Homework #4

Problems 15.3.2, 15.4.4, 19.2.2, 19.2.6 and 19.3.1 are due on Tuesday, March 28.

15.3.2 Consider a two-person, two-good exchange economy. Endowments are $\omega^1 = (1, 1)$ and $\omega^2 = (2, 1)$. The utility functions are

$$u_1(x^1) = \min\{x_1^1, 2x_2^1\} \quad \text{and} \quad u_2(x^2) = 2x_1^2 + x_2^2.$$

Find all Walrasian equilibrium prices and allocations.

Answer: If either price is zero, consumer two will be unable to maximize utility. We can safely take good 1 as numéraire and set $p_2 = p$. Since both prices are positive, consumer one chooses a point where $x_1^1 = 2x_2^1$.

Consumer one's income is $1 + p$. Using the budget constraint, we find

$$x^1(p) = \left(\frac{1 + p}{2 + p}\right)(2, 1).$$

Consumer two's income is $2 + p$. Consumer two will buy only good 1 if $p < 1/2$, only good 2 if $p > 1/2$ and will be happy with any point on the budget constraint if $p = 1/2$.

If $p < 1/2$, demand for good 2 is $(1 + p)/(2 + p)$ which must equal the endowment of good 2. But then $p = -3$, which is impossible. This cannot be an equilibrium.

If $p > 1/2$, demand for good 1 is $2(1 + p)/(2 + p)$ which must equal 1, the endowment of good 1. But then $p = 0$, which is impossible. This is also not an equilibrium.

That leaves $p = 1/2$. Then $x^1 = (6/5, 3/5)$. Market clearing yields $x^2 = (9/5, 7/5)$. As that is on the budget constraint for consumer two, it maximizes utility and we have found the equilibrium.

15.4.4 Consider a two-agent, two-good, one-firm production economy where utility is $u_1(x^1) = (x_1^1)^{1/2}(x_2^1)^{1/2}$ and $u_2(x^2) = (x_1^2)^{1/2}(x_2^2)^{1/2}$, and endowments are $\omega^1 = (4, 0)$ and $\omega^2 = (4, 0)$.

There is one firm with production set

$$Y = \left\{\mathbf{y} : y_1 \le 0, y_2 \le \sqrt{-y_1}\right\}.$$

Each agent will receive half of the profits of the firm.

Find the equilibrium prices, equilibrium demands by individuals, and the firm's equilibrium net output.

Answer: We start by considering profit, $p_1 y_1 + p_2\sqrt{-y_1}$. The first-order condition for profit maximization is $p_1 = (p_2/2)(1/\sqrt{-y_1})$, so the net output is

$$y = \begin{pmatrix} -p_2^2/4p_1^2 \\ p_2/2p_1 \end{pmatrix}.$$

The resulting profit function is $\pi(p) = p_2^2/4p_1$.

Each consumer receives half of the profit and has endowment income $4p_1$, so each consumer has income $m = 4p_1 + p_2^2/8p_1$. Since consumers have identical preferences, and income, they will consume the same amount. Utility is equal-weighted Cobb-Douglas, and market demand is

$$2m\left(\frac{1}{2p_1}, \frac{1}{2p_2}\right) = m\left(\frac{1}{p_1}, \frac{1}{p_2}\right).$$

Adding the endowment to the firm's supply yields $(8 - p_2^2/4p_1^2, p_2/2p_1)$. Both goods are demanded in equilibrium, and prices must be strictly positive. We normalize prices so that $p_1 = 1$ and $p_2 = p$.

Then market clearing for good two says

$$\frac{p}{2} = \frac{m}{p} = \left(\frac{1}{8p}\right)(32 + p^2).$$

Thus $4p^2 = 32 + p^2$ implying $p = \pm\sqrt{32/3}$. Only the positive root makes economic sense, and the equilibrium has $p = (1, 4\sqrt{2/3})$, $y = (-8/3, \sqrt{8/3})$, $m = 16/3$, and $x^1 = x^2 = (8/3, \sqrt{2/3})$.

19.2.2 Suppose there are three goods and utility has the Cobb-Douglas forms

$$u_1(x^1) = (x_1^1)^{1/3}(x_2^1)^{1/3}(x_3^1)^{1/3} \quad \text{and} \quad u_2(x^2) = (x_1^2)^{1/3}(x_2^2)^{1/3}(x_3^2)^{1/3},$$

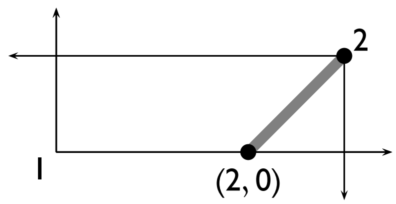
and total endowment $\omega = (5, 2, 3)$. Find all Pareto optimal allocations.

Answer: We appeal to Example 19.2.5. Each consumer receives a share of the endowment. The Pareto set is $\{(x^1, x^2) : x^1 = t(5, 2, 3), x^2 = (1 - t)(5, 2, 3), 0 \le t \le 1\}$.

19.2.6 Suppose $u_1(x^1) = x_1^1 + 2x_2^1$ and $u_2(x^2) = \min\{x_1^2, x_2^2\}$, with $\omega = (3, 1)$. Find all Pareto optimal allocations. Be careful on the boundary.

Answer: Here $\text{MRS}_{12}^1 = 1/2$ while MRS_{12}^2 can be interpreted as anything when $x_1^2 = x_2^2$. The interior Pareto optimal allocations run from the upper right corner of the box to $(2, 0)$. Only two boundary points are included. Note that $(x, 0)$ for $x < 2$ is not Pareto optimal as $(2, 0)$ is a Pareto improvement (consumer 1 is better off, consumer 2 is indifferent). The set of Pareto optima is the heavy line in the diagram,

$$\left\{(x_1, x_2) : x_1 - 2 = x_2, x_1 \geq 2\right\}.$$



19.3.1 An economy has two goods and two identical Cobb-Douglas consumers with $u_i(x^i) = \sqrt{x_1^i x_2^i}$. The total endowment is $(0, 6)$. There is one constant returns to scale firm that produces good 1 and uses good 2 as its only input. The production function is $f(z) = 2z$.

Find all Pareto optimal allocations of goods and the corresponding net output vector.

Answer: The net output vector will have the form $y = (-2y_2, y_2)$ with $y_2 \leq 0$. Here the marginal rate of transformation is $MRT_{12} = 1/2$. This must also be the marginal rate of substitution at any interior Pareto optimum. Note that since utility is zero if there is no production, the production technology must be used.

Now $MRS_{12}^i = x_2^i/x_1^i = 1/2$, so $2x_2^i = x_1^i$. Summing over both consumers, aggregate consumption obeys $2x_2 = x_1$. Since good 1 can only be obtained from the production sector, $x_1 = -2y_2$ and $x_2 = 6 + y_2$. Thus $x_2 = -y_2$ and $x_2 = 6 + y_2$. It follows that $x_2 = 3$ and $y_2 = -3$, so $x_1 = 6$.

Since both consumers will consume in the same proportions, $x_1 = t(6, 3)$ and $x_2 = (1 - t)(6, 3)$ for some t, $0 \leq t \leq 1$. Also, $y = (6, -3)$. These are the Pareto optimal allocations.