10. Welfare, Indices, and Aggregates

Feb. 7, 2023

Homework: Problems 7.4.1, 9.1.5, 9.1.8, 9.2.2, and 9.4.2 from my manuscript are due on Tuesday, February 14.

Chapter Outline

1. Consumer's Surplus
2. Compensating and Equivalent Variations
3. Price Indices
4. Quantity Indices

NB: As in the previous two chapters, our default commodity space will be $\mathbb{R}^m$ with consumption set $\mathfrak{X} = \mathbb{R}^m_+$. The consumer has continuous preferences $\succsim$ which can be represented by a continuous utility function u using Debreu's Representation Theorem. We will often assume utility is not just continuous, but also differentiable.

10.0.1 Surpluses and the Expenditure and Cost Functions

The analysis of economic policy is grounded on welfare calculations—determining the gains and losses of everyone involved. But how do we measure the gains and losses? To be consistent with economic theory, the gains and losses should represent utility changes. Moreover, a common method of measuring the gains and losses would allow us to add them up.

In a monetized economy, the simplest method is to reduce everything to monetary terms. In undergraduate economics this is usually accomplished by calculating consumer's and producer's surpluses.

10.1 Consumer's Surplus

The *consumer's surplus* is the difference between the monetary value of consumption, how much the consumer is willing to pay for it, and the amount the consumer actually pays for it.

We start by considering a single consumption good. To find the value of a quantity x, we interpret the demand price as the marginal value or marginal willingness to pay.

To see that this makes sense, we start with the first-order condition

$$\lambda p_k = MU_k$$

here λ is the Lagrange multiplier associated with income. By the Envelope Theorem, λ is the marginal utility of income, $MU_\$$.

Assuming we are using dollars, we find the marginal dollar value of good k by dividing the marginal utility of k by the marginal utility of a dollar, $\lambda = MU_\$$. Then the equation

$$p_k = MU_k \,/\, MU_\$$$

determines the demand curve and expresses the marginal dollar value of x_k at the same time.

10.1.1 Calculating Consumer's Surplus

To find the dollar value of consuming a total amount x, we add up the area under the demand curve. The consumer's surplus is then the difference between the dollar value and the dollar cost of x, px. This is the area under the demand curve and above the price. In other words, the *consumer's surplus* from consuming $x(p^1)$ at price p^1, $CS(p^1)$, is defined by

$$CS(p^1) = \int_{p^1}^{\infty} x(p)\, dp.$$

This is illustrated in Figure 10.1.1.[1]

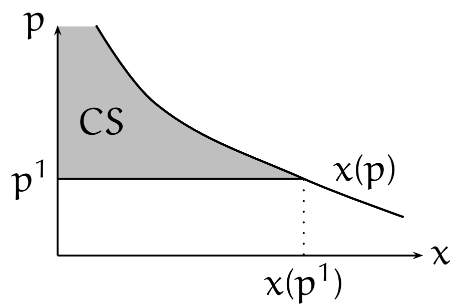

Figure 10.1.1: The quantity demanded at price p is $x(p)$. In the left panel, the shaded region is the consumer's surplus at price p^1 and quantity $x(p^1)$. It is the area under the demand curve, above the price line, and right of the vertical axis.

[1] Because p is the independent variable, the area lies between the demand curve and the vertical axis, not the horizontal axis.

10.1.2 Changes in Consumer's Surplus

One important way that policy changes affect consumer's surplus is by changing the price of a good. When the price of our good falls from p^0 to p^1, the change in consumer's surplus is

$$\begin{aligned}\Delta CS &= CS(p^1) - CS(p^0) \\ &= \int_{p^1}^{\infty} x(p)\,dp - \int_{p^0}^{\infty} x(p)\,dp \\ &= \int_{p^1}^{p^0} x(p)\,dp \\ &= -\int_{p^0}^{p^1} x(p)\,dp.\end{aligned}$$

This is illustrated in Figure 10.1.2.

