9. Measuring Demand

Chapter Outline

1. Ordinary Elasticities
2. Elasticity of Substitution
3. Linear Expenditure Systems
4. Separability and Demand

In Chapters 4 and 5, we defined both Marshallian and Hicksian demand and established their basic properties, together with properties of the related indirect utility and expenditure functions. Now that we have the basic theory under control, we can connect this theory to some common methods of measuring demand.[1]

We start by looking at elasticity in section one. After establishing some basic properties of elasticity, section two uses the elasticity of substitution to distinguish substitutes and complements.

Section three considers two demand systems: the linear expenditure system and the addilog system. Finally, section four focuses on the relation between demand and separability, both direct and indirect.

NB: Our default commodity space in this chapter is $\mathbb{R}^m$ with consumption set $\mathfrak{X} = \mathbb{R}^m_+$. The consumer has continuous preferences $\succsim$ which can be represented by a continuous utility function u using Debreu's Representation Theorem. We will often assume utility is not just continuous, but also differentiable.

[1] Some of this chapter also applies to supply. This is partly due to the fact the consumer's expenditure minimization problem and the producer's cost minimization problem are identical from a mathematical point of view. The big difference is that utility is ordinal while production is cardinal. As a result, certain things make economic sense for supply but not for demand. Compare marginal product and marginal utility. The former is economically meaningful, the latter is not. Marginal utility's main importance is that it is used to find the marginal rate of substitution. Another important difference between demand and supply is that demand is based on a budget constraint, which has no analog for supply.

9.1 Elasticities

Elasticities measure how responsive quantities are to changes in price, income, or anything else that can change demand or supply. Of course, there is an obvious such measure. We could use the slope of the demand curve, drawn in mathematical fashion with price or any other variable of interest on the horizontal axis. In the case of price, this is a departure from the traditional demand curve with quantity on the horizontal axis and price on the vertical axis.

Using the slope introduces problems due to the fact that it is dependent on both how we measure quantities and how we measure prices. In the case of a good's own price, we get a number measured in units used to measure the quantity of that good, times the units used in pricing, divided by the monetary unit. Thus if we are considering quantity demanded measured in short tons and prices in dollars per ounce, the slope is measured in ounce-tons per dollar. The meaning of units such as these is not transparent. It is also an obstacle if we wish to make comparisons across goods (or countries) of response to own-price changes.

An easy way around this to measure the changes in percentage terms. This can be done by graphing the logarithm of quantity demanded versus the logarithm of price. The resulting slope is the elasticity. Any changes in units used simply offsets the curve. The change in units does not affect the slope.

Elasticity is independent of the units used to measure demand. Whether demand for milk is measured in quarts, gallons, liters, or cubic furlongs makes no difference as far as elasticity is concerned. Similarly, whether prices and income are measured in dollars, euros, yuan, rupees, or pesos has no effect on elasticity.

9.1.1 Elasticity of Demand

We formalize this using derivatives. The *price elasticity of demand* for good i with respect to the price of j is

$$\epsilon_{ij} = \frac{\partial \ln x_i}{\partial \ln p_j} = \frac{p_j}{x_i}\frac{\partial x_i}{\partial p_j}$$

where x_i is the Marshallian demand for good i. The elasticity is the percentage rate of change of prices divided by the percentage rate of change in quantity. Empirical studies of demand usually transform the quantities and prices into logarithms. This means that the estimated coefficients are elasticities.

The basic method of calculating elasticities is always the same: divide the percentage change in quantity by the percentage change in the variable of interest. Sometimes it is useful to adjust the calculation, typically by an income or cost share, but the basic idea is the same. When a function f depends on a variable x, its elasticity has the form

$$\epsilon = \frac{\partial \ln f}{\partial \ln x} = \frac{x}{f}\frac{\partial f}{\partial x}.$$

The different names for various elasticities reflect what f is and what x is.

One such case is the *compensated price elasticity of demand* defined by

$$\epsilon^*_{ij} = \frac{\partial \ln h_i}{\partial \ln p_j} = \frac{p_j}{h_i}\frac{\partial h_i}{\partial p_j}$$

where h is the Hicksian or compensated demand.

9.1.2 Elasticity of Linear Demand

Linear demand curves provide a simple example of elasticity.

Example 9.1.1: Elasticity of Linear Demand A simple example where we can compute the elasticity is a linear demand curve, with quantity demanded $q = a - bp$ where p is price and a and b are parameters. The quantity demanded goes to zero at the choke price $p = a/b$, and the formula only applies when $0 \leq p \leq a/b$.

We calculate the ordinary elasticity of demand as typically used in introductory microeconomics.[2] The elasticity of demand is

$$\begin{aligned} \epsilon &= \frac{p}{q}\frac{dq}{dp} = -b\frac{p}{q} \\ &= \frac{-bp}{a - bp}. \end{aligned}$$

It ranges from 0 to $-\infty$ as p goes from 0 to a/b. At the midpoint of the demand curve $p = a/2b$ and the elasticity is -1. In this case demand is *unit elastic*, meaning that $|\epsilon| = 1$. For $0 \leq p < a/2b$ demand is *inelastic*: $|\epsilon| < 1$. In contrast, for $a/2b < p \leq a/b$, demand is *elastic*: $|\epsilon| > 1$.[3] As the price increases, elasticity decreases and demand becomes more elastic, with quantity demanded more responsive to changes in price.

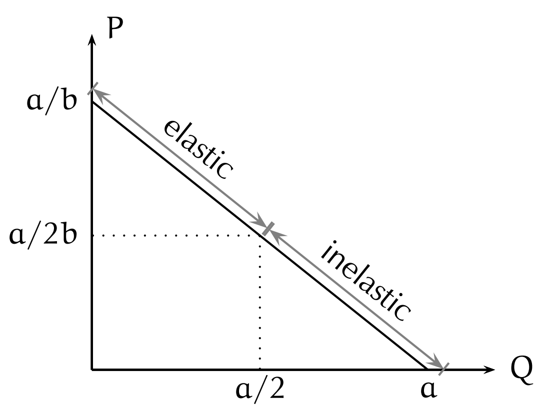

Figure 9.1.2: The elasticity of demand changes as we move along a linear demand curve. Absolute elasticity is high at high prices (low quantities) and low at low prices (high quantities).

◀

[2] Some authors prefer to use the absolute value.

[3] Here we use absolute value to facilitate comparison with elasticity of supply, which is positive.

9.1.3 More Elasticities

Example 9.1.1 is a case where $i = j$, where the quantity and price both refer to the same good. This is called the *own-price elasticity*. Normally, $\epsilon_{ii} \leq 0$ (Law of Demand), but elasticity will be positive when i is a Giffen good. Corollary 5.2.6 shows that the compensated own-price elasticity ϵ^*_{ii} is never positive.

Cross-Price Elasticity. When $i \neq j$, ϵ_{ij} is called the *cross-price elasticity*.

One way of classifying substitutes and complements is to use the cross-price elasticity. A natural way to define them is to call two goods substitutes if the cross-price elasticity is positive, meaning that an increase in the price of one leads to an increase in quantity demanded for the other. In other words, it increases demand, shifting the demand curve to the right. Similarly, two goods would be complements if the cross-price elasticity were negative. Unfortunately, due to the income effect, ϵ_{ij} and ϵ_{ji} may have different signs as we saw in section 5.2.

