Homework #1

3.1.6 Let u and v be equivalent utility functions on $\mathbb{R}^m_+$.

a) Suppose u and v are both homogeneous of degree one. Show that $v = Cu$ for some $C > 0$.

b) Suppose u and v are homogeneous of degree β and γ, respectively. Show that $u = Cv^{(\beta/\gamma)}$ for some $C > 0$.

Answer:

a) **Method 1:** Let $t > 0$ and $x \in \mathbb{R}^L_+$ be arbitrary. Since the utility functions are equivalent, there is an increasing function φ with $v(x) = \varphi\big(u(x)\big)$. We now appeal to homogeneity

$$\begin{aligned} t\varphi\big(u(x)\big) &= tv(x) \\ &= v(tx) \\ &= \varphi\big(u(tx)\big) \\ &= \varphi\big(tu(x)\big). \end{aligned}$$

This implies that φ itself is homogeneous of degree 1. Since $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(z) = Cz$ for some C. Moreover, since φ is increasing, $C > 0$.

Method 2: When the functions are differentiable, an alternative method is to use Euler's formula. Since the utilities are equivalent, there is an increasing function φ with $v(x) = \varphi\big(u(x)\big)$. Differentiate to obtain $dv = \varphi' du$. Take the dot product with x and apply Euler's formula. This yields

$$v(x) = dv\cdot x = \varphi'(du\cdot x) = \varphi'\big(u(x)\big)u(x).$$

Since u and v are homogeneous of degree one in x, we can conclude that φ' is homogeneous of degree zero in u. This implies φ' is some constant $C > 0$ and that $u(x) = Cv(x)$.

b) To apply part (a), we first convert the functions to homogeneous of degree one function. Consider $\psi(x) = [u(x)]^{1/\beta}$ and $\phi(x) = [v(x)]^{1/\gamma}$. Now apply part (a) to find a constant A so that $\psi = A\phi$. Then raise it to the β power to get $u = \psi^\beta = A^\beta v^{(\beta/\gamma)}$. Set $C = A^\beta$ to obtain the result. There is also an alternative method as in part (a).

3.2.2 Show that $u(x, y) = (1 + x)(1 + y) + y^{1/2}$ does not have an additive separable representation on $\mathbb{R}^2_+$.

Answer: Suppose there is a φ so that $v = \varphi \circ u$ is additive separable. We now compute $\partial v/\partial x = (1 + y)\varphi(u)$ and

$$\begin{aligned}\frac{\partial^2 v}{\partial x\,\partial y} &= \varphi'(u) + (1 + y)[1 + x + \frac{1}{2}y^{-1/2}]\varphi''(u)\\ &= \varphi'(u) + \left[u - \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right]\varphi''(u).\end{aligned}$$

For v to be additive separable, we must have

$$0 = \varphi'(u) + \left[u - \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right]\varphi''(u).$$

However, the presence of y indicates that φ is not solely a function of u. Thus it is impossible to find a monotonic function φ of u that yields the required condition ($\partial^2 v/\partial x \partial y = 0$).

3.3.6 Suppose utility on $\mathbb{R}^3_+$ is given by $u(\mathbf{x}) = (x_1 + 1)x_2(x_3 + 5)$.

a) Is there a monotonic transformation that transforms u into an additive separable utility function?

Answer: Yes. Let $v = \ln u$. Then $v(\mathbf{x}) = \ln(x_1 + 1) + \ln x_2 + \ln(x_3 + 5)$, which is in additive separable form.

If you can't quickly guess it, one way to find the right transformation is to consider $v(\mathbf{x}) = \phi\big(u(\mathbf{x})\big)$. The second cross partial derivatives of v must be zero. Now $\partial v/\partial x_1 = \phi' x_2(x_3 + 5)$, and so

$$\frac{\partial^2 v}{\partial x_2\,\partial x_1} = \phi'(x_3 + 5) + \phi''(x_1 + 1)x_2(x_3 + 5)^2 = 0.$$

This can be written as

$$\phi'(x_3 + 5) + \phi'' u(x_3 + 5) = 0.$$

Then ϕ obeys the differential equation $\phi' + \phi'' u = 0$.