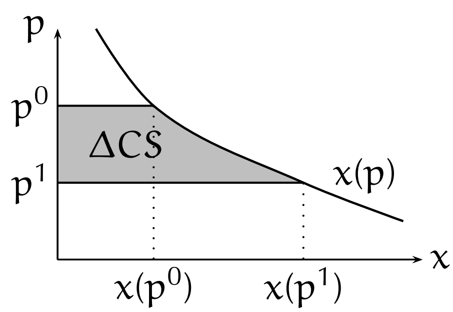

Figure 10.1.2: Suppose the price falls from p^0 to p^1. The shaded area is the magnitude of change in consumer's surplus due to the price change and the change in consumer's. Here the price fell, so the consumer buys more and pays a lower price, increasing the surplus. The change in surplus is the shaded area. Had the price risen, the shaded area would be $-\Delta CS$ instead of ΔCS.

10.1.3 Consumer's Surplus with Multiple Price Changes

Complications arise when multiple prices change simultaneously. An obvious way to compute the change in consumer's surplus is to change one price first, and then the other. However, the order you change the prices in can change the answer!

We illustrate this phenomenon with an example.

Example 10.1.3: The problems caused by multiple price changes already show up if only two prices change. Consider the utility function $u(\mathbf{x}) = x_1 + \sqrt{x_2}$. In Example 4.2.4, we found that the Marshallian demand functions are:

$$x_1(\mathbf{p}, m) = \begin{cases} 0 & \text{if } m \leq p_1^2/4p_2 \\ \frac{m}{p_1} - \frac{p_1^2}{4p_2} & \text{if } m > p_1^2/4p_2. \end{cases}$$

and

$$x_2(\mathbf{p}, m) = \begin{cases} m/p_2 & \text{if } m \leq p_1^2/4p_2 \\ \frac{p_1^2}{4p_2^2} & \text{if } m > p_1^2/4p_2. \end{cases}$$

Now suppose $m = 2$ and $\mathbf{p} \in [1,2] \times [1,2]$. In that case the demand function simplifies to

$$\mathbf{x}(\mathbf{p}) = \begin{pmatrix} \frac{2}{p_1} - \frac{p_1^2}{4p_2} \\ \frac{p_1^2}{4p_2^2} \end{pmatrix}.$$

where we have omitted the income argument from $\mathbf{x}$ to simplify notation.

10.1.4 Multiple Price Changes II

We now compute the change in consumer's surplus as prices change from $\mathbf{p}^0 = (2,2)$ to $\mathbf{p}^1 = (1,1)$ in two different ways. In the first, we hold p_1 constant at 2 while reducing p_2 to 1, then hold p_2 constant while reducing p_1 to 1. The change in consumer's surplus is then

$$\begin{aligned}\Delta CS_1 &= \int_2^1 x_2(2,p_2)\,dp_2 + \int_2^1 x_1(p_1,1)\,dp_1\\ &= \int_2^1 \frac{dp_2}{p_2^2} + \int_2^1 \left(\frac{2}{p_1} - \frac{p_1^2}{4}\right)\,dp_1.\end{aligned}$$

The other method is to change p_1 first, followed by p_2. That yields

$$\begin{aligned}\Delta CS_2 &= \int_2^1 x_1(p_1,2)\,dp_1 + \int_2^1 x_2(1,p_2)\,dp_2\\ &= \int_2^1 \left(\frac{2}{p_1} - \frac{p_1^2}{8}\right)\,dp_1 + \int_2^1 \frac{dp_2}{4p_2^2}.\end{aligned}$$

Then $\Delta CS_1 = 7/12 - 3\ln 2 \neq \Delta CS_2 = 4/24 - 2\ln 2$. ◀

10.1.5 CS May Depend on How Prices are Changed

That is rather embarrassing. The change in consumer's surplus depends on the order in which prices are changed. Other methods of changing the prices in Example 10.1.3 can yield additional numbers for the change in consumer's surplus.

So why does this happen? We are calculating

$$\int_\gamma \sum_i x_i(\mathbf{p}, m)\, dp_i,$$

the integral of the form $d\mathbf{x} = \sum_i x_i(\mathbf{p}, m)\, dp_i$ along some piecewise smooth path γ from $\mathbf{p}^0$ and $\mathbf{p}^1$. Stokes' Theorem tells us that this integral is independent of the path provided $d\mathbf{x}$ is a closed differentiable form. Now

$$0 = d^2\mathbf{x} = \sum_{k<i} \left(\frac{\partial x_j}{\partial p_i} - \frac{\partial x_i}{\partial p_j} \right) dx_j \wedge dx_i.$$

The form $d\mathbf{x}$ is closed if and only if the Marshallian reciprocity relations

$$\frac{\partial x_j}{\partial p_i} = \frac{\partial x_i}{\partial p_j}$$

are satisfied whenever $i \neq j$.