Income Elasticity. The *income elasticity of demand* for good i is defined in similar fashion, by

$$\eta_k = \frac{\partial \ln x_i}{\partial \ln m} = \frac{m}{x_i}\frac{\partial x_i}{\partial m}.$$

The income elasticity is the percentage change in quantity demanded divided by the percentage change in income.

The income elasticity can be used to distinguish three types of goods, depending on how demand responds to changes in income. The quantity demanded can fall. This means we buy less as we become richer. Such goods are called *inferior goods* and have $\eta_i < 0$.

Alternatively, purchases can rise with income. This is the usual case. Such goods have $\eta_i > 0$ and are called *normal goods*.

Finally, demand can strongly increase with income. These goods have $\eta_i > 1$ and are called *superior* or *luxury* goods. Superior goods are so strongly normal that the fraction of income devoted to them increases as income increases. In the United States, both education and health care are luxury goods, but housing is a normal good that is not a luxury good.

9.1.4 Example: Homothetic Preferences

Example 9.1.3: Homothetic Preferences When preferences are homothetic, Corollary 4.3.6 tells us that $x(\mathbf{p}, m) = mx(\mathbf{p}, 1)$. Then $d_m x(\mathbf{p}, m) = x(\mathbf{p}, 1)$. When $x \gg 0$, all goods are normal when preferences are homothetic. In fact, the income elasticity is always 1 when preferences are homothetic.[4] As a result, both Cobb-Douglas and CES preferences yield an income elasticity of 1 for every good. ◀

[4] This is part of Exercise 9.1.3.

9.1.5 Basic Properties of Elasticities

Two basic properties of demand, homogeneity of degree zero and Walras' Law, can be expressed in terms of elasticities.

We start with homogeneity of degree zero. By Theorem 4.3.4, $\mathbf{x}(\mathbf{p}, m)$ is homogeneous of degree zero in $(\mathbf{p}, m)$. Applying Euler's Theorem, we find

$$0 = m\frac{\partial x_i}{\partial m} + \sum_j p_j \frac{\partial x_i}{\partial p_j}.$$

Dividing by x_i and writing in terms of elasticities yields

$$0 = \eta_i + \sum_{j=1}^{m} \epsilon_{ij}. \tag{9.1.1}$$

Equation 9.1.1 expresses degree-zero homogeneity of demand.

We now turn to the adding-up condition, Walras' Law. Let $s_i = p_i x_i / m$ be the budget share of good i. Start with the budget constraint $\mathbf{p}\cdot\mathbf{x} = m$ and take the derivative with respect to m, yielding

$$\sum_i p_i \frac{\partial x_i}{\partial m} = 1.$$

Now substitute $p_i = s_i(m/x_i)$. This allows us to rewrite Walras' Law in terms of budget shares and income elasticities so that:

$$\sum_{i=1}^{m} s_i \eta_i = 1. \tag{9.1.2}$$

Of course the shares obey $0 \le s_i \le 1$ and $\sum_{i=1}^{m} s_i = 1$.

9.1.6 Homogeneity and Adding Up with Cobb-Douglas Utility

Cobb-Douglas utility provides a simple example.

Example 9.1.4: Cobb-Douglas and Elasticity Suppose the utility function has the Cobb-Douglas form

$$u(\mathbf{x}) = A \prod_{i=1}^{m} x_i^{\gamma_i}$$

where each $\gamma_i > 0$ and $\sum_i \gamma_i = 1$. We know that demand is

$$\mathbf{x}(\mathbf{p}, m) = m \left(\frac{\gamma_1}{p_1}, \dots, \frac{\gamma_m}{p_m} \right).$$

We can now compute the elasticities. We find $\epsilon_{ij} = 0$ for $i \neq j$ and $\epsilon_{ii} = -1$. Then $\sum_{j=1}^{m} \epsilon_{ij} = -1$. Since the income elasticity is $\eta_i = 1$, equation 9.1.1 is satisified.

Now the budget shares are $s_i = \gamma_i$, and $\eta_i = 1$. Then adding up (equation 9.1.2) is satisfied as $\sum_i \gamma_i = 1$. ◄

9.1.7 Elasticity Chain Rule

Besides the properties that follow from utility maximization, elasticities also behave well under differentiable transformations. We can write the elasticities of a composite function as a product of elasticities.

Proposition 9.1.5. *Suppose* $\phi\colon \mathbb{R} \to \mathbb{R}$ *and* $f\colon \mathbb{R}^m \to \mathbb{R}$ *are* $\mathcal{C}^1$ *functions. Define* $g = \phi \circ f(\mathbf{x})$*. Then* $\epsilon_i^g = \epsilon_f^\phi\, \epsilon_i^f$ *where* ϵ^ϕ*,* ϵ^f*, and* ϵ^g *denote the elasticities of* ϕ*,* f*, and* g*, respectively.*

Proof. By the chain rule, $\partial g/\partial x_i = (d\phi/df)\,(\partial f/\partial x_i)$. Then

$$\begin{aligned}\epsilon_i^g &= \frac{x_k}{g}\frac{\partial g}{\partial x_k} = \frac{x_i}{\phi \circ f}\frac{d\phi}{df}\frac{\partial f}{\partial x_i}\\ &= \left(\frac{f}{\phi \circ f}\frac{d\phi}{df}\right)\left(\frac{x_i}{f}\frac{\partial f}{\partial x_i}\right)\\ &= \epsilon_f^\phi\, \epsilon_i^g.\end{aligned}$$

□

One consequence of this is that if $\phi \circ f(\mathbf{x}) = x_i$, then $\epsilon_f^\phi\, \epsilon_i^f = 1$. When one function inverts another, the elasticities are inverses of each other.

9.2 Elasticities, Substitution, and Complementarity

2/2/23

In section 5.5, we saw that the Hicksian and Marshallian demands are related by the Slutsky equation

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial m} \tag{5.6.7}$$

for each k and j. This applies when $m = e(\mathbf{p}, \bar{u})$ or equivalently, $\bar{u} = v(\mathbf{p}, m)$.

Recall that compensated and ordinary demands are equal when $m = e(\mathbf{p}, \bar{u})$ or $\bar{u} = v(\mathbf{p}, m)$. That is $h_i\big(\mathbf{p}, v(\mathbf{p}, m)\big) = x_i(\mathbf{p}, m)$ and $h_i(\mathbf{p}, \bar{u}) = x_i\big(\mathbf{p}, e(\mathbf{p}, \bar{u})\big)$. We can use this to rewrite the Slutsky equation in elasticity terms as

$$\epsilon_{ij} = \epsilon^*_{ij} - s_j \eta_i$$

where ϵ^*_{ij} denotes the elasticity of Hicksian demand for i with respect to the price of j. Notice that the elasticity of Hicksian demand will be less than the (Marshallian) elasticity of demand for normal goods, and greater than the (Marshallian) elasticity of demand for inferior goods. This makes intuitive sense. If the price of j rises, it has two effects: the substitution effect captured by ϵ^*_{ij} (which reduces demand), and the income effect. Real income has fallen and in the case of a normal good this makes demand fall (hence the negative sign), while in the case of an inferior good demand rises.

