Homework #7

Problems 25.2.6, 25.2.7, 25.5.2, 27.2.2, and 27.3.2 are due on Thursday, April 20.

25.2.6 Suppose a consumer discounts at rate $\rho > 0$, so the discount factor is $\delta = (1+\rho)^{-1}$. There is one good and its price at time t is $p_t = (1+r)^{-t}$ where $r > 0$ is the interest rate. The consumer has felicity function $u(c_t) = c_t^{1-\sigma}/(1-\sigma)$ where $\sigma > 0$ and $\sigma \neq 1$. The consumer has wealth W.

a) How fast does an optimal consumption path grow?

b) Let $\beta = 1 + g$ be the growth factor with g the growth rate. How must g and r be related in order for the sum $\sum_t p_t c_t$ to converge.

c) Are there cases where $\rho < 0$ makes sense?

Answer:

a) The first order conditions are

$$\frac{p_{t+1}}{p_t} = \delta \frac{u'(c_{t+1})}{u'(c_t)}.$$

Using $\delta = (1+\rho)^{-1}$, $u'(c) = c^{-\sigma}$, and $p_t = (1+r)^{-t}$, we obtain

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r}{1+\rho}\right)^{1/\sigma}.$$

Consumption grows (or shrinks) by the factor $\beta = \big((1+r)/(1+\rho)\big)^{1/\sigma}$ Since $\sigma > 0$, consumption grows when $r > \rho$ and shrinks when $r < \rho$.

b) Here $p_t c_t = c_0 \beta^t (1+r)^{-t}$. The sum $\sum_t p_t c_t$ converges if and only if $1+g = \beta < 1+r$, that is, if $g < r$.

c) The condition $\beta < 1 + r$ means $(1+r)^{1-\sigma} < 1+\rho$. Negative values of ρ will obey this for certain values of r and σ. E.g., if $\sigma = 2$ and $r = 0.1$, we find any $\rho > 1/1.1 - 1 \approx -0.091$ works, such as $\rho = -0.05$.

25.2.7 Suppose a consumer has discount factor $0 < \delta < 1$ and period utility function $u(c) = (1+c)^{1/2}$. The consumer has wealth $W > 0$ and faces prices $p_t = p > 0$ for all times t.

a) Use the Kuhn-Tucker Theorem to show that if $c_t = 0$ on the optimal path, then $c_{t+1} = 0$.

b) Show that there is a T with $c_t = 0$ for $t > T$.

Answer:

a) Because the problem concerns the constraints $c_t \geq 0$, the relevant Lagrangian is

$$\mathcal{L} = \sum \delta^t (1 + c_t)^{1/2} - \lambda \left(\sum_t p c_t - W \right) + \sum_t \mu_t c_t$$

and the first order conditions are

$$\left(\frac{1}{2}\right) \delta^t (1 + c_t)^{-1/2} + \mu_t = \lambda p.$$

If $c_t = 0$, we have $\frac{1}{2}\delta^t = \lambda p - \mu_t \leq \lambda p$, while if $c_t > 0$, $\mu_t = 0$ by complementary slackness. Then $\frac{1}{2}\delta^t = \lambda p(1 + c_t)^{1/2} > \lambda p$. It follows that if $c_t = 0$, then $\frac{1}{2}\delta^{t+1} < \frac{1}{2}\delta^t \leq \lambda p$, implying that $c_{t+1} = 0$ also.

b) If $c_t > 0$ for all t, then $\mu_t = 0$ by complementary slackness. Then $\delta^t = 2\lambda p\sqrt{1 + c_t} \geq 2\lambda p$ for all t. Letting $t \to +\infty$, we find $\lambda = 0$. This contradicts the first order conditions, which require $\lambda > 0$. There must be a T with $c_T = 0$. Then by part (a), $c_t = 0$ for all $t > T$.

25.5.2 Consider the following Ramsey problem. A consumer has utility $U(\mathbf{c}) = \sum_{t=0}^{\infty} \delta^t u(c_t)$ where $0 < \delta < 1$ and the felicity function is $u(c) = c^{1/2}$. The production function is $f(a) = \beta a$ where $\beta > 1$. The initial endowment is $b > 0$. Consider an optimal path with $c_t > 0$ for every t (in fact, all optimal paths obey this).

a) Use the Euler equations to find c_t in terms of initial consumption c_0.

b) Does consumption grow over time? If so, what is the growth factor?

c) What are the corresponding time-zero prices p_t? The interest rate at time t is given by $1 + r_t = p_t/p_{t+1}$. Calculate r_t.

