Micro I Final: April 27, 2023

1. Suppose an exchange economy has endowment $\omega = (3,5)$ and there are two consumers with utility functions $u_1(\mathbf{x}^1) = x_1^1 + \frac{4}{3}(x_2^1)^{1/2}$ and $u_2(\mathbf{x}^2) = x_1^2 + \frac{1}{2}(x_2^2)^{2/3}$. Find all Pareto optima.

   Answer: We start by finding the interior Pareto optima. The marginal utilities of good two are $MU_2^1 = \frac{2}{3}(x_2^1)^{-1/2}$ and $MU_2^2 = \frac{1}{3}(x_2^2)^{-1/3}$. This means the marginal rates of substitution are $MRS_{12}^1 = \frac{3}{2}(x_2^1)^{1/2}$ and $MRS_{12}^2 = 3(x_2^2)^{1/3}$. Note that these depend only on consumption of the second good, as is typical of quasi-linear utility.

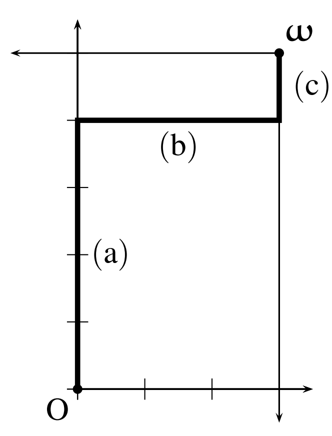
   We equate the marginal rates of substitution, obtaining

$$\frac{3}{2}(x_2^1)^{1/2} = 3(x_2^2)^{1/3} \quad \text{so} \quad x_2^2 = \frac{1}{2^6}(x_2^1)^3. \tag{1}$$

   It's clear that total consumption of good 2 is increasing in consumer one's consumption of good 2, so there is a unique x_2^1 where $x_2^1 + x_2^2 = 5$. In fact, $x_2^1 = 4$ and $x_2^2 = 1$ are the unique solution to equation (1).

   If $x_2^1 < 4$, the marginal rate of substitution for consumer one will fall, and it will rise for consumer two. As a result of the gap between marginal rates of substitution, consumer one will not consume good two, while consumer two will not consume good one. This reverses when $x_2^1 > 4$.

   It follows that the set of Pareto optima (illustrated in the Edgeworth box below) is the union of the sets of $(\mathbf{x}^1, \mathbf{x}^2)$ with (a) $\mathbf{x}^1 = (0, x_2^1)$ and $\mathbf{x}^2 = (3, 5 - x_2^1)$ for $0 \leq x_2^1 \leq 4$; with (b) $\mathbf{x}^1 = (x_1^1, 4)$, $\mathbf{x}^2 = (3 - x_1^1, 1)$ for $0 \leq x_1^1 \leq 3$; and with (c) $\mathbf{x}^1 = (3, x_2^1)$ and $\mathbf{x}^2 = (0, 5 - x_2^1)$ for $4 \leq x_2^1 \leq 5$.



2. Consider a two-person two-good production economy where good one is produced from good two using the production function $f(z_2) = 3(z_2)^{1/3}$. Endowments are $\boldsymbol{\omega}^1 = (0, 2)$ and $\boldsymbol{\omega}^2 = (0, 3)$ and preferences are described by the Cobb-Douglas utility functions $u_i(\mathbf{x}^i) = \ln x_1 + \ln x_2$.

   Find all competitive equilibria.

   Answer: Opps! I forgot to give you profit shares. They were intended to be $\theta^1 = 2/5$, $\theta^2 = 3/5$.

   There must be positive consumption of both goods due to the logarithmic Cobb-Douglas utility, so we can take good one as numéraire and write the price vector as $\mathbf{p} = (1, p)$.

   Profit is $3(z_2)^{1/3} - pz_2$. The first order condition for profit maximization is

$$(z_2)^{-2/3} = p \quad \text{with net supply} \quad \mathbf{y} = \left(3p^{-1/2}, -p^{-3/2}\right).$$

   Maximized profit is $1 \cdot 3p^{-1/2} + p \cdot (-p^{-3/2}) = 2p^{-1/2}$.

   Aggregate income is $5p + 2p^{-1/2}$, so demand is

$$\frac{5p + 2p^{-1/2}}{2}\left(1, \frac{1}{p}\right).$$

   Setting supply equal to demand for good one, we find $5p/2 = 2p^{-1/2}$, so $p = (4/5)^{2/3}$. The equilibrium price vector is then $\hat{\mathbf{p}} = \left(1, (4/5)^2\right)$.

   Then $z_2 = 5/4 < \omega_2 = 5$ is feasible. Net output is $\hat{\mathbf{y}} = \left(3z_2^{1/3}, -z_2\right) = \left(3(5/4)^{1/3}, -5/4\right)$ and total consumption is $\boldsymbol{\omega} + \hat{\mathbf{y}} = \left(3(5/4)^{1/3}, 15/4\right) = 15\left((1/100)^{1/3}, 1/4\right)$.