9.2.1 A Naive Approach to Substitutes and Complements

The Slutsky equation also shows why we may get inconsistent results if we try to characterize substitutes and complements via Marshallian elasticities. Although the Hicksian elasticities ϵ^*_{ij} and ϵ^*_{ji} always have the same sign, the income term $(s_j\eta_i)$ may affect the Marshallian elasticities differently, depending on whether we are considering ϵ_{ij} or ϵ_{ji}.

As we saw in Example 5.5.2, the two types of demands may give different answers as to whether goods are substitutes or complements. This has led to the development of a variety of methods to measure substitutability and complementarity. One simple attempt is to look at the direct effect of consumption of one good on the marginal utility of another. That is, consider the cross-partial derivatives

$$\frac{\partial^2 u}{\partial x_i\,\partial x_j}.$$

The idea is that increasing consumption of a substitute lowers the marginal value of the original good ($\partial^2 u/\partial x_i \partial x_j < 0$) while increasing consumption of a complement raises the marginal value of the original good ($\partial^2 u/\partial x_i \partial x_j > 0$).

While this might make sense for production, where marginal product is a cardinal quantity with an objective meaning, it fails in the utility setting because marginal utility is ordinal and subjective.[5] Different representations of the same preference order can give quite different answers to whether a pair of goods are substitutes or complements, as in the following example.

[5] Hicks (1970) defines q-substitutes and q-complements. For increasing production functions, i and j are q-*substitutes* if $\partial^2 f/\partial x_i \partial x_j > 0$ and q-*complements* if $\partial^2 f/\partial x_i \partial x_j < 0$.

9.2.2 Cobb-Douglas: Substitutes or Complements?

Example 9.2.1: Cobb-Douglas: Substitutes or Complements? Consider the Cobb-Douglas form $u(\mathbf{x}) = \ln x_1 + \ln x_2$. A quick computation shows that

$$\frac{\partial^2 u}{\partial x_1\, \partial x_2} = 0,$$

indicating the goods are neither substitutes nor complements.

What if we use the equivalent form $u(\mathbf{x}) = \sqrt{x_1 x_2}$ where

$$\frac{\partial^2 u}{\partial x_k\, \partial x_\ell} = \frac{1}{4\sqrt{x_1 x_2}} > 0.$$

We must conclude they are substitutes.

The little-used form $u(\mathbf{x}) = -1/(x_1 x_2)$ is also equivalent, and yields

$$\frac{\partial^2 u}{\partial x_k\, \partial x_\ell} = \frac{-1}{x_1^2 x_2^2} < 0.$$

Goods one and two must be complements! ◄

9.2.3 Developing the Elasticity of Substitution

It is pretty clear that this naive definition fails to do the job when dealing with consumer demand. It fails because it is not an ordinal definition. When characterizing separability, we replaced the cardinal condition that $\partial^2 u/\partial x_i \partial x_j = 0$ with an ordinal condition based on the marginal rate of substitution. That is what we will do here also, but in a different way suggested by Hicks (1932).

Hicks introduced the elasticity of substitution in a two-factor model of production. To quote Blackorby and Russell (1989), "The elasticity of substitution was originally introduced by John R. Hicks (1932) for the purpose of analyzing changes in the income shares of labor and capital in a growing economy. Hicks' key insight was to note that the effect of changes in the capital/labor ratio (or the factor price ratio) on the distribution of income (for a given output) can be completely characterized by a scalar measure of curvature of the isoquant."[6]

[6] The elasticity of substitution was discovered independently by Robinson (1933). Lerner (1933) and Pigou (1934) helped clarify the definition, leading to its current form.

9.2.4 Elasticity of Substitution

The *elasticity of substitution* σ pertains to both consumption and production. It is defined by

$$\sigma_{ij} = -\left.\frac{\partial \ln \mathrm{MRS}_{ij}}{\partial \ln(x_i/x_j)}\right|_u \qquad \text{or} \qquad \frac{1}{\sigma_{ij}} = -\left.\frac{\partial \ln(x_i/x_j)}{\partial \ln \mathrm{MRS}_{ij}}\right|_u$$

The elasticity of substitution is the percentage change in the marginal rate of substitution (or prices of goods or inputs) divided by the percentage change in the proportions of the corresponding goods (inputs) as we move along the indifference curve u (or along an isoquant in the production case).

Sometimes this is written using $\ln(x_j/x_i)$ instead of $\ln(x_j/x_i)$, in which case the minus sign disappears. Goods are *substitutes* when $\sigma_{ij} > 1$ and *complements* when $\sigma_{ij} < 1$. The elasticity of substitution is the same for all equivalent utility representations because it is defined in terms of the marginal rate of substitution.

Further, it is easy to see that $\sigma_{ij} = \sigma_{ji}$. Reversing the order merely flips the sign of both logarithms, leaving the ratio unchanged.

9.2.5 More on Elasticity of Substitution

If there are two goods, the elasticity of substitution is clear-cut. We hold utility constant (output for production functions). Then once we know x_1/x_2, we have specified a particular point on the given indifference curve. When there are three (or more) goods, it is not enough to know utility and x_1/x_2. We need to hold something else constant to determine where and how to take the partial derivative. So what is it? x_2? x_3? x_1/x_3? x_2/x_3? Or do we alter the amounts of the other goods to a new optimal point on the indifference curve or isoquant. It can make a difference and whatever we choose has to be applied systematically.

We do not have these problems when utility or production is both homothetic and additive separable. In that case, additive separability implies that the marginal rate of substitution between i and j depends only on x_i and x_j. By homotheticity, the marginal rate of substitution is homogeneous of degree zero. The combination of additive separability and homotheticity means that the marginal rate of substitution between i and j depends only on x_i/x_j. This eliminates any ambiguity about how to define the define the marginal rate of substitution. We do not need to constrain ourselves to a particular indifference curve when preferences are additive separable and homothetic. Even without the restriction on utility, the elasticity of substitution is well-defined.

For most utility functions the marginal rate of substitution between x_i and x_j does not depend solely on x_i/x_j. Hicks and Allen (1934) suggested two possible ways of handling this problem. One is to hold consumption of all goods other than i and j constant, this is the *Hicks elasticity of substitution*, and we will call it the *elasticity of substitution*. Hicks and Allen also proposed what they called the *partial elasticity of substitution*, later known as the *Allen-Uzawa elasticity of substitution*, which involves other choices about what varies and how. Neither concept is entirely successful at generalizing the two good case. In particular, they don't give information about factor or spending shares except in the case where utility or production is both homothetic and additive separable. In that case, Bergson's Separability Theorem applies and utility or the production function must have the CES form (or equivalent).[7]

[7] Blackorby and Russell argue that yet another elasticity of substitution, the Morishima (1967) elasticity of substitution is the proper generalization.

9.2.6 Cobb-Douglas Elasticity of Substitution

Things are a bit more complex when utility or production is homothetic but not separable. In that case the marginal rate of substitution is still homogeneous of degree zero (Theorem 3.1.2). We can write $MRS(\mathbf{x}) = MRS(x_1/x_j, \dots, x_m/x_j)$ and an obvious choice when computing σ_{ij} is to hold the other x_i/x_j constant as well as utility. But we then have to ask how σ_{ji} relates to σ_{ij}.

