Homework #6

Problems 22.4.3, 22.6.3, 22.6.4, 23.1.4, and 23.3.4 are due on Thursday, April 14.

22.4.3 Suppose $u(w) = w + \sqrt{w}$.

a) Find the expected utility of the lottery that pays 1 with probability p and 0 with probability $(1 - p)$.

b) Find the expected utility of the lottery that pays 3 with probability p and 1 with probability $(1 - p)$.

Answer:

a) Expected utility is $Eu(L) = pu(1) + (1 - p)u(0) = 2p$.

b) Expected utility is $Eu(L) = pu(3) + (1 - p)u(1) = p(3 + \sqrt{3}) + 2(1 - p) = 2 + p(1 + \sqrt{3})$.

22.6.3 Suppose a lottery has probability density $f(x) = 3x^2$ for $0 \le x \le 1$ and zero otherwise. Let $u(x) = x^{1/4}$. Compute the expected utility.

Answer: The expected utility is

$$\begin{aligned} Eu &= \int_0^1 u(x)f(x)\,dx = \int_0^1 x^{1/4}\,(3x^2)\,dF(x) \\ &= \int_0^1 3x^{9/4}\,dx = \left[\frac{12}{13}x^{13/4}\right]_0^1 \\ &= 12/13. \end{aligned}$$

22.6.4 Suppose the random variable X has distribution F which is described by a probability density function $f(x) = \beta e^{-\alpha x}$ for $x \ge 0$ and $f(x) = 0$ for $x < 0$ where $\alpha, \beta > 0$.

a) What must β be (in terms of α) for f to be a probability density function?

b) What is the mean of X?

c) Compute the variance of X.

d) Suppose $u(x) = x^2$. Find $Eu(X)$.

Answer:

a) This must obey $\beta \int_0^\infty e^{-\alpha x}\,dx = 1$ to be a probability density function. Evaluating the integral, we obtain β/α, so $\beta = \alpha$.

b) The mean is $\mu = \int_0^\infty \alpha x e^{-\alpha x}\,dx$. This may be integrated by parts to find

$$\mu = \frac{1}{\alpha}\left[-ue^{-u} - e^{-u}\right]_0^{+\infty} = \frac{1}{\alpha}.$$

c) The variance is $\text{var}(X) = E(X^2) - \mu^2 = E(X^2) - \alpha^{-2}$. Now

$$\begin{aligned} E(X^2) &= \alpha \int_0^\infty x^2 e^{-\alpha x}\,dx \\ &= \frac{1}{\alpha^2}\int_0^\infty u^2 e^{-u}\,du \\ &= \frac{1}{\alpha^2}\left[-u^2e^{-u} - 2ue^{-u} - 2e^{-u}\right]_0^\infty = \frac{2}{\alpha^2}. \end{aligned}$$

So $\text{var}(X) = E(X^2) - \mu^2 = 1/\alpha^2$.

d) This is just $E(X^2)$ from part (c), which is $Eu(F) = 2/\alpha^2$.

23.1.4 Suppose F is uniformly distributed over $[1, a]$ for $a > 1$. Calculate the risk premium for the following utility functions.

a) $u(x) = x^3$.

b) $u(x) = x^{1/2}$.

c) $u(x) = \ln x$.

Answer: Note that the probability density for all three parts is $1/(a - 1)$, and that this density has expected value of $EX = (1 + a)/2$.

a) The expected utility is

$$Eu(F) = \frac{1}{a-1}\left(\int_1^a x^3\,dx\right) = \frac{a^4 - 1}{4(a-1)}.$$

This has certainty equivalent $c(u, F) = [(a^4 - 1)/4(a - 1)]^{1/3}$, so the risk premium is $R(u, F) = EX - c(u, F) = (1 + a)/2 - [(a^4 - 1)/4(a - 1)]^{1/3}$.

b) The expected utility is

$$Eu(F) = \frac{1}{a-1}\left(\int_1^a x^{1/2}\,dx\right) = \frac{2(a^{3/2} - 1)}{3(a-1)}.$$

This has certainty equivalent $c(u, F) = [2(a^{3/2} - 1)/3(a - 1)]^2$, so the risk premium is $R(u, F) = (1 + a)/2 - [2(a^{3/2} - 1)/3(a - 1)]^2$.

c) The expected utility is

$$Eu(F) = \frac{1}{a-1}\left(\int_1^a \ln x\,dx\right) = \frac{a\ln a - a + 1}{a-1}.$$

This has certainty equivalent

$$c(u,F) = \exp\left[\frac{a\ln a - a + 1}{a-1}\right] = \exp\left[\frac{a\ln a}{a-1} - 1\right] = \left(\frac{1}{e}\right) a^{\frac{a}{a-1}},$$

so the risk premium is

$$R(u,F) = \frac{1+a}{2} - \left(\frac{1}{e}\right) a^{\frac{a}{a-1}}.$$

23.3.4 A firm with cost function $C(q) = 2q$ faces an uncertain price p. The firm chooses the production level q before the price is revealed.

a) Write expected profit in terms of expected price $\mathrm{E}p$ and the chosen quantity of output q.

b) Suppose the firm maximizes expected profit. At what expected prices can the firm maximize expected profit?

c) When expected profit can be maximized, what quantity q maximizes expected profit?

d) Suppose the owner of the firm maximizes *expected utility* of income where $u(m) = \ln m$. The owner has \$10 of other income in addition to profit income. There is 50% chance that the price is \$1 and 50% chance that the price is \$3. Find the profit maximizing quantity q.

Answer:

a) Profit is $\pi(q) = pq - C(q) = pq - 2q$. Expected profit is $\mathrm{E}\big(\pi(q)\big) = \mathrm{E}(pq - 2q) = q(\mathrm{E}p - 2)$.

b) If $\mathrm{E}p > 2$, expected profit cannot be maximized because $\lim_{q\to\infty} \mathrm{E}\pi = +\infty$. If $\mathrm{E}p = 2$, profit is also zero, which is the maximum. If $\mathrm{E}p < 2$, profit is negative for $q > 0$, and maximum profit is zero (at $q = 0$).

c) When $\mathrm{E}p = 2$, any $q \geq 0$ maximizes profit. When $\mathrm{E}p < 2$, $q = 0$ maximizes profit.

d) The owner's income is $10 + (p-2)q$. The expected utility of income is

$$\mathrm{E}u = \frac{1}{2}\ln(10+q) + \frac{1}{2}\ln(10-q).$$

We differentiate with respect to q to find the first-order conditions.

$$\frac{1}{10+q} - \frac{1}{10-q} = 0.$$

It follows that expected profit is maximized when $q = 0$.