Homework #2

7.4.1 Let $g(x) = f(x - a)$ for some $a \in \mathbb{R}^m$. Show that $g^*(p) = p\cdot a + f^*(p)$.

Answer: We show this by using the definition. Then the concave conjugate g^* is

$$\begin{aligned} g^*(p) &= \inf\{p\cdot x - g(x)\} \\ &= \inf_x\{p\cdot x - f(x-a)\} \\ &= \inf_x\{p\cdot(x+a) - f(x)\} \\ &= p\cdot a + \inf_x\{p\cdot x - f(x)\} \\ &= p\cdot a + f^*(p). \end{aligned}$$

In the third line we replaced x by $x + a$.

9.1.5 Compute the income and price elasticities for quasi-linear utility of Example 4.2.4

Answer: Based on Example 4.2.4, when $m \ge p_2^{\alpha/(\alpha-1)} p_1^{-1/(\alpha-1)} \alpha^{1/(1-\alpha)}$, demand is

$$x_1(p, m) = \frac{m}{p_1} - \alpha^{1/(1-\alpha)} \left(\frac{p_1}{p_2}\right)^{\alpha/(1-\alpha)}$$

and

$$x_2(p, m) = \alpha^{1/(1-\alpha)} \left(\frac{p_1}{p_2}\right)^{1/(1-\alpha)}$$

while if $m \le p_2^{\alpha/(\alpha-1)} p_1^{-1/(\alpha-1)} \alpha^{1/(1-\alpha)}$ demand is

$$x(p, m) = \begin{pmatrix} 0 \\ m/p_2 \end{pmatrix}.$$

In the first case, we have

$$\begin{aligned} \epsilon_{11} &= \frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1} = -\frac{1}{x_1}\left[\frac{m}{p_1} - \frac{\alpha^{(2-\alpha)/(1-\alpha)}}{1-\alpha}\left(\frac{p_1}{p_2}\right)^{\alpha/(1-\alpha)}\right], \\ \epsilon_{12} &= \frac{p_2}{x_1}\frac{\partial x_1}{\partial p_2} = \frac{1}{x_1}\frac{\alpha^{(2-\alpha)/(1-\alpha)}}{(1-\alpha)}\left(\frac{p_1}{p_2}\right)^{\alpha/(1-\alpha)}, \\ \epsilon_{21} &= \frac{p_1}{x_2}\frac{\partial x_2}{\partial p_1} = 1/(1-\alpha), \\ \epsilon_{22} &= \frac{p_2}{x_2}\frac{\partial x_2}{\partial p_2} = -1/(1-\alpha), \\ \eta_1 &= (m/p_1)/x_1 > 1, \\ \eta_2 &= 0. \end{aligned}$$

In the second case, the x_1 elasticities are $\epsilon_{11} = \epsilon_{12} = \eta_1 = 0$ and the x_2 elasticities are $\epsilon_{21} = 0$, $\epsilon_{22} = -1$, and $\eta_2 = +1$.

9.1.8 Demand functions of the form $\ln x_\ell = a_\ell + \eta_\ell \ln m + \sum_k \epsilon_{\ell k} \ln p_k$ are sometimes used to estimate demand.

a) Show that η_ℓ is the income elasticity of demand and $\epsilon_{\ell k}$ are the cross-price elasticities of demand.

b) Show that Walras' Law requires $\eta_\ell = 1$ for all ℓ.

c) If $\eta_\ell = 1$, show that $\epsilon_{\ell k} = 0$ for $\ell \neq k$ and $\epsilon_{kk} = -1$ for all k.

Answer:

a) Direct calculation shows that the income elasticity of demand is $\frac{\partial \ln x_\ell}{\partial \ln m} = \eta_\ell$ and the cross-price elasticities are $\frac{\partial \ln x_\ell}{\partial \ln p_k} = \epsilon_{\ell k}$.

b) Now $p_\ell x_\ell = p_\ell e^{a_\ell} \prod_k p_k^{\epsilon_{\ell k}} m^{\eta_\ell}$ By Walras' Law, $\mathbf{p}\cdot\mathbf{x} = m$, so we can write $m = \sum_\ell a_\ell(\mathbf{p}) m^{\eta_\ell}$.

We now show every $\eta_\ell = 1$ by contradiction. Because the various powers of m are linearly independent, the coefficients on the m^{η_ℓ} terms must sum to zero for $\eta_\ell \neq 1$. With this form, that can only happen if there are no such terms. In that case, the sum would be zero, which is impossible for $m > 0$. Thus all terms have $\eta_\ell = 1$.

c) Walras' Law now yields

$$\sum_\ell p_\ell e^{a_\ell} \prod_k p_k^{\epsilon_{\ell k}} = 1.$$

Since the right-hand side does not depend on any prices, and since we are taking positive linear combinations, the left-hand side cannot depend on any of the prices either. That implies $\epsilon_{kk} = -1$ for all k and $\epsilon_{k\ell} = 0$ for $k \neq \ell$.

9.2.2 For $k \neq \ell$, compute the elasticity of substitution $\sigma_{k\ell}$ for the Cobb-Douglas utility $u(\mathbf{x}) = \prod_k x_k^{\gamma_k}$.

Answer: We start by finding $\text{MRS}_{k\ell} = (\gamma_k/\gamma_\ell)(x_k/x_\ell)^{-1}$. This is solely a function of x_k/x_ℓ. Then $\frac{\partial\,\text{MRS}}{\partial(x_k/x_\ell)} = -(\gamma_k/\gamma_\ell)(x_k/x_\ell)^{-2}$. The elasticity of substitution is

$$\sigma_{k\ell} = -\frac{x_k/x_\ell}{\text{MRS}_{k\ell}} \frac{\partial\,\text{MRS}}{\partial(x_k/x_\ell)} = \frac{(x_k/x_\ell)^2}{(\gamma_k/\gamma_\ell)} \frac{(\gamma_k/\gamma_\ell)}{(x_k/x_\ell)^2} = 1.$$

9.4.2 Suppose utility has the Cobb-Douglas form $u(\mathbf{x}) = \sum_k \gamma_k \ln x_k$ where each $\gamma_k > 0$ and $\sum_k \gamma_k = 1$. Show that indirect utility is additive separable.

Answer: The Cobb-Douglas demands are $x_k = \gamma_k m/p_k$. We can calculate indirect utility by substituting back into $u(x)$

$$v(\mathbf{p}, m) = \left(\sum_k \gamma_k \ln \gamma_k\right) + \sum_k \gamma_k \ln\left(\frac{m}{p_k}\right),$$

which is also in Cobb-Douglas form, albeit with a different constant term. It is also an additive separable indirect utility function.

To have the same constant term would require $\sum_k \gamma_k \ln \gamma_k = 0$, or in other words $\prod_k \gamma_k^{\gamma_k} = 1$, which cannot happen as all γ_k obey $0 < \gamma_k < 1$.