Example 9.2.2: Cobb-Douglas Elasticity of Substitution Let's apply this to the Cobb-Douglas utility, $u(\mathbf{x}) = (x_1 x_2)^{1/2}$. This utility function is particularly well-behaved as far as the elasticity of substitution is concerned. Here $MRS_{12} = (x_2/x_1) = (x_1/x_2)^{-1}$, which is solely a function of x_1/x_2. Then $\partial\, MRS_{12}/\partial(x_1/x_2) = -(x_1/x_2)^{-2}$. It follows that

$$\sigma_{12} = -\frac{x_1/x_2}{MRS_{12}} \cdot \frac{\partial\, MRS_{12}}{\partial(x_1/x_2)} = -\frac{x_1/x_2}{MRS_{12}} \cdot \frac{-1}{(x_1/x_2)^2} = \frac{(x_1/x_2)^2}{(x_1/x_2)^2} = 1$$

so $\sigma_{12} = 1$. In the Cobb-Douglas case, goods are neither substitutes nor complements.

As we saw in Example 5.5.2, the cross-price elasticity for Marshallian demand $x_k = \gamma_k m/p_k$ is zero: $\epsilon_{12} = \epsilon_{21} = 0$. This occurs because the income effect cancels the substitution effect. In contrast, we saw that these goods are substitutes for Hicksian demand: $\epsilon^*_{12} = \epsilon^*_{21} = 1/2 > 0$.

Before we close this example, we consider an alternative method of calculating σ_{12}. The point is that utility is being held constant, so $u^2 = x_1 x_2$. Then $x_2 = u^2/x_1$ and $x_1/x_2 = x_1^2/u^2$. Also $MRS_{12} = x_2/x_1 = u^2/x_1^2$. Substituting all this, we obtain

$$\sigma_{12} = -\frac{x_1/x_2}{MRS_{12}} \cdot \frac{\partial\, MRS_{12}}{\partial(x_1/x_2)} = -\frac{x_1^4}{u^4} \cdot \frac{\partial(u^2/x_1^2)}{\partial(x_1^2/u^2)} = +\frac{x_1^4}{u^4} \cdot \frac{u^4}{x_1^4} = +1.$$

This second method of calculation can be used even when the utility function is not homothetic. ◀

9.2.7 Constant Elasticity of Substitution

When there are two goods, it is fairly easy to show that any increasing utility function with constant elasticity of substitution utility must be equivalent to a CES function. We've already done the hard work.

For σ_{12} to make sense, utility has to be at least twice continuously differentiable and $du \gg 0$. Now suppose u has elasticity of substitution $\sigma_{12} = \sigma$. Then $\ln \mathrm{MRS}_{12} = C - (1/\sigma)\ln(x_1/x_2)$. This implies that MRS_{12} is homogeneous of degree zero, so utility is homothetic by Theorem 3.1.4. Further, $\ln \mathrm{MRS}_{12}$ is additive separable, so utility is also additive separable. Bergson's Separability Theorem tells us that utility must have a CES form, where we include the Cobb-Douglas case as $\rho = -1$. Moreover, if $u(x_1, x_2)$ has the CES form, it is not hard to show that $\sigma = 1/(1 + \rho)$.

9.3 Demand Systems

For empirical purposes, it is helpful to have demand equations that are both based on utility maximization, and which can represent a variety of consumer behaviors.

When estimating demand in a single market, we do not have to worry about how demand in that market fits with estimates of demand in other markets. However, when doing more comprehensive studies of consumer expenditure, it is important to realize that the markets are related. This has implications. It limits the form that a demand system may take.

Demand systems write demand as a function of prices and income. Under weak hypotheses about consumer choice, that they spend all of their income,[8] and that demand is a function of the budget set, we obtain two fundamental restrictions concerning demand. One is the adding-up condition, Walras' Law. Consumer expenditures on all goods must equal consumer income. In terms of elasticities and income shares, $\sum_i s_i \eta_i = 1$. The other is zero-degree homogeneity in prices and income. In elasticity terms, $\eta_i + \sum_j \epsilon_{ij} = 0$.

If we think of the demand system of being generated directly from utility maximization, as when we have a representative consumer, there are two more restrictions.[9] The Slutsky matrix must be symmetric ($\epsilon^*_{ij} = \epsilon^*_{ji}$ for $i \neq j$), and also negative semidefinite. We will sometimes strengthen the last condition to require that the Slutsky matrix be negative definite.

[8] For this, it is sufficient to posit local non-satiation.

[9] See Chapter 12 for a discussion of the properties of market demand, and when it can be generated by a representative consumer.

9.3.1 Linear Expenditure System

The linear expenditure system is one of the simpler demand systems.[10]

A system of demand equations is a *linear expenditure system* if can be written

$$p_i x_i = \gamma_i m + \sum_{j=1}^{m} \beta_{ji} p_j \tag{9.3.3}$$

for every $i = 1, \ldots, m$ with $m > 1$.

It is called a linear expenditure system because for each good, expenditure $p_i x_i$ is a linear function of prices and income. Needless to say, it will be necessary to impose restrictions on $(\mathbf{p}, m)$ to ensure that the demand equations make sense. For example, we don't want to allow price-income pairs that yield negative expenditure or demand. We also want the system to be derived from utility maximization. As such, demand should be homogeneous of degree zero in $(\mathbf{p}, m)$, obey Walras' Law and Slutsky symmetry, and the substitution matrix should be negative semi-definite.

Demand is homogeneous of degree zero if and only if expenditure is homogeneous of degree one. This is automatically guaranteed by the linearity of the linear expenditure system. The requirements of Walras' Law, Slutsky symmetry, and a negative definite substitution matrix will impose further conditions on the parameters γ_i and β_{ji}.

There are parameters that work. It is easy to see that Cobb-Douglas utility generates a linear expenditure system (Problem 9.3.1). Moreover, the resulting linear expenditure system is valid whenever prices and income are strictly positive.

[10] The linear expenditure system was introduced by Klein and Rubin (1947-48). This and related demand systems, including the almost ideal demand system, are discussed in Deaton and Muellbauer (1980). See Barnett and Serletis (2008) for a more recent survey of the consumer demand systems literature.

9.3.2 Stone-Geary Expenditure System

Another utility function that generates a linear expenditure system is the Stone-Geary utility function.

Example 9.3.1: Stone-Geary Expenditure System Consider the Stone-Geary utility function $u(\mathbf{x}) = \prod_{i=1}^{m}(x_i - a_i)^{\gamma_i}$ where each $\gamma_i > 0$ and $\sum_i \gamma_i = 1$. Stone-Geary utility is defined on the consumption set $\mathfrak{X} = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geq \mathbf{a}\}$. Demand is only valid when prices and income obey $\mathbf{p}\cdot\mathbf{a} \leq m$.[11]

We will assume that $\mathbf{a}$ and m obey $m \geq \mathbf{p}\cdot\mathbf{a}$. By Exercise 4.2.4 the demand function for good i is

$$x_i = a_i + \frac{\gamma_i}{p_i}\big(m - \mathbf{p}\cdot\mathbf{a}\big).$$

We can write this in terms of expenditures as

$$p_i x_i = p_i a_i + \gamma_i(m - \mathbf{p}\cdot\mathbf{a}). \tag{9.3.4}$$

The Stone-Geary expenditure system can now be rewritten in the linear expenditure form

$$p_i x_i = \gamma_i m + \sum_{j=1}^{m} \beta_{ji} p_j$$

by setting $\beta_{ji} = -\gamma_i a_j$ for $j \neq i$, and $\beta_{ii} = (1 - \gamma_i)a_i$.

