Homework #3

Problems 10.3.2, 6.1.2, 6.2.2, 13.4.2, and 13.5.2 are due on Thursday, February 23

10.3.2 Suppose utility on $\mathbb{R}^m_+$ is defined by $u(x) = \big(\min\{x_i/\alpha_i\}\big)^\gamma$ where $\gamma > 0$ and each $\alpha_i > 0$. Compute the Konüs true cost-of-living index for utility level $\bar{u} > 0$ and $\mathbf{p}^0, \mathbf{p}^1 > 0$.

Answer: Expenditure is minimized by consuming the minimum required to attain utility $\bar{u}$, so $x_i = \bar{u}^{1/\gamma}\alpha_i$. It follows that $e(\mathbf{p}, \bar{u}) = \bar{u}^{1/\gamma}(\mathbf{p}\cdot\boldsymbol{\alpha})$.

The true cost-of-living index is then

$$\frac{e(\mathbf{p}^1, \bar{u})}{e(\mathbf{p}^0, \bar{u})} = \frac{\bar{u}^{1/\gamma}(\mathbf{p}^1\cdot\boldsymbol{\alpha})}{\bar{u}^{1/\gamma}(\mathbf{p}^0\cdot\boldsymbol{\alpha})}. = \frac{\mathbf{p}^1\cdot\boldsymbol{\alpha}}{\mathbf{p}^0\cdot\boldsymbol{\alpha}}.$$

6.1.2 Suppose $f(z) = (1 + e^5)/(1 + e^{-(z-5)}) - 1$. Maximize profit.

Answer: We compute

$$f'(z) = \frac{e^{-(z-5)}(1 + e^5)}{(1 + e^{-(z-5)})^2} = \frac{1 + e^5}{(e^{(z-5)/2} + e^{-(z-5)/2})^2} > 0.$$

The first order conditions for profit maximization are $f'(z) = w/p$. It's clear that f' is bounded above. In fact, it is less than $(1 + e^5)$. If w/p is too large, profit cannot be maximized. So how big can f' be?

Profit is $pf(z) - wz$, and its second derivative is pf''. We compute f''.

$$f''(z) = \frac{-2(1 + e^5)}{(e^{(z-5)/2} + e^{-(z-5)/2})^3}\left[\frac{1}{2}e^{(z-5)/2} - \frac{1}{2}e^{-(z-5)/2}\right].$$

Now $f'' > 0$ when $z < 5$ and $f'' < 0$ when $z > 5$. The profit curve is S-shaped with an inflection point at $z = 5$. That inflection point has negative profit. Maximum profit (if it exists) must lie to the right of the inflection point (with $f'' < 0$).

We return to the first order conditions. Profit is $pf(z) - wz$. The first-order condition is $pf'(z) = w$. The second derivative of profit will only be satisfied if $z > 5$ (if $z < 5$ we have a local minimum of profit).

Now

$$p\frac{1 + e^5}{(e^{(z-5)/2} + e^{-(z-5)/2})^2} = w.$$

so

$$\frac{p(1 + e^5)}{4w} = \left(\frac{1}{2}e^{(z-5)/2} + \frac{1}{2}e^{-(z-5)/2}\right)^2 = \cosh^2\left(\frac{z-5}{2}\right)$$

where cosh is the hyperbolic cosine. It follows that

$$z^* = 5 + 2\cosh^{-1}\left(\frac{p(1+e^5)}{4w}\right)^{-1/2}.$$

This only has a solution when $p(1+e^5) \geq 4w$ because cosh ≥ 1. Moreover, because $\cosh(x) = \cosh(-x)$, there are two inverse hyperbolic cosines, one positive and one negative. We need the positive one since $z^* \geq 5$ by the second-order condition. It has formula $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)$, so

$$z^* = 5 + 2\ln\left(x + \sqrt{x^2-1}\right)$$

where $x = \frac{p(1+e^5)}{4w}$.

6.2.2 Consider a firm facing a capacity constraint $\overline{Q}$ with production function $f(z) = az$, $a > 0$. We can incorporate the capacity constraint into the production function by writing

$$f(z) = \begin{cases} az & \text{when } 0 \leq z \leq \overline{Q}/a \\ \overline{Q} & \text{when } \overline{Q}/a \leq z \end{cases}$$

Find the cost function $c(w, q)$ and conditional factor demand $z(w, q)$.

