Homework #5

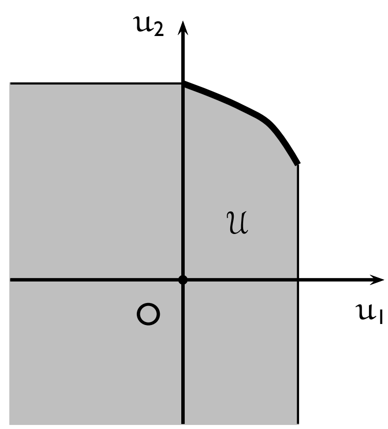
Problems 20.4.2, 20.4.5, 21.2.1, 21.2.6, and 22.4.2 are due on Thursday, April 6.

20.4.2 Consider the two-person exchange economy where utility is $u_1(x) = x_1$ and $u_2(x) = x_2 + \sqrt{x_1}$. Endowments are $\omega^1 = \omega^2 = (1, 1)$. Find the utility possibility set.

Answer: We start by finding the Pareto frontier. It is clear that good two is valueless to consumer one. If consumer one has a positive amount of good two, we can make a Pareto improvement by transferring all of good two to consumer two. It follows that Pareto optimal allocations must have the form $(x_1^1, 0)$, $(2 - x_1^1, 2)$. Moreover, any such allocation is Pareto optimal because increasing x_1^1 decreases consumer two's utility, and decreasing x_1^1 decreases consumer one's utility.

The corresponding utility levels are $u_1 = x_1^1$ and $u_2 = 2 + \sqrt{2 - x_1^1}$. The utility possibility set is the comprehensive hull of such points,

$$\mathcal{U} = \left\{(u_1, u_2) : u_1 \leq x \text{ and } u_2 \leq 2 + \sqrt{2 - x} \text{ for some } x, 0 \leq x \leq 1\right\}.$$



Utility Possibility Set: Here $\mathcal{U}$ is the utility possibility set for Problem 20.4.2. The heavy line indicates the Pareto frontier.

20.4.5 Suppose there are two consumers and two goods in an exchange economy $\mathcal{E}$. Both consumers have identical utility $u(x) = \sqrt{x_1 x_2}$ with consumption set $\mathbb{R}^2_+$. Consumer one has endowment $\omega^1 = (1, 3)$ and consumer two has endowment $\omega^2 = (1, 5)$. The social welfare function is $W(\mathbf{u}) = u_1^\alpha u_2^{1-\alpha}$ for some α, $0 < \alpha < 1$.

a) Find all social welfare maxima.

b) For each social welfare maximum, find prices and transfers that make it a quasi-equilibrium with taxes and transfers. Is it a Walrasian equilibrium with taxes and transfers?

c) Is there an α where the transfers are zero?

Answer:

a) The aggregate endowment is $\omega = (2, 8)$. By Example 19.2.5, the Pareto set is $\{(u_1, u_2) \in \mathbb{R}^2_+ : u_1 + u_2 = \sqrt{16} = 4\}$. We must maximize W over the Pareto set. Setting

$$\mathcal{L} = u_1^{\alpha}u_2^{1-\alpha} + \lambda(4 - u_1 - u_2) + \mu_1 u_1 + \mu_2 u_2,$$

we obtain the first-order conditions

$$\alpha u_1^{\alpha-1}u_2^{1-\alpha} + \mu_1 = \lambda \quad \text{and} \quad (1-\alpha)u_1^{\alpha}u_2^{-1\alpha} + \mu_2 = \lambda.$$

If $u_1, u_2 > 0$, this yields $\alpha u_2 = (1-\alpha)u_1$ so $\alpha(4 - u_1) = (1-\alpha)u_1$. Thus $u_1 = 4\alpha$ and $u_2 = 4 - 4\alpha$. The corner solutions are the cases $\alpha = 0$ and $\alpha = 1$. By Example 19.2.5, the corresponding allocation of goods is $x^1 = \alpha\omega$ and $x^2 = (1-\alpha)\omega$.

b) By Example 19.2.5, the common $\text{MRS}_{12} = x_2/x_1 = \omega_2/\omega_1 = 8/2 = 4$. It follows that the equilibrium price ratio is $p_1/p_2 = \text{MRS}_{12} = 4$. Choosing good two as the numéraire we obtain $p = (4, 1)$. Aggregate wealth is $p\cdot\omega = 16$. The corresponding wealth levels are

$$m^1 = 16\alpha \quad \text{and} \quad u_2 = 16(1-\alpha).$$

As long as $0 < \alpha < 1$, both consumers will satisfy the cheaper point condition.

By Corollary 20.3.3, this Pareto optimum can be written as a Walrasian equilibrium with taxes and transfers. Further, since one of the consumers will consume nothing when $\alpha = 0$ or $\alpha = 1$, those cases are easily seen to be price equilibria with taxes and transfers.

c) The transfer will be zero if consumer i's income from their endowment is m^i. For consumer one, that means $7 = 16\alpha$ and for consumer two, $9 = 16 - 16\alpha$. This only happens if $\alpha = 7/16$.