Returning to equation 9.3.4, we can break the Stone-Geary expenditure on good i into subsistence expenditure, $p_i a_i$, plus income dependent expenditure. The latter is given by good i's marginal budget share γ_i times disposable income $(m - \mathbf{p}\cdot\mathbf{a})$.

The income-expenditure relation (Engel curve) is linear for the Stone-Geary utility function as well as in the linear expenditure system. With Stone-Geary utility, equation 9.3.4 shows that the Engel curves do not go through the origin unless subsistence is zero. The own-price elasticities are all between 0 and -1 while the cross-price elasticities are negative (Problem 9.1.6). ◄

As we will see, any reasonable linear expenditure system is generated by a Stone-Geary utility function. This includes the Cobb-Douglas case, which is a Stone-Geary utility function where the subsistence level is zero ($\mathbf{a} = \mathbf{0}$).

[11] The Stone-Geary utility is due to Geary (1950), Stone (1954), and Samuelson (1947-48). Although Samuelson noted its relationship to the linear expenditure system first, Stone and Geary made it useful for empirical work. In fact, the basic idea was first used in a production context by Tinbergen (1942).

9.3.3 Walras' Law and the Linear Expenditure System

Next we will show that any linear expenditure system derived from a utility function must be generated by a Stone-Geary utility function. We know that the linear expenditure system obeys homogeneity. We now turn to the requirement that it obey Walras' Law and Slutsky symmetry, and that the substitution matrix be negative semi-definite.

Our first target is Walras' Law.

Lemma 9.3.2. *Suppose the set S of allowable price-income pairs has an interior point. If Walras' Law holds for the linear expenditure system, then*

$$\sum_i \gamma_i = 1 \text{ and } \sum_i \beta_{ji} = 0 \tag{9.3.5}$$

for every $j = 1, \dots, m$.

Proof. Summing equation 9.3.3 over all i shows that Walras' Law holds if and only if

$$\begin{aligned} m = \sum_i p_i x_i &= \left(\sum_i \gamma_i\right) m + \sum_{ji} \beta_{ji} p_j \\ &= \left(\sum_i \gamma_i\right) m + \sum_j \left(\sum_i \beta_{ji}\right) p_j \end{aligned} \tag{9.3.6}$$

for all $(\mathbf{p}, m)$. Let $(\mathbf{p}^0, m^0)$ be an interior point of S. Then there is an $\varepsilon > 0$ so that equation 9.3.6 holds with $(\Delta\mathbf{p}, \Delta m)$ substituted for $(\mathbf{p}, m)$ provided that $\|(\Delta\mathbf{p}, \Delta m)\| < \varepsilon$. Linearity allows us to extend this to all $(\Delta\mathbf{p}, \Delta m) \in \mathbb{R}^{m+1}$. Since this must be true for every choice of Δm and $\Delta\mathbf{p}$, the coefficient on m must be 1 and the coefficients on each p_j must be zero. In other words, equations 9.3.5 hold. □

9.3.4 The Substitution Matrix in the Linear Expenditure System

The remaining restrictions on the linear expenditure system that reduce it to the Stone-Geary system derive from the substitution matrix. Recall that the terms of the substitution matrix are given by

$$s_{ji} = \frac{\partial x_j}{\partial p_i} + x_i \frac{\partial x_j}{\partial m}. \tag{5.5.5}$$

When $i \neq j$, we can make this easier to work with by multiplying each substitution term s_{ji} by $p_j p_i$. This yields

$$\begin{aligned} p_j p_i s_{ji} &= p_i \frac{\partial (p_j x_j)}{\partial p_i} + p_i x_i \frac{\partial (p_j x_j)}{\partial m} \\ &= p_i \beta_{ij} + \gamma_j p_i x_i \\ &= \gamma_j \gamma_i m + p_i \beta_{ij} + \gamma_j \left(\sum_j \beta_{ji} p_j \right). \end{aligned} \tag{9.3.7}$$

for $i \neq j$. Here we have used the linear expenditure system (equation 9.3.3) to replace the derivatives in second line and $p_i x_i$ in the third line.

This doesn't work quite the same way when $i = j$. When $i = j$, we must take into account that $p_i \partial x_i / \partial p_i = -x_i + \partial (p_i x_i)/\partial p_i$. Then

$$\begin{aligned} p_i^2 s_{ii} &= -p_i x_i + p_i \frac{\partial (p_i x_i)}{\partial p_i} + p_i x_i \frac{\partial (p_i x_i)}{\partial m} \\ &= p_i \beta_{ii} - (1 - \gamma_i) p_i x_i \\ &= p_i \beta_{ii} - (1 - \gamma_i) \gamma_i m - (1 - \gamma_i) \left(\sum_j \beta_{ji} p_j \right) \end{aligned} \tag{9.3.8}$$

If m can be increased without bound, as happens in the Stone-Geary case, a negative definite substitution matrix implies $(1 - \gamma_i)\gamma_i > 0$. From that, it follows that $0 < \gamma_i < 1$.

9.3.5 Slutsky Symmetry in the Linear Expenditure System

Lemma 9.3.3. *Suppose the set S of allowable price-income pairs has an interior point. If Slutsky symmetry holds for a linear expenditure system then for all $i \neq j$,*

$$(1-\gamma_i)\beta_{ij} = -\gamma_j\beta_{ii}$$

and for any distinct i, j, and k.

$$\gamma_j\beta_{ki} = \gamma_i\beta_{kj}.$$

Proof. By Slutsky symmetry, $p_i p_j s_{ij} = p_i p_j s_{ji}$. We employ equation 9.3.7 and cancel the $\gamma_i\gamma_j m$ terms to obtain

$$p_i\beta_{ij} + \gamma_j\left(\sum_k \beta_{ki}p_k\right) = p_j\beta_{ji} + \gamma_i\left(\sum_k \beta_{kj}p_k\right). \tag{9.3.9}$$

As in the proof of Lemma 9.3.2, the coefficients on each p_k must be equal, so for $i \neq j$, $\beta_{ij} + \gamma_j\beta_{ii} = \gamma_i\beta_{ij}$ while for distinct i, j, and k, $\gamma_j\beta_{ki} = \gamma_i\beta_{kj}$. The first of these can be rewritten as $(1-\gamma_i)\beta_{ij} = -\gamma_j\beta_{ii}$. □

9.3.6 Characterizing the Linear Expenditure System

We saw earlier that Stone-Geary demand functions form a linear expenditure system. We can now prove the converse.