10.1.6 Slutsky Symmetry and Reciprocity

The Slutsky equation (5.6.7) is useful here.

$$\frac{\partial x_j}{\partial p_i} = \frac{\partial h_j}{\partial p_i} - x_i \frac{\partial x_j}{\partial m}.$$

In the unusual cases where there is no income effect ($\partial x_j/\partial m = 0$), Slutsky symmetry shows that $\mathbf{dx}$ is closed.

More generally, applying the Slutsky equation to reciprocity yields

$$\frac{\partial h_j}{\partial p_i} - x_i \frac{\partial x_j}{\partial m} = \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial m}.$$

Then, by Slutsky symmetry

$$x_i \frac{\partial x_j}{\partial m} = x_j \frac{\partial x_i}{\partial m} \tag{10.1.1}$$

if and only if Marshallian reciprocity holds.

When preferences are homothetic, $x_j(\mathbf{p}, m) = m x_j(\mathbf{p}, 1)$ and so $\partial x_j/\partial m = x_j(\mathbf{p}, 1)$. But then, equation 10.1.1 is satisfied and so is Marshallian reciprocity. This implies that $\mathbf{dx}$ is a closed form.

Conversely, if $\mathbf{dx}$ is closed, equation 10.1.1 is satisfied. It implies all the income elasticities are the same. Then Walras' Law, in its adding-up form of equation 9.1.2, implies the elasticities are all 1. But then, demand is linear in income. This implies that the marginal rates of substitution are constant along rays though the origin, and that preferences are homothetic by Theorem 3.1.4.

That is why we used non-homothetic preferences to create Example 10.1.3. The example would not have worked had the preferences been homothetic.

10.2 Dual Measures of Welfare

Consumer's surplus is not the only tool we can use to measure consumer gains and losses in monetary terms. We can also use money metric utility, or its alter ego, the expenditure function.

Here we focus on the effect of price changes on consumers. We start with income m and price $\mathbf{p}^0$. The consumer chooses the optimal consumption bundle $\mathbf{x}^0$, which yields utility $u^0 = v(\mathbf{p}^0, m) = u(\mathbf{x}^0)$. Suppose instead prices were $\mathbf{p}^1$. Then the consumer would choose $\mathbf{x}^1$ and get utility $u^1 = v(\mathbf{p}^1, m) = u(\mathbf{x}^1)$. Under these conditions, income can be expressed in terms of the expenditure functions $m = e(\mathbf{p}^0, u^0) = e(\mathbf{p}^1, u^1)$ or in terms of the minimum income functions $m = m(\mathbf{p}, \mathbf{x}^0) = m(\mathbf{p}, \mathbf{x}^1)$.

We can now put a dollar value on the utility change from u^0 to u^1 by asking what income change has the same effect on utility as the price change. There are two ways to do this calculation: *ex ante*, using the original prices, or *ex post*, using the new prices.

10.2.1 Equivalent Variation

Let $u^0 = v(\mathbf{p}^0, m) = u(\mathbf{x}^0)$ denote the original utility and $u^1 = v(\mathbf{p}^1, m) = u(\mathbf{x}^1)$ the utility after the price change occurs. Of course, the $\mathbf{x}^i$ are the consumer's utility maximizers at $(\mathbf{p}^i, m)$. Define the *equivalent variation*, which is measured at the **original** prices $\mathbf{p}^0$, by

$$\mathrm{EV}(\mathbf{p}^0, \mathbf{p}^1; m) = e(\mathbf{p}^0, u^1) - e(\mathbf{p}^0, u^0). \tag{10.2.2}$$

The price change has been transformed to a utility change in the right-hand side of equation (10.2.2).

When prices remain at $\mathbf{p}^0$, giving this amount of income to the consumer has the same effect on utility as if prices had actually changed to $\mathbf{p}^1$. In that sense, it is equivalent to the price change.

We can also express the equivalent variation in terms of the consumption bundles using the minimum income function. Then

$$\mathrm{EV}(\mathbf{p}^0, \mathbf{p}^1; m) = m(\mathbf{p}^0, \mathbf{x}^1) - m(\mathbf{p}^0, \mathbf{x}^0).$$

This lets us interpret the equivalent variation as the difference between the money metric utility of $\mathbf{x}^1$ and $\mathbf{x}^0$ at the original prices $\mathbf{p}^0$.