To solve this differential equation, set $\psi = \phi'$. The equation becomes $\psi + \psi' u = 0$. In other words, $d\psi/\psi = -du/u$. Its solution is $\psi(u) = A/u$ for some constant A which may be of either sign.

Now $\phi' = \psi = A/u$. This has general solution $\phi = B + A \ln u$ for some constants A and B. Because ϕ is increasing, $A > 0$. One such function is $\phi(u) = \ln u$. We don't have to worry about the other cross partial derivatives as ϕ converts u into the additive separable form $v(\mathbf{x}) = \ln(x_1 + 1) + \ln x_2 + \ln(x_3 + 5)$.

b) Does u induce a preference order on each commodity subgroup of $\{1, 2, 3\}$?

Answer: Yes. There are six commodity subgroups to consider and we consider each of them (we ignore the empty set and whole set). (1)–(3) It induces the same preference order defined by the utility function x_i on $\{i\}$. (4) On $\{1, 2\}$, it induces the order defined by the utility function $\ln(x_1 + 1) + \ln x_2$. (5) On $\{1, 3\}$ it induces $\ln(x_1 + 1) + \ln(x_3 + 5)$. (6) On $\{2, 3\}$ it induces $\ln x_2 + \ln(x_3 + 5)$.

3.4.4 Let $u \in \mathcal{C}^2$ be a utility function on $\mathbb{R}^2_+$ with $\partial u/\partial x_1, \partial u/\partial x_2 > 0$. Show that u is completely separable. This implies that Corollary 3.4.7 fails when $L = 2$.

Answer: Since u is increasing in each argument, it induces an order on $\{1\}$ and $\{2\}$. Since the only possible partitions of $\{1, 2\}$ are $\{\{1\}, \{2\}\}$ and $\{1, 2\}$, it is strongly separable on $\{1, 2\}$ relative to the partition of singletons. This means it is completely separable. However, as shown in Exercise 3.2.2, such functions need not be additively separable.

3.4.5 Let $u(x) = x_1^2 + 2x_1x_2x_3 + x_2^2x_3^2$. Is u separable on $\mathbb{R}^3_{++}$ relative to any partition? If so, is u strongly separable relative to that partition? Does u have a quasi-linear representation?

Answer: There are 4 non-trivial partitions to consider: $\big\{\{1\}, \{2\}, \{3\}\big\}$, $\big\{\{1, 2\}, \{3\}\big\}$, $\big\{\{1, 3\}, \{2\}\big\}$, and $\big\{\{1\}, \{2, 3\}\big\}$.

Since u is increasing, it is separable relative to the partition $\big\{\{1\}, \{2\}, \{3\}\big\}$. For the rest, it will be helpful to calculate the marginal rates of substitution. They are $\text{MRS}_{12} = 1/x_3$, $\text{MRS}_{13} = 1/x_2$ and $\text{MRS}_{23} = x_3/x_2$. Since MRS_{12} depends on x_3, which is not in $\{1\} \cup \{2\}$, u is not strongly separable relative to $\big\{\{1\}, \{2\}, \{3\}\big\}$.

The same marginal rate of substitution also tells us that u is not separable relative to $\big\{\{1, 2\}, \{3\}\big\}$. It is also not separable relative to $\big\{\{1, 3\}, \{2\}\big\}$ because $\text{MRS}_{13} = 1/x_2$. The lack of separability in the last two cases implies there are no quasi-linear representations relative to x_2 or x_3

That leaves $\big\{\{1\}, \{2, 3\}\big\}$, which passes the marginal rate of substitution test. In fact, $u(x) = (x_1 + x_2x_3)^2$, and is equivalent to $v(x) = \sqrt{u(x)} = x_1 + x_2x_3$. Regardless of the value of x_1, the ranking of (x_1, x_2, x_3) and (x_1, y_2, y_3) only depends on whether x_2x_3 is bigger than y_2y_3. These preferences are both separable and strongly separable relative to $\big\{\{1\}, \{2, 3\}\big\}$. Moreover, we have quasi-linear representation relative to x_1, $v(x) = x_1 + x_2x_3$.