21.2.1 An exchange economy has two consumers with utility $u_1(x^1) = (x_1^1)^{1/3}(x_2^1)^{2/3}$ and $u_2(x^2) = (x_1^2)^{1/3}(x_2^2)^{2/3}$. Their endowments are $\omega^1 = (7, 1)$ and $\omega^2 = (3, 1)$. Find the core.

Answer: Since the consumers have identical Cobb-Douglas preferences, the Pareto set is the diagonal of the Edgeworth box. The aggregate endowment is $\omega = (10, 2)$ and total utility is $10^{1/3}2^{2/3} = 40^{1/3}$. Individual rationality requires $u_1 \geq u_1(\omega^1) = 7^{1/3}$ and $u_2 \geq u_2(\omega^2) = 3^{1/3}$.

Thus the core is

$$\mathbf{C}(\mathcal{E}) = \left\{ \big(u_1(10,2), (1-u_1)(10,2)\big) : u_1 \geq (7/40)^{1/3}, u_2 \geq 3^{1/3}, u_1 + u_2 = 1 \right\},$$

which can be written

$$\mathbf{C}(\mathcal{E}) = \left\{ \big(u_1(10,2), (1-u_1)(10,2)\big) : (7/40)^{1/3} \leq u_1 \leq 1 - (3/40)^{1/3} \right\}.$$

The limits on u_1 are approximately 0.559 and 0.578.

21.2.6 An exchange economy has two consumers with utility $u_1(x) = x_1 + x_2$ and $u_2(x) = \sqrt{x_1 x_2}$. Their endowments are $\omega^1 = (2,0)$ and $\omega^2 = (1,2)$.

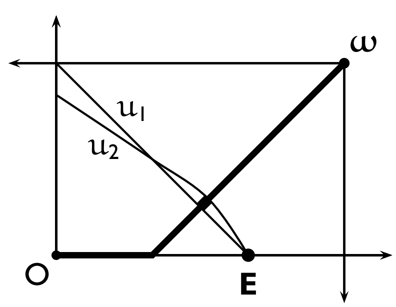
a) Find all Pareto optimal allocations.

b) Find the core.

Answer:

a) We start by finding the interior Pareto optimum allocations. Consumer one has $\text{MRS}^1 = 1$ while consumer two has $\text{MRS}^2 = x_2^2/x_1^2$. It follows that $x_1^2 = x_2^2$ at all interior Pareto optima. Since the aggregate endowment is $\omega = (3,2)$, the interior optima must have the form $\mathbf{x}^1 = (1+x, x)$, $\mathbf{x}^2 = (2-x, 2-x)$ for $0 < x < 2$. (note that $\mathbf{x}^1 + \mathbf{x}^2 = \omega$).

The rest of the Pareto optima are the cases $x = 0$, $x = 2$, and the line segment $\mathbf{x}^1 = (x, 0)$, $\mathbf{x}^2 = (3-x, 2)$ with $0 < x < 1$. This is illustrated by the medium heavy line on the diagram.



b) For two person exchange economies, the core consists of the individually rational Pareto optimal allocations. Now $u_1(\omega^1) = 2$ and $u_2(\omega^2) = \sqrt{2}$. Individually rational allows obey $u_1(\mathbf{x}^1) \geq 2$ and $u_2(\mathbf{x}^2) \geq \sqrt{2}$. This area is shown between the utility curves on the diagram. The core consists of the Pareto optimal points between the utility curves, indicated by the short heavy line on the diagram.

The u_1 indifference curve $x_1 + x_2 = 2$ intersects the Pareto optimal line $x_2 = -1 + x_1$ at $x_1 = 3/2$, while the u_2 indifference curve intersects the same line at $(3 - \sqrt{2}, 2 - \sqrt{2})$. It follows that the core is

$$\mathbf{C}(\mathcal{E}) = \left\{ \big((x_1, -1 + x_1), (3 - x_1, 3 - x_1)\big) : 3/2 \le x_1 \le 3 - \sqrt{2} \right\}.$$

22.4.2 Suppose there are 3 states, $s = 1, 2, 3$. Define lotteries $L_1 = (1/2, 1/2, 0)$, $L_2 = (0, 1/3, 2/3)$ and $L_3 = (2/3, 0, 1/3)$. Can $(1/3, 1/3, 1/3)$ be written as a compound lottery based on L_1, L_2, and L_3? If so, demonstrate how.

Answer: We look for a solution to

$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

It is easily verified that $x = (2/5, 2/5, 1/5)$ is the solution. Thus

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \frac{2}{5}L_1 \oplus \frac{2}{5}L_2 \oplus \frac{1}{5}L_3.$$