The equivalent variation will be positive if and only if the consumer is better off at u^1, if $u^1 > u^0$.

10.2.2 Equivalent Variation: Alternate Form

Sometimes it is useful to write the equivalent variation another way, by exploiting the fact that $e(\mathbf{p}^0, u^0) = m = e(\mathbf{p}^1, u^1)$. That yields

$$\mathrm{EV}(\mathbf{p}^0, \mathbf{p}^1; m) = e(\mathbf{p}^0, u^1) - e(\mathbf{p}^1, u^1). \tag{10.2.3}$$

This allows us to interpret the equivalent variation as an integral by using the Shephard-McKenzie Lemma, $D_{\mathbf{p}}e(\mathbf{p}, u^1) = \mathbf{h}(\mathbf{p}, u^1)$. Thus

$$\mathrm{EV}(\mathbf{p}^0, \mathbf{p}^1; m) = \int_\gamma D_{\mathbf{p}}e(\mathbf{p}, u^1)\, d\mathbf{p} = \int_\gamma \mathbf{h}(\mathbf{p}, u^1)\, d\mathbf{p}$$

where γ is any path connecting $\mathbf{p}^0$ and $\mathbf{p}^1$.

Since Hicksian demand is the differential of the expenditure function, the differential form $\mathbf{h}\, d\mathbf{p}$ is exact, therefore closed. Stokes' Theorem tells us that it doesn't matter which price path we use to compute the integral provided the expenditure function is $\mathcal{C}^2$.

In particular, if $\mathbf{p}^0$ and $\mathbf{p}^1$ differ at only one coordinate, say k, we can rewrite the integral as

$$\mathrm{EV}(\mathbf{p}^0, \mathbf{p}^1; m) = \int_{p_k^1}^{p_k^0} h_k(\mathbf{p}, u^1)\, dp_k. \tag{10.2.4}$$

This means that the equivalent variation is the area under the Hicksian demand curve with utility u^1, measured along the price axis between p_k^0 and p_k^1. That is, it is the change in the Hicksian consumer's surplus at utility level u^1 caused by the change in prices. More generally, the equivalent variation adds the changes in Hicksian consumer's surplus across all goods, taking all price changes into account.

10.2.3 Compensating Variation

The equivalent variation is not the only way to use the expenditure function to measure consumer gains and losses. Another is to ask about the amount of income that must be taken away after the price change in order to restore the consumer to the original utility level. This is the *compensating variation*. It is measured at the **new** price level $\mathbf{p}^1$ and is defined by

$$CV(\mathbf{p}^0, \mathbf{p}^1; m) = e(\mathbf{p}^1, u^1) - e(\mathbf{p}^1, u^0). \qquad (10.2.5)$$

The compensating variation compensates for the price change. Once prices become $\mathbf{p}^1$, taking the compensating variation from income leaves the consumer exactly as well off as they were prior to the price change. This too can be written using the minimum income function. Then

$$CV(\mathbf{p}^0, \mathbf{p}^1; m) = m(\mathbf{p}^1, \mathbf{x}^1) - m(\mathbf{p}^1, \mathbf{x}^0),$$

so that the compensating variation is the difference between the money metric utility at $\mathbf{x}^1$ and $\mathbf{x}^0$ measured at the new prices $\mathbf{p}^1$.

As with the equivalent variation, the compensating variation is positive if and only if the consumer is better off after the price change.

10.2.4 Compensating Variation: Alternate Form

Further, we can again exploit the fact that $e(\mathbf{p}^0, u^0) = m = e(\mathbf{p}^1, u^1)$. That yields

$$\mathrm{CV}(\mathbf{p}^0, \mathbf{p}^1; m) = e(\mathbf{p}^0, u^0) - e(\mathbf{p}^1, u^0), \tag{10.2.6}$$

which again allows us to interpret the equivalent variation as an integral by using the Shephard-McKenzie Lemma. We obtain

$$\mathrm{CV}(\mathbf{p}^0, \mathbf{p}^1; m) = \int_\gamma D_{\mathbf{p}} e(\mathbf{p}, u^0)\, d\mathbf{p} = \int_\gamma \mathbf{h}(\mathbf{p}, u^0)\, d\mathbf{p}$$

where γ is any path connecting $\mathbf{p}^0$ and $\mathbf{p}^1$. Once again, Hicksian reciprocity implies that the integral is independent of the path chosen. Unlike consumer's surplus, both variations always work well when there are multiple price changes.