Proposition 9.3.4. *Suppose the set S of allowable price-income pairs for a linear expenditure system has an interior point and has no upper limit on income for each* $\mathbf{p}$*. If a linear expenditure system obeys Walras' Law and Slutsky symmetry, and has a negative semidefinite substitution matrix, it can be derived from a Stone-Geary utility function.*

Proof. The negative semidefinite substitution matrix implies each γ_i obeys $0 \leq \gamma_i \leq 1$. Moreover, by Lemma 9.3.2, $\sum \gamma_i = 1$, implying we can find an index ℓ with $\gamma_\ell > 0$.

Now whenever i, j, and ℓ are distinct, $\gamma_i \beta_{j\ell} = \gamma_\ell \beta_{ji}$ by Lemma 9.3.3. We rearrange this to show

$$\beta_{ji} = \gamma_i \frac{\beta_{j\ell}}{\gamma_\ell} \tag{9.3.10}$$

whenever i, j, and our special index ℓ are distinct. Now define

$$a_j = -\frac{\beta_{j\ell}}{\gamma_\ell}$$

so that $\beta_{ji} = -\gamma_i a_j$ for all $j \neq i$.

Next we consider β_{ii} for every $i \neq \ell$. Lemma 9.3.3 also tells us that $(1-\gamma_i)\beta_{i\ell} = -\gamma_\ell \beta_{ii}$ for $i \neq \ell$. It follows that $\beta_{ii} = (1-\gamma_i)a_i$ for $i \neq \ell$.

That leaves $\beta_{\ell\ell}$. If there is a $k \neq \ell$ with $\gamma_k > 0$, we can rearrange equation 9.3.10 to show that $a_j = -\beta_{jk}/\gamma_k$. Then $(1-\gamma_\ell)\beta_{\ell k} = -\gamma_k \beta_{\ell\ell}$, which implies $\beta_{\ell\ell} = (1-\gamma_\ell)a_\ell$.

If there is no such k, then $\gamma_\ell = 1$. By equation 9.3.5, we find $\beta_{\ell\ell} = -\sum_{i\neq\ell} \beta_{\ell i} = \sum_{i\neq\ell} \gamma_i a_\ell$. But since $\gamma_\ell = 1$, all the other γ_i are zero, showing that $\beta_{\ell\ell} = 0$.

By Example 9.3.1, the Stone-Geary utility defined by γ_i and $\mathbf{a}$ generates our linear expenditure system □

The case where the substitution matrix is only negative semi-definite is similar, but some extra care must be taken to handle the possibility that some of the γ_i are zero.

9.3.7 Direct and Indirect Addilog Utility

One alternative to the linear expenditure system is the addilog system. Although both these demand systems are fairly simple to use, the addilog system is more flexible. The linear expenditure system requires that Engel curves always be straight lines, and does not allow for inferior goods, goods with elastic demand, or goods that are gross substitutes ($\epsilon_{ij} < 0$). The indirect addilog system allows for all of these.[12] [13]

Suppose indirect utility has the *addilog* form

$$v(\mathbf{p}, m) = \sum_{i=1}^{m} a_i \left(\frac{m}{p_i}\right)^{b_i}$$

where $a_i, b_i > 0$. Because we have written m/p_i instead of p_i/m, $b_i > 0$ ensures that v is increasing in income m and decreasing in prices $\mathbf{p}$. We use Roy's Identity to calculate demand

$$x_j = -\frac{\partial v/\partial p_j}{\partial v/\partial m} = \frac{a_j b_j \left(\frac{m}{p_j}\right)^{b_j}}{\sum_i a_i b_i \left(\frac{1}{p_i}\right)\left(\frac{m}{p_i}\right)^{b_j}}. \tag{9.3.11}$$

[12] The indirect addilog utility was introduced independently by Konüs (1939) and Leser (1941). Houthakker (1960) showed it could be derived from an addilog indirect utility function.

[13] The origin of the name "addilog" is a little obscure. Houthakker (1960) describes the addilog indirect utility as a sum of logarithmic functions, even though it isn't. A year later, in Houthakker (1961), it comes from the fact that the logarithm of the ratio of expenditures on any two goods is the sum of logarithmic functions.

9.3.8 Estimation of Addilog Demand Systems

Equation 9.3.11 is in a form that is quite unsuitable for estimation via linear regression. This can fixed by computing the expenditure ratios and taking the logarithm. The result is equation 9.3.12 below, which **is** suitable for linear regression.

$$\frac{p_j x_j}{p_i x_i} = \frac{\partial v/\partial p_j}{\partial v/\partial p_i} = \frac{a_j b_j \left(\frac{m}{p_j}\right)^{b_j}}{a_i b_i \left(\frac{m}{p_i}\right)^{b_i}}$$

It follows that the logarithm of the expenditure ratio is

$$\ln\left(\frac{p_j x_j}{p_i x_i}\right) = \ln\left(\frac{a_j b_j}{a_i b_i}\right) + b_j \ln\left(\frac{m}{p_j}\right) + b_i \ln\left(\frac{m}{p_i}\right) \tag{9.3.12}$$

For comparison, if direct utility is in addilog form

$$u(\mathbf{x}) = \sum_{i=1}^{m} \alpha_i x_i^{\beta_i}$$

with $\alpha_i, \beta_i > 0$, we can combine first-order conditions to show that

$$\ln\left(\frac{m}{p_j}\right) - \ln\left(\frac{m}{p_i}\right) = \ln\left(\frac{\alpha_i \beta_i}{\alpha_j \beta_j}\right) + (1 - \beta_j) \ln x_j - (1 - \beta_i) \ln x_i$$

which is broadly similar to equation 9.3.12, but with the roles of quantities and prices reversed. You will notice that we have avoided calculating the indirect utility function. There is a reason for that, which you find out if you try!

9.4 Separability and Demand

Demand derived from additive separable utility has some special properties. A few will be covered now. Others will be apparent once we study duality (Chapter 5). The first result shows that for any pair of goods, the ratios of price derivatives of demand and income derivatives of demand are equal. This is a key step in showing that every good must be normal when utility is additive separable.[14]

Proposition 9.4.1. *Suppose preferences can be represented by an additive separable utility function* $u(\mathbf{x}) = \sum_i u_i(x_i)$ *on* $\mathbb{R}^m_+$*, that each* u_i *is twice continuously differentiable with* $u_i' > 0$ *and* $u_i'' < 0$*, and that all solutions to the consumer's utility maximization problem have* $\mathbf{x} \gg \mathbf{0}$*. Then for every* i, j *and* $k \neq i, j$*, the Marshallian demand functions obey*

$$\frac{\partial x_j/\partial p_k}{\partial x_i/\partial p_k} = \frac{\partial x_j/\partial m}{\partial x_i/\partial m} \tag{9.4.13}$$

provided $\partial x_i/\partial p_k \neq 0$*. Stated in elasticity terms,* $\epsilon_{jk}/\epsilon_{ik} = \eta_j/\eta_i$*.*

Proof. Consider the first-order conditions for the consumer's utility maximization problem, $\lambda p_i = u_i'(x_i)$. We can eliminate λ by considering goods i and j. This yields

$$p_i u_j'(x_j) = p_j u_i'(x_i). \tag{9.4.14}$$

Differentiating equation 9.4.14 with respect to income m yields

$$p_i u_j'' \frac{\partial x_j}{\partial m} = p_j u_i'' \frac{\partial x_i}{\partial m}. \tag{9.4.15}$$