   Without the correct profit shares, you couldn't see that consumer one has 2/5 of income, and so consumption. Consumer two has 3/5 of both. Equilibrium consumption is

$$\hat{\mathbf{x}}^1 = 6\left(\left(\frac{1}{100}\right)^{1/3}, \frac{1}{4}\right) \quad \text{and} \quad \hat{\mathbf{x}}^2 = 9\left(\left(\frac{1}{100}\right)^{1/3}, \frac{1}{4}\right).$$

3. Is the utility function $u(x_1, x_2, x_3) = x_1 + x_1x_2 + x_1x_2x_3$ additive separable on $\mathbb{R}^3_{++}$? Explain.

Answer: No, it is not additive separable on $\mathbb{R}^3_{++}$. We compute $MU_1 = 1 + x_2 + x_2x_3$ and $MU_2 = x_1 + x_1x_3$. Then $MRS_{12} = (1 + x_2 + x_2x_3)/x_1(1 + x_3)$. Since MRS_{12} depends $x3$, u is not additive separable. In fact, MRS_{12} depends on all three variables.

If the x_3 dependence obvious, we can compute x_3-derivative to reveal it.

$$\begin{aligned}\frac{\partial\, MRS_{12}}{\partial x_3} &= \frac{x_2}{x_1(1+x_3)} - \frac{1 + x_2 + x_2x_3}{x_1(1+x_3)^2}\\ &= \frac{x_2(1+x_3) - (1 + x_2 + x_2x_3)}{x_1(1+x_3)^2}\\ &= \frac{-1}{x_1(1+x_3)^2} < 0.\end{aligned}$$

Since the x_3-derivative is not zero, MRS_{12} depends on x_3.

4. Let $0 \leq a < b$ and define

$$f(x) = \begin{cases} \beta x^3 & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

a) What value of β makes f a probability density function.

Answer: To be a probability density, we need

$$1 = \beta \int_a^b x^3\, dx = \beta\, \frac{1}{4}x^4 \Big|_a^b = \beta \frac{1}{4}(b^4 - a^4).$$

It follows that $\varphi = 4/(b^4 - a^4)$.

b) Calculate the expectation of x in terms of a and b.

Answer: The expectation is

$$\frac{4}{b^4 - a^4} \int_a^b x^4\, dx = \frac{4}{5(b^4 - a^4)}\, x^5\Big|_a^b = \frac{4(b^5 - a^5)}{5(b^4 - a^4)}.$$

5. Consider a contingent goods exchange economy with two consumers, one good and two states. Endowments are $\boldsymbol{\omega}^1 = (2,0)$ and $\boldsymbol{\omega}^2 = (0,2)$. Consumer one has utility $u_1(\mathbf{x}^1) = 0.6\ln x_1^1 + 0.4\ln x_2^1$ while consumer two has utility $u_2(\mathbf{x}^1) = 0.2\ln x_1^2 + 0.8\ln x_2^2$

   a) Find all Arrow-Debreu equilibria.

   Answer: In this Cobb-Douglas case, both goods will be demanded in equilibrium, so both prices will be strictly positive. We can pick good one as numéraire, and let $\mathbf{p} = (1, p)$.

   Then consumer one has income 2 and consumer two has income $2p$. Demands are

$$\mathbf{x}^1(p) = \left(1.2, \frac{0.8}{p}\right) \quad \text{and} \quad \mathbf{x}^2(p) = p\left(0.4, \frac{1.6}{p}\right)$$

   so market demand is

$$\mathbf{x}(p) = \left(1.2 + 0.4p, \frac{1.6 + 0.8p}{p}\right)$$

   and supply is $\boldsymbol{\omega} = (2,2)$. Setting $1.2 + 0.4p = 2$, meaning $p = 2$. The price vector is $\hat{\mathbf{p}} = (1,2)$ and the resulting allocation of goods is

$$\hat{\mathbf{x}}^1 = (1.2, 0.4) \quad \text{and} \quad \hat{\mathbf{x}}^2 = (0.8, 1.6)$$

   The other Arrow-Debreu equilibria have as price vector any positive multiple of $\mathbf{p}$. I.e., $\nu p = \lambda\hat{\mathbf{p}} = \lambda(1,2)$ for $\lambda > 0$. Such prices yield the same equilibrium allocations.

   b) Find all Arrovian securities equilibria.

   Answer: The easiest way to find this is to use the Arrovian Equivalence Theorem. Then $\hat{\mathbf{q}} = (1,2)$, $\hat{\mathbf{p}} = (1,1)$, $\hat{\mathbf{x}}^1 = (1.2, 0.4)$, $\hat{\mathbf{x}}^2 = (0.8, 1.6)$, $\hat{\mathbf{z}}^1 = (-0.8, 0.4)$, and $\hat{\mathbf{z}}^2 = (0.8, -0.4)$.