/2p_2$.

Profit can be maximized all non-negative prices.

c) Net output at price vector $\mathbf{p}$ is

$$\mathbf{y}(\mathbf{p}) = \left(-\frac{p_3^2}{4p_1^2}, -\frac{p_3^2}{4p_2^2}, \frac{p_3}{2p_1} + \frac{p_3}{2p_2}\right)$$

It follows that the profit function is

$$\pi(\mathbf{p}) = \frac{p_3^2}{4p_1} + \frac{p_3^2}{4p_2}.$$

4. Homogenize the production function $f(z_1, z_2) = z_1^{1/2} + z_2^{1/3}$ by adding an extra factor of production.

Answer: Recall the homogenized production function $F\colon \mathbb{R}_+^{L+1} \to \mathbb{R}_+$ is defined by

$$F(z, t) = \begin{cases} tf\big(t^{-1}z\big) & \text{when } t > 0 \\ \lim_{t\downarrow 0} tf\big(t^{-1}z\big) & \text{when } t = 0. \end{cases}$$

So for $t > 0$, $F(z_1, z_2, , t) = t[(z_1/t)^{1/2} + (z_2/t)^{1/3}] = (tz_1)^{1/2} + t^{2/3}z_2^{1/3}$. The limit as $t \to 0$ is zero, so

$$F(z_1, z_2, , t) = (tz_1)^{1/2} + t^{2/3}z_2^{1/3}.$$