Similarly, we may differentiate equation 9.4.14 with respect to p_k for $k \neq i, j$, obtaining

$$p_i u_j'' \frac{\partial x_j}{\partial p_k} = p_j u_i'' \frac{\partial x_i}{\partial p_k}. \tag{9.4.16}$$

The conditions on utility ensure $\partial x_i/\partial m > 0$, while the demand hypotheses require $\partial x_i/\partial p_k \neq 0$. This allows us to combine equations 9.4.15 and 9.4.16, obtaining

$$\frac{p_j u_i''}{p_i u_j''} = \frac{\partial x_j/\partial p_k}{\partial x_i/\partial p_k} = \frac{\partial x_j/\partial m}{\partial x_i/\partial m}. \tag{9.4.17}$$

The elasticity version follows immediately. □

The extra assumption that $\partial x_i/\partial p_k \neq 0$ is not a trivial one. It rules out Cobb-Douglas utility where the left-hand side of equation 9.4.13 is 0/0 for $k \neq i, j$ while the right-hand side is $x_j/x_i = (\gamma_j/p_j)/(\gamma_i/p_i)$ with γ_j denoting the budget shares.

Theorem 9.4.1 does not require that utility be in an additive separable form, merely that it be equivalent to one. That makes sense as it only imposes conditions on demand, which will be the same for any equivalent utility.

[14] The sections on separability and indirect separability draw on Houthakker (1960).

9.4.1 Separability Implies Normality

The proof of Theorem 9.4.1 has an important corollary. When preferences can be represented by a well-behaved additive separable utility function, every good is normal.

Corollary 9.4.2. *Under the main hypotheses of Proposition 9.4.1, demand for every good is normal when prices are strictly positive.*

Proof. Since prices are positive and each $u_k'' < 0$, the ratios in equation 9.4.17 are all positive. Then $\partial x_i/\partial m$ and $\partial x_j/\partial m$ have the same sign. Now differentiate Walras' Law, $\sum_i p_i x_i = m$ with respect to income. That yields

$$\sum_i p_i \frac{\partial x_i}{\partial m} = 1.$$

Since all of the income derivatives have the same sign, this equation implies they must all be positive. □

The corollary applies a bit more broadly than equation 9.4.17, which also requires that all price derivatives be non-zero. In particular, it works with Cobb-Douglas utility, which we already know to be normal by direct calculation of demand.

9.4.2 Indirect Separability

SKIPPED

There are actually two ways utility can be additive separable. The utility function itself can be additive separable. We will sometimes call such utility functions *directly separable*. Alternatively, utility may be *indirectly separable* meaning that indirect utility is additive separable. In other words, there are indirect subutility functions φ_i with $v(\mathbf{p}, 1) = \sum_{i=1}^{m} \varphi_i(p_i)$.

We will focus on the case where indirect utility is twice continuously differentiable, implying that each $\varphi_i \in \mathcal{C}^2$. To make the calculations more compact and readable, we use the notation v_i for $\partial v/\partial p_i = \partial \varphi_i/\partial p_i$ for $i = 1, \ldots, m$, and $v_0 = \partial v/\partial m$. Notice that the income derivative is v_0, while the price derivatives are v_i for $i = 1, \ldots, m$.

We use a similar notation for second partials: v_{ij} denotes $\partial^2 v/\partial p_i \partial p_j$ for $k = 1, \ldots, m$, and v_{i0} denotes $\partial^2 v/\partial p_i \partial m$. Thus v is indirectly separable if and only if $v_{ij} = 0$ whenever $i \neq j$ and $i, j \geq 1$.[15]

[15] Recall that we use η for income elasticities, thus we would write η_i rather than ϵ_{i0}.

9.4.3 Indirectly Separable Demand

SKIPPED

One important result for indirectly additive utility involves the slope of the demand curve. If $i, j \neq k$, the ratio of the slopes of demand for i and j with respect to the price of k is the same as the ratio of demand itself. This is more concisely stated in elasticity terms as as $\epsilon_{ik} = \epsilon_{jk}$.

Proposition 9.4.3. *Let $v(\mathbf{p}, m)$ be equivalent to an indirect utility function u that is additive separable. Suppose both $u, v \in \mathcal{C}^2$ with $u_i' > 0$ and $u_i'' < 0$, and that all solutions to the consumer's utility maximization problem have $\mathbf{x} \gg \mathbf{0}$. Then the demand functions obey*

$$\frac{\partial x_i/\partial p_k}{\partial x_j/\partial p_k} = \frac{x_i}{x_j} \tag{9.4.18}$$

for all goods $i, j \neq k$ obeying $\partial x_j/\partial p_k \neq 0$. In elasticity terms, $\epsilon_{ik} = \epsilon_{jk}$.

Proof. Since we will be dealing with demand functions, any equivalent indirect utility function will give the same results. Without loss of generality, we assume that v is additive separable.

Our calculations begin with Roy's Identity, now written $x_i = -v_i/v_0$. Taking the p_k derivative of Roy's Identity, and then substituting it in the result, we find

$$\begin{aligned}\frac{\partial x_i}{\partial p_k} &= \frac{-v_{ik} + v_{k0}(v_i/v_0)}{v_0}\\ &= \frac{-v_{ik} - x_i v_{k0}}{v_0}\end{aligned} \tag{9.4.19}$$

We obtain the following sequence of equations by first taking the p_k derivative of Walras' Law. Then substitute equation 9.4.19 into Walras' Law. Step 3 uses the separability condition that $v_{ik} = 0$ for $i \neq k$. Finally, we apply Walras' Law itself in the last line. $x_k + \sum_i p_i(\partial x_i/\partial p_i) = 0$.

$$\begin{aligned}0 &= x_k + \sum_{i=1}^m p_i \frac{\partial x_i}{\partial p_k}\\ &= x_k + \sum_{i=1}^m p_i \frac{-v_{ik} - x_i v_{k0}}{v_0}\\ &= x_k - p_k \frac{v_{kk}}{v_0} - \sum_{i=1}^m p_i x_i \frac{v_{k0}}{v_0}\\ &= x_k - p_k \frac{v_{kk}}{v_0} - m\frac{v_{k0}}{v_0}.\end{aligned}$$

(Proof continues on next page...)

9.4.4 Proof of Proposition 9.4.3, Part II

SKIPPED

Remainder of Proof. The final line is the important part, which can be rearranged to obtain

$$\frac{v_{k0}}{v_0} = \frac{x_k}{m} - \frac{p_k}{m}\frac{v_{kk}}{v_0} \tag{9.4.20.}$$

Now for $i \neq k$, equation 9.4.19 becomes

$$\begin{aligned}\frac{\partial x_i}{\partial p_k} &= -\frac{x_i v_{k0}}{v_0} \\ &= x_i\left(\frac{x_k}{m} - \frac{p_k}{m}\frac{v_{kk}}{v_0}\right)\end{aligned} \tag{9.4.21}$$

which implies

$$\frac{\partial x_i/\partial p_k}{\partial x_j/\partial p_k} = \frac{x_i}{x_j} \tag{9.4.22}$$

for all $i, j \neq k$. Keep in mind that this only makes sense because the denominator $\partial x_i/\partial p_k$ is not zero (as we assume).