Finally, if u is homogeneous of degree one, the expenditure function has the form $\bar{e}(\mathbf{p})u$. In that case, we can write the equivalent variation as $\mathrm{EV} = \bar{e}(\mathbf{p}^0)(u^1 - u^0)$ and the compensating variation as $\mathrm{CV} = \bar{e}(\mathbf{p}^1)(u^1 - u^0)$, which emphasizes the dependence on utility at the initial and final prices.

10.2.5 The Income Effect and the Two Variations

By using either the compensating or equivalent variation we have avoided any dependence of our welfare measure on the order of price changes. However, there is a cost. These measures are not the same. They depend on a set of reference prices, or more to the point, a reference utility level. Although this potentially opens up a plethora of different welfare measures, they all have in common that positive values of the variation translate to increases in utility.

Let's focus on the difference between compensating and equivalent variation. It turns out that this difference is driven by the income effect. In those cases where there is no income effect, the two variations are identical.

Let's consider the case where only good one good changes price, so $\mathbf{p}^0 = (p_1^0, p_2^0, \ldots, p_m^0)$ and $\mathbf{p}^1 = (p_1^1, p_2^0, p_3^0, \ldots, p_m^0)$. The equivalent variation is then given by equation (10.2.4) as

$$EV(\mathbf{p}^0, \mathbf{p}^1; m) = \int_{p_1^1}^{p_1^0} h_1(\mathbf{p}, u^1)\, dp_1.$$

Notice that we integrate Hicksian demand with utility level u^1.

Similarly, the compensating variation is

$$CV(\mathbf{p}^0, \mathbf{p}^1; m) = \int_{p_1^1}^{p_1^0} h_1(\mathbf{p}, u^0)\, dp_1,$$

where the utility level u^0 is used. The different reference level of utility is the only difference between the two variations.

10.2.6 Comparing Surplus and Variations I: Normality

We want to compare changes in the Marshallian consumer's surplus to the two variations, compensating and equivalent. The change in Marshallian consumer surplus is

$$\Delta CS = \int_{p_1^1}^{p_1^0} x_1(\mathbf{p}, m)\, dp_1.$$

We will use the Slutsky equation to compare all three quantities. Here, we need the expression for $\partial h_1/\partial p_1$. By equation (5.6.7), it is

$$\frac{\partial h_1}{\partial p_1} = \frac{\partial x_1}{\partial p_1} + x_1 \frac{\partial x_1}{\partial m}.$$

When good one is normal, $0 > \partial h_1/\partial p_1 > \partial x_1/\partial p_1$.

Since the slope of the demand curve is the inverse of the price derivative, the Marshallian demand curve is flatter than the Hicksian demand curves. By duality, the Marshallian demand curve crosses the u^0 Hicksian demand at p_1^0 where $x_1(p_1^0, m) = h_1(p_1^0, u^0)$, while it crosses the u^1 Hicksian demand at p_1^1 when $x_1(p_1^1, m) = h_1(p_1^1, u^1)$.

The situation is shown in Figure 10.2.1 for the case where $p_1^1 < p_1^0$ (meaning $u^1 > u^0$). Figure 10.2.1 also shows that $EV > \Delta CS > CV$ in this case.

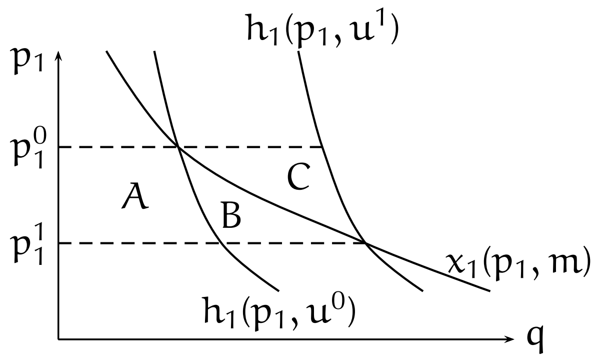

Figure 10.2.1: The region A is the compensating variation (using u^0), $A + B$ is the change in consumer's surplus, based on the Marshallian demand $x_1(p_1, m)$, and $A + B + C$ is the equivalent variation (using u^1). It follows that $0 < CV < \Delta CS < EV$ when demand is normal and the price of single good decreases from p_1^0 to p_1^1.