Answer: When $q \leq \overline{Q}$, $f(z) \geq q$ whenever $z \geq q/a$. It follows that the minimum cost occurs when $z^* = q/a$. Then $c(w, q) = wz^* = wq/a$. When $q > \overline{Q}$, there are no z with $f(z) \geq \overline{Q}$, so $c(w, q) = +\infty$.

Thus

$$c(w, q) = \begin{cases} wq/a & \text{when } 0 \leq q \leq \overline{Q} \\ +\infty & \text{when } q > \overline{Q} \end{cases}$$

and

$$z(w, q) = \begin{cases} q/a & \text{when } 0 \leq q \leq \overline{Q} \\ \emptyset & \text{when } q > \overline{Q}. \end{cases}$$

13.4.2 Suppose there are two inputs and one output with the linear production function $f(z_1, z_2) = 2z_1 + z_2$. The output price is $p > 0$ and the input prices are $w_\ell > 0$.

a) Find all profit-maximizing net output vectors.

b) Calculate the profit function.

c) Show directly that the Law of Supply holds.

Answer:

a) Because this is a linear technology (CRS), maximum profit is either zero or infinite. To maximize profits, output must be $q = 2z_1 + z_2$ with profit

$$p(2z_1 + z_2) - w_1 z_1 - w_2 z_2.$$

Profit will be infinite if either $2p > w_1$ or $p > w_2$. We can also see that if both $2p < w_1$ and $p < w_2$, only 0 maximizes profit.

Finally, if $2p = w_1$ and $p < w_2$, any amount of input 1 is optimal while if $p = w_2$ and $p < w_1/2$, any amount of input 2 is optimal. If both $2p = w_1$ and $p = w_2$, then any $(-z_1, -z_2, 2z_1 + z_2)$ with $z \in \mathbb{R}^2_+$ is a net output.

b) Based on the answer to (a), the profit function is

$$\pi(\mathbf{p}) = \begin{cases} +\infty & \text{if } 2p > w_1 \text{ or } p > w_2 \\ 0 & \text{if } 2p \le w_1 \text{ and } p \le w_2. \end{cases}$$

c) Profit maximization is only possible when $2p \le w_1$ and $p \le w_2$. In that case, we write

$$\begin{aligned} \Delta\mathbf{p}\cdot\Delta\mathbf{y} = {} & [2(p' - p) - w_1' + w_1](z_1' - z_1) \\ & + [p' - p - w_2' + w_2](z_2' - z_2). \end{aligned}$$

Notice that the w_1 and w_2 terms act independently. We will consider the w_1 terms, but similar arguments apply to the w_2 terms.

There are 4 cases for the w_1 terms. (1) If $2p' = w_1'$ and $2p = w_1$, the term is zero. (2) If $2p' = w_1'$ and $2p < w_1$, $z_1 = 0$, so we have $(-2p + w_1)z_1' \ge 0$. (3) If $2p' < w_1'$ and $2p = w_1$, $z_1' = 0$ and we have $(2p' - w_1')(-z_1) \ge 0$. (4) If $2p' < w_1'$ and $2p < w_1$, then $z_1' = z_1 = 0$ and the term is zero.

13.5.2 Suppose production is described a linear activity model with basic activities $a^1 = (3, 2, 1, -1)^T$, $a^2 = (2, 2, 1, -1)^T$, $a^3 = (4, 0, -1, -1)^T$ and $(0, 4, -1, -1)^T$. Find all efficient net output vectors.

Answer: The first thing to note is that while a^1 and a^2 both use the same input, a^1 produces more output. This means that the use of a^2 is never efficient. None of the other activity vectors dominate one another.

We must consider non-negative linear combinations of the form $z_1 a^1 + z_3 a^3 + z_4 a^4 = (3z_1 + 4z_3, 2z_1 + 4z_4, z_1 - z_3 - z_4, -z_1 - z_3 - z_4)$. In fact, increasing any one of the z_i will decrease some component of the vector, so all choices with $z_2 = 0$ and $z_i \ge 0$ for $i \ne 2$ are efficient.