Answer:

a) The Euler equations are

$$\delta f'(a_t) u'(c_{t+1}) = u'(c_t)$$

yielding

$$\delta \beta c_{t+1}^{-1/2} = c_t^{-1/2}.$$

It follows that $c_{t+1} = (\delta\beta)^2 c_t$. Then $c_t = (\delta\beta)^{2t} c_0$ by induction.

b) Consumption grows by the growth factor $(\delta\beta)^2$ when $\delta\beta > 1$, is constant if $\delta\beta = 1$, and shrinks if $\delta\beta < 1$, all of which are possible with $\delta < 1$ and $\beta > 1$.

c) The time-zero prices are given by $p_t = \partial U/\partial c_t = \delta^t u'(c_t) = \delta^t/2c_t^{1/2}$.

Substituting $c_t = (\delta\beta)^{2t}c_0$, we obtain $p_t = 1/2\beta^t c_0^{1/2}$.

The equilibrium interest rate at time t is $r_t = p_t/p_{t+1} - 1 = \beta - 1 > 0$.

27.2.2 Consider a contingent goods exchange economy with two consumers, one good and two states. Endowments are $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (0, 2)$. Both consumers have identical utility function $u(\mathbf{x}) = \pi \ln x_1 + (1-\pi)\ln x_2$ where $0 < \pi < 1$ is the probability of state one.

a) Find all Arrow-Debreu equilibria.

b) How does the equilibrium price of good two relative to good one relate to the probability π?

Answer:

a) Let $\mathbf{p} \gg \mathbf{0}$ be the price vector. Note that zero price is not allowed in equilibrium due to the Cobb-Douglas preferences. Since preferences are Cobb-Douglas and identical for both consumers, so demand is

$$\mathbf{x}^i = m^i\left(\frac{\pi}{p_1}, \frac{1-\pi}{p_2}\right)$$

where $m^i = \mathbf{p}\cdot\boldsymbol{\omega}^i$ is the income of consumer i. It follows that market demand is $m\big(\pi/p_1, (1-\pi)/p_2\big)$ where $m = m^1 + m^2$.

Market supply is $\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2 = (2, 2)$. Setting demand equal to supply we find $\pi m/p_1 = (1-\pi)m/p_2 = 2$. Using good one as numéraire, $m = 2/\pi$, the equilibrium prices are $\mathbf{p} = \big(1, (1-\pi)/\pi\big)$. Individual incomes are $m^1 = 2$ and $m^2 = 2(1-\pi)/\pi$ and the corresponding allocation is $\mathbf{x}^1 = \big(2\pi, 2\pi\big)$ and $\mathbf{x}^2 = \big(2(1-\pi), 2(1-\pi)\big)$.

Any positive scalar multiple of $\mathbf{p}$ is also an equilibrium price vector with the same allocation.

b) From part (a), the relative price of good two is $p_2/p_1 = (1-\pi)/\pi$.

27.3.2 Consider a contingent goods exchange economy with one good and two states. Endowments are $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (0, 5)$. Both consumers have utility $u(\mathbf{x}) = \ln x_1 + \ln x_2$.

a) Find the Arrow-Debreu equilibrium.

b) Are the consumers fully insured?

Answer:

a) With Cobb-Douglas utility, both prices must be strictly positive. We take good one as numéraire and set $\mathbf{p} = (1, p)$. Then demand is

$$\mathbf{x}^1 = (1, 1/p) \quad \text{and} \quad \mathbf{x}^2 = \frac{5p}{2}(1, 1/p).$$

Setting supply equal to demand for good one we obtain $1 + 5p/2 = 2$, so $p = 2/5$. Equilibrium prices are $\mathbf{p} = (1, 2/5)$. Then $\mathbf{x}^1 = (1, 5/2)$ and $\mathbf{x}^2 = (1, 5/2)$.

b) Here there is aggregate uncertainty and full insurance for both consumers is not possible. In fact, neither consumers is fully insured since $x_1^i \neq x_2^i$.