10.2.7 Comparing Surplus and Variations II: Inferiority

What if good one is inferior? We return to equation (5.6.7):

$$\frac{\partial h_1}{\partial p_1} = \frac{\partial x_1}{\partial p_1} + x_1 \frac{\partial x_1}{\partial m},$$

but now $\partial h_1/\partial p_1 < \partial x_1/\partial p_1$. We focus on the case where our inferior good is not Giffen, so $\partial h_1/\partial p_1 < \partial x_1/\partial p_1 < 0$. The Marshallian demand curve is now steeper than the Hicksian demand curves. This reverses the result, with the compensating variation now the largest and the equivalent variation the smallest, as shown in Figure 10.2.2.

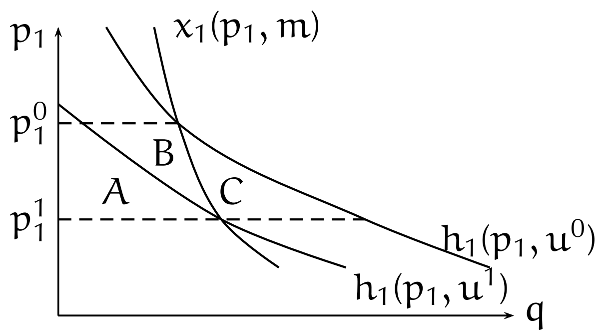

Figure 10.2.1: The region $A + B + C$ is the compensating variation (using u^0), $A + B$ is the change in consumer's surplus, based on the Marshallian demand $x_1(p_1, m)$, and A is the equivalent variation (using u^1). It follows that $0 < EV < \Delta CS < CV$ when demand is inferior and the price of single good decreases from p_1^0 to p_1^1.

10.2.8 What About Producer's Surplus?

So what about producer's surplus? How do we measure producer welfare? Producer's surplus commonly involves a distinction between short-run and long-run production. One definition of *producer's surplus* is revenue minus variable costs. An equivalent definition is the area below the price and above the marginal cost curve.

We do not make a distinction between short and long run behavior here—our models do not involve fixed costs. In that case, all costs are variable and producer's surplus is simply profit, revenue minus costs. What could be a more natural measure of producer welfare? Our consumer welfare measures revolve around utility, producer welfare around profit.

So using profit as the criterion, let's try to construct the equivalent and compensating variations for producers. So we have two sets of prices, $(p^0, \boldsymbol{w}^0)$ and $(p^1, \boldsymbol{w}^1)$. Corresponding to those are the quantities the firm would choose, q^0 and q^1.

There are two ways to calculate the equivalent variation for consumers. One is given in equation (10.2.5). It uses the minimum expenditure at the old prices to attain the new and old utility levels. The other method is equation (10.2.3). It uses the new utility level at both old and new prices.

10.2.9 CV and EV for Firms

We will use the second method to calculate the equivalent variation based on profit. We use q^1 to calculate profit in both cases. We also need to recall that expenditure is a negative for the consumer, while profit is a positive for the firm. Because of this, we flip the sign.

$$\begin{aligned} EV &= \left[p^1 q^1 - c(\boldsymbol{w}^1, q^1)\right] - \left[p^0 q^1 - c(\boldsymbol{w}^0, q^1)\right] \\ &= (p^1 - p^0)q^1 + \left[c(\boldsymbol{w}^0, q^1) - c(\boldsymbol{w}^1, q^1)\right] \end{aligned}$$

The p term captures the effect of changes in output prices, while the cost terms measure the effect of changes in input prices. If the output price is constant ($p^0 = p^1$), we obtain

$$EV = c(\boldsymbol{w}^0, q^1) - c(\boldsymbol{w}^1, q^1),$$

which is equation (10.2.3) with expenditure and utility replaced by price and cost.