Finally, equation 9.4.22 can be written in elasticity terms as $\epsilon_{ik} = \epsilon_{jk}$ for any goods $i, j \neq k$. □

This tells us that the ratio of cross-price derivatives of demand is equal to the ratio of demand when utility is indirectly additive separable. In elasticity terms, it says that the cross-price elasticities with respect to k are the same for all goods (other than k). That is $\epsilon_{ik} = \epsilon_{jk}$ for $i, j \neq k$.

9.4.5 Combining Direct and Indirect Separability

SKIPPED

It is interesting to compare equation 9.4.18 with Equation 9.4.13, which leads to the following theorem.

Theorem 9.4.4. *Suppose utility is $\mathcal{C}^2$ on $\mathbb{R}^m_+$ and both directly and indirectly additive separable. If also $\partial x_i/\partial p_k \neq 0$ for all $i \neq k$, then Marshallian demand is linear in income, $\mathbf{x}(\mathbf{p}, m) = m\mathbf{x}(\mathbf{p}, 1)$ and both direct utility $u(\mathbf{x})$ and indirect utility $v(\mathbf{p}/m, 1)$ are homothetic.*

Proof. Combining equations 9.4.18 and 9.4.13, we find that

$$\frac{\partial x_i/\partial p_k}{\partial x_j/\partial p_k} = \frac{\partial x_i/\partial m}{\partial x_j/\partial m} = \frac{x_i}{x_j}.$$

This implies the income elasticities are all the same, so by Equation 9.1.2, the income elasticities are all 1. But then, demand is linear in income and can be written $\mathbf{x}(\mathbf{p}, m) = m\mathbf{x}(\mathbf{p}, 1)$. The relationship between quantity demanded of a good and income is the *Engel curve*. In this case, all of the Engel curves are straight lines through the origin.[16]

Because demand is linear in income (holding prices constant), the marginal rate of substitution is constant along rays through the origin, hence homogeneous of degree zero in $\mathbf{x}$. Theorem 3.1.4 shows that utility is homothetic. □

We can go a little further. If the utility function is separable and homothetic, all of subutilities must have the same homotheticity. By Bergson's Separability Theorem, when there are two or more commodity goods, the only cases where the sum is homothetic are the homogeneous or logarithmic forms.[17]

In the homogeneous case, we normally require that the power $0 < \gamma < 1$ so that utility is increasing and concave. In this case, the forms for homogeneous direct and indirect utility are $u(\mathbf{x}) = \sum_i \alpha_i f(x_i)$ and $v(\mathbf{p}, m) = \sum_i a_i f(m/p_i)$ where $f(x) = x^\gamma$ or $f(x) = \log x$. However, Theorem 9.4.4 only gives necessary conditions for utility to be both directly and indirectly additive separable. These are not sufficient conditions. Moreover, they are satisfied if direct or indirect utility is merely equivalent to an additive separable form. We should not expect the same additive separable function to serve as both direct and indirect utility. In fact, in the cases covered by the theorem, it doesn't. However, if we start with Bergson utility, we obtain indirect utility that is equivalent to the Bergson form as in Exercise 9.4.3.

The case where Theorem 9.4.4 fails, $\partial x_i/\partial p_k = 0$, occurs when utility is Cobb-Douglas. In that case, utility is both directly and indirectly additive. For example, if $u(\mathbf{x}) = \sum_i \gamma_i \ln x_i$, demand is $x_i = \gamma_i m/p_i$ and $v(\mathbf{p}, m) = \sum_i \gamma_i\big(\ln \gamma_i + \ln(m/p_i)\big)$.

---

[16] Engel studied the relation between income and consumption of various goods. He is best known for *Engel's Law*, which states that poorer families spend a larger fraction of their income on food (Engel, 1857). See Chai and Moneta (2010) for more on Engel and Engel curves.

[17] The homogeneous case is equivalent to CES utility. In the older literature it is sometimes called the Bergson family of utility, after Bergson (1936).

9.4.6 Hicks' Example

SKIPPED

This is not the end of the story. There are still more possibilities in the $\partial x_i/\partial p_k = 0$ case. Hicks (1969) discovered that utility functions of the form $u(x_1) + \alpha \ln x_2$ where u is any reasonable utility function on $\mathbb{R}_+$ are also both directly and indirectly additive separable. In fact, Hicks commented that it also works if there are many Cobb-Douglas goods.

Example 9.4.5: Hicks' Indirect Additivity Example Suppose utility on $\mathbb{R}^m_+$ is defined by $u(\mathbf{x}) = f(x_1) + \sum_{i=2}^m \alpha_i \ln x_i$ where $f\colon \mathbb{R}_+ \to \mathbb{R}$ is $\mathcal{C}^2$ with $f' > 0$, $f'' < 0$ and $f'(0+) = +\infty$. Clearly u is additive separable. We can show that it is also indirectly additive separable.

The conditions on f imply that the first-order conditions are necessary and sufficient to solve the consumer's problem. It follows that

$$\lambda p_1 = f'(x_1) \quad \text{and} \quad \lambda p_i = \frac{\alpha_i}{x_i} \text{ for } i = 2, \dots, m$$

where λ is the Lagrange multiplier associated with the budget constraint.

We exploit the zero degree homogeneity of the consumer's problem to solve it when $m = 1$. The $m \neq 1$ case is then handled by substituting $\mathbf{p}/m$ for $\mathbf{p}$. Multiplying the first-order conditions by x_i and summing yields

$$\lambda = x_1 f'(x_1) + \sum_{i=2}^m \alpha_i.$$

Writing $\alpha = \sum_{i=2}^m \alpha_i$, we have $x_1 f'(x_1) + \alpha = \lambda$. This shows that λ depends only on x_1. By the first-order conditions we also have $\lambda = f'(x_1)/p_1$, so

$$\frac{1}{p_1} = x_1 + \frac{\alpha}{f'(x_1)}.$$

The right-hand side is continuous and increasing in x_1. By considering the limits and 0 and $+\infty$, we see that the range of the right hand side is $(0, +\infty)$. A solution exists, and is unique because $1/f'$ is increasing.

We denote the unique solution by $x_1(p_1)$. Since the right-hand side is larger than x_1, $x_1(p_1) < 1/p_1$. It follows that not all income is spent on good one. The remainder $1 - p_1 x_1$ must be spent on the remaining goods with spending shares α_i/α. It follows that $x_i = \alpha_i(1 - p_1 x_1)/\alpha p_i$. Indirect utility becomes

$$v(\mathbf{p}, 1) = f^*(p_1) + \sum_{i=2}^m \big[\ln(\alpha_i/\alpha) - \alpha_i \ln p_i\big]$$

where $f^*(p_1) = f\big(x_1(p_1)\big) + \alpha \ln\big(1 - p_1 x_1(p_1)\big)$. This shows that utility is indirectly additive separable.[18] ◀

February 6, 2023

[18] This example is adapted from Hicks (1969) and further discussed by Samuelson (1969b).