Micro I Midterm, March 7, 2023

1. Consider an activity analysis model generated by the basic activities $a^1 = (1, -2)$ and $a^2 = (-2, 1)$. Let Y be the production set it generates.

   a) Show that this activity model satisifies the five conditions that characterize a production set.

   b) Find a transformation function that describes this production set.

   Answer:

   a) This production set Y was illustated in 13.1.3.

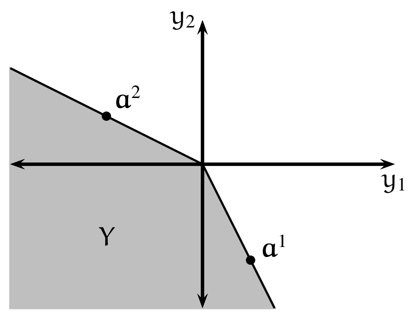

Figure 13.1.3: Let $Y = \{(y_1, y_2) : 2y_1 + y_2 \leq 0 \text{ and } y_1 + 2y_2 \leq 0\}$. This constant returns production set satisfies all 12 production conditions. Note that either good can be used as an input to produce the other good as an output.

   It is $Y = \{(y_1, y_2) : 2y_1 + y_2 \leq 0 \text{ and } y_1 + 2y_2 \leq 0\}$. Then Y consists of all non-negative linear combinations of a^1 and a^2. To see this, consider the system $y = Az$. Then $\det A = 5$ and $z_1 = (y_1 + 2y_2)/5 \geq 0$ and $z_2 = (2y_1 + y_2)/5 \geq 0$.

   As a linear activity analysis model it is a closed convex cone that obeys no free lunch and free disposal. If $y \in Y$ with $y \geq 0$, non-negatively of the activity levels imply $2y_1 + y_2 \leq 0$ and $y_1 + 2y_2 \leq 0$, which is only possible if $y = 0$. This shows there is no free lunch.

   b) The function $T(y) = \min\{2y_1 + y_2, y_1 + 2y_2\}$. It follows from our characterization of Y in part (a).

2. Consider the utility function $u(x_1, x_2, x_3) = x_1x_2x_3 + x_2x_3$. Is it equivalent to an additive separable utility function on $\mathbb{R}^3_{++}$?

   Answer: Method 1: We rewrite $u(x) = (1 + x_1)x_2x_3$. Since this is a product of functions of x_i, we take any logarithm, such as $\ln u(x) = \ln(1 + x_1) + \ln x_2 + \ln x_3$. This transforms the function into an additive separable form.

Method 2: Compute the marginal rates of substitution.

$$\begin{aligned}
\mathrm{MRS}_{12} &= x_2x_3/(x_1x_3+x_3) = x_2/(1+x_1)\\
\mathrm{MRS}_{13} &= x_2x_3/(x_1x_2+x_2) = x_3/(1+x_1)\\
\mathrm{MRS}_{23} &= (x_1x_3+x_3)/(x_1x_2+x_2) = x_3/x_2.
\end{aligned}$$

As each MRS is independent of the other variable, we can define induced orders on $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$. We can also define induced orders on every singleton, because the utility function is increasing. It follows that we have induced orders on every possible commodity group. This means it is completely separable. By Debreu's Separability Theorem, it has a additive separable representation.

Method 3: Suppose there is a φ so that $v = \varphi \circ u$ is additive separable. We compute the ij cross partial derivative. First,

$$\frac{\partial v}{\partial x_j} = \varphi' \frac{\partial u}{\partial x_j}$$

and

$$0 = \frac{\partial^2 v}{\partial x_i\,\partial x_j} = \varphi'' \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \varphi' \frac{\partial^2 u}{\partial x_i\,\partial x_j}(1+x_1)x_3$$

for all $i \neq j$. Due to symmetry of the second partials, we could potentially need to consider $ij = 12$, 13, and 23. However, we are in luck and $ij = 12$ will do the job. Rewrite $u(\mathbf{x}) = (1+x_1)x_2x_3$. Then

$$0 = (x_1+1)x_2x_3^2\varphi'' + x_3\varphi' = u\varphi'' + x_3\varphi'.$$

We substitute $\psi = \varphi'$, obtaining

$$0 = u\psi' + \psi$$

with solution $\varphi' = \psi = C/u$. It follows that

$$\varphi(u) = C\ln u + C'.$$

The constant C' can be set to zero, and $C\ln u$ is any logarithm of u. In particular, setting $C = 1$ gives the solution of method 1,

$$v\big(u(\mathbf{x})\big) = \ln(1+x_1) + \ln x_2 + \ln x_3,$$

which is additive separable.

3. Answer both parts.

   a) Suppose a utility function is homogeneous of degree 1. Show that the Marshallian demand has an income elasticity of 1.

   b) Does the same result hold when utility is homothetic but not necessarily homogeneous? Explain.

Answer:

   a) Corollary 4.3.6 applies, yielding $\mathbf{x}(\mathbf{p}, m) = m\mathbf{x}(\mathbf{p}, 1)$. Then $\partial x_\ell/\partial m = x_\ell(\mathbf{p}, 1)$, and the income elasticity is $\big(m/x_\ell(\mathbf{p}, m)\big)x_\ell(\mathbf{p}, 1) = 1$.

   b) Yes. Corollary 4.3.6 applies whenever utility is homothetic.

4. Let $A = \{(x, y) \in \mathbb{R}^2_+ : x^2 + y^2 \le 25\}$ and $\mathbf{x}_0 = (3, 4)$.

   a) Show that A is a convex set.

   b) Show that $\mathbf{x}_0 \notin \operatorname{int} A$.

   c) Use Support Property II from section 31.4.1 to find a vector $\mathbf{p}$ as in Separation Theorem D.

Answer:

   a) Now A is a lower contour set for the function $x^2 + y^2$. This function is convex as its Hessian

$$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

   is positive definite, so A is a convex set.

   b) $\operatorname{int} A = \{(x, y) : x^2 + y^2 < 25\}$. Since $3^2 + 4^2 = 25$, $\mathbf{x}_0 \notin \operatorname{int} A$.

   c) Let $f(x, y) = -x^2 - y^2$ so $A = \{(x, y) : f(x, y) \ge f(\mathbf{x}_0) = -25\}$. The Support Property Theorem II now tells us that

$$Df(\mathbf{x}_0)\cdot\mathbf{x} \ge Df(\mathbf{x}_0)\cdot\mathbf{x}_0$$

   for all $\mathbf{x} \in A$. Here $Df = (-2x, -2y)$, so $\mathbf{p} = Df(\mathbf{x}_0) = (-6, -8)$ and $\mathbf{p}\cdot\mathbf{x}_0 = -18 - 32 = -50$, so

$$\mathbf{p}\cdot\mathbf{x} \ge -50$$

   for all $\mathbf{x} \in A$ where $\mathbf{p} = (-6, -8)$.