Doing the same thing for compensating variation, we obtain

$$\begin{aligned} CV &= \left[p^1 q^0 - c(\boldsymbol{w}^1, q^0)\right] - \left[p^0 q^0 - c(\boldsymbol{w}^0, q^0)\right] \\ &= (p^1 - p^0)q^0 + \left[c(\boldsymbol{w}^0, q^0) - c(\boldsymbol{w}^1, q^0)\right]. \end{aligned}$$

When output prices are constant, this yields

$$CV = c(\boldsymbol{w}^0, q^0) - c(\boldsymbol{w}^1, q^0)$$

which is the analogue of equation (10.2.6).

These expressions are not used as often as the consumer's compensating and equivalent variations, largely because firm profits are often included in consumer income, as is typical of general equilibrium models. Additionally, models sometimes involve constant returns to scale production, when equilibrium profit is zero.

10.3 Price Indices

The compensating and equivalent variations are not the only way to measure the welfare effect of price changes. Another method is use a price index.

How is a price index a welfare measure? We use price indices to decide whether real income has gone up or down. The idea is that increases in real income correspond to welfare gains, and decreases to welfare losses.

The proper construction of price and quantity indices, measuring general changes in prices and quantities, has been a long-standing problem in economics. If all prices changed in the same proportion, there would be no problem of determining the price level. It would also change in the same proportion. The problem arises when prices change by different amounts and possibly even in different directions. The price of a particular type of television falls while the price of a year of college rises. How do we determine what happened to prices **in general** in the face of such changes? This is the problem of price indices.

10.3.1 A Fixed Basket of Goods

Diewert (1993) discusses the history of such indices and examines several of the methods that have been proposed. One important method, still in use today, is to measure price changes by comparing the cost of a fixed basket of goods. This method dates back at least to 1707, when it was used by William Fleetwood, the Bishop of Ely.

To use this technique, we must first decide on a fixed consumption vector $\bar{\mathbf{x}}$. Then we compute its cost at the initial prices $\mathbf{p}^0$ and the new prices $\mathbf{p}^1$. The price index is the ratio $\mathbf{p}^1 \cdot \bar{\mathbf{x}} / \mathbf{p}^0 \cdot \bar{\mathbf{x}}$.

An important question remains unanswered. How do you choose the basket of goods? At about the same time that marginal utility theory was originally being developed, Laspeyres and Paasche provided two answers. Laspeyres (1871) suggested using the initial bundle of goods, $\mathbf{x}^0$, while Paasche (1874) preferred using the new basket, $\mathbf{x}^1$.

Some economists preferred to split the difference by using the average of the two bundles. For example, Fisher's ideal index (1922) is the geometric mean of the Laspeyres and Paasche indices. Shortly thereafter, Konüs (1924) introduced his true cost-of-living index that was grounded in utility theory.[2]

We continue to use these indices today, but now follow Konüs, basing our indices on modern utility theory (for consumers) and cost theory (for firms). These indices will later play a role in the theory of aggregation.

[2] Konüs is sometimes spelled Konyus.

10.3.2 Laspeyres and Paasche Price Indices

The Laspeyres index is based on actual consumption in the initial period when prices are $\mathbf{p}^0$. We denote that consumption by $\mathbf{x}^0$. The *Laspeyres price index* is defined by

$$P_L(\mathbf{p}^0, \mathbf{p}^1; \mathbf{x}^0) = \frac{\mathbf{p}^1 \cdot \mathbf{x}^0}{\mathbf{p}^0 \cdot \mathbf{x}^0}.$$

The Paasche index is based on the consumption vector $\mathbf{x}^1$ that a consumer chooses once the price change to $\mathbf{p}^1$ occurs. The *Paasche price index* is defined by

$$P_P(\mathbf{p}^0, \mathbf{p}^1; \mathbf{x}^1) = \frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{\mathbf{p}^0 \cdot \mathbf{x}^1}$$

where $\mathbf{x}^1$ is the bundle the consumer chooses at prices $\mathbf{p}^1$. Since the bundle is constant for either index, we are also holding utility constant.

In both cases we form the index by dividing the cost of the bundle at the new prices by the cost of the same bundle at the old prices.

With both Laspeyres and Paasche price indices, multiplying all prices by the factor $t > 0$ ($\mathbf{p}^1 = t\mathbf{p}^0$) multiplies the price index by t. These indices are also homogeneous of degree 1 in the initial prices, homogeneous of degree -1 in the final prices, and have the property that reversing the order of the price changes inverts the index.[3]

[3] Some economists have approached price indices in an axiomatic manner, where the above properties would be required. Diewert (1993, sec. 4) has a discussion of this.

10.3.3 The Konüs True Cost-of-Living Index

An alternative to using a basket of goods to measure price changes is to appeal to utility theory. We can ask how much it costs to maintain our utility level before and after prices change. In other words, we can use the expenditure function, as we just did for measuring welfare changes.

Konüs (1924) did exactly this when he introduced his *true cost-of-living index*. The Konüs index tells us how much income would have to expand (or contract) in order to maintain a given utility level in the face of the price change. Formally, the Konüs index P_K is defined by

$$P_K\big(\mathbf{p}^0, \mathbf{p}^1; u\big) = \frac{e(\mathbf{p}^1, u)}{e(\mathbf{p}^0, u)}$$

where u is a reference level of utility. We could write the Konüs index in terms of a consumption vector $\mathbf{x}$ by using the utility function, $P_K\big(\mathbf{p}^0, \mathbf{p}^1; u(\mathbf{x})\big)$.

10.3.4 Homotheticity and the Konüs Index

When preferences are homothetic, the Konüs index is independent of the utility level.

Theorem 10.3.1. *Suppose u is homothetic, continuous and monotonic. Then u can be written $u(\mathbf{x}) = \varphi\big(v(\mathbf{x})\big)$ where v is homogeneous of degree one and φ is increasing. Further, the Konüs index is independent of u with*

$$P_K\big(\mathbf{p}^0, \mathbf{p}^1; u\big) = \frac{e\big(\mathbf{p}^1, \varphi^{-1}(1)\big)}{e\big(\mathbf{p}^0, \varphi^{-1}(1)\big)}$$

for all $u \in$ ran u.

Proof. By the Homothetic Representation Theorem, we can write $u(\mathbf{x}) = \varphi\big(v(\mathbf{x})\big)$ for some increasing function φ and homogeneous of degree one function v. Using Corollary 5.1.7, we find $e(\mathbf{p}, \bar{u}) = \varphi(\bar{u})e\big(\mathbf{p}, \varphi^{-1}(1)\big)$.

It follows that the Konüs index is

$$\begin{aligned}
P_K\big(\mathbf{p}^0, \mathbf{p}^1; u\big) &= \frac{e(\mathbf{p}^1, u)}{e(\mathbf{p}^0, u)} \\
&= \frac{\varphi(u)\, e\big(\mathbf{p}^1, \varphi^{-1}(1)\big)}{\varphi(u)\, e\big(\mathbf{p}^0, \varphi^{-1}(1)\big)} \\
&= \frac{e\big(\mathbf{p}^1, \varphi^{-1}(1)\big)}{e\big(\mathbf{p}^0, \varphi^{-1}(1)\big)},
\end{aligned}$$

which is independent of u. □

10.3.5 Producer Price Index

We can also construct a producer's form of the Konüs index by replacing the expenditure function with the cost function. For producers,

$$P_K\left(\boldsymbol{w}^0, \boldsymbol{w}^1; q\right) = \frac{c(\boldsymbol{w}^1, q)}{c(\boldsymbol{w}^0, q)}.$$

Here q is a reference level of output, $\boldsymbol{w}$ are input price vectors, and c is the cost function.

If production is homogeneous of degree $\gamma > 0$, we can write $c(\boldsymbol{w}, q) = q^{1/\gamma} b(\boldsymbol{w})$ where $b(\boldsymbol{w}) = c(\boldsymbol{w}, 1)$ is the unit cost function.[4] In that case, the Konüs index is independent of output and can be written as a ratio of unit cost functions:

$$P_K\left(\boldsymbol{w}^0, \boldsymbol{w}^1; q\right) = \frac{b(\boldsymbol{w}^1)}{b(\boldsymbol{w}^0)}.$$

[4] Similar results hold for the homothetic case, as we saw for consumers. However, the fact that production is cardinal, not ordinal, means that such transformations usually fundamentally change the production problem. One case where such transformations are useful is when production is homogeneous.

10.3.6 Paasche and Laspeyres Style Konüs Index I

The true cost-of-living index has one big problem. It is generally unobservable. In contrast, the well-known price indices of Paasche and Laspeyres are based entirely on observable quantities.

When preferences or costs are not homothetic, the reference utility or quantity can make a difference. In that case, we combine Konüs's ideas with those of Paasche and Laspeyres. The case $u = u^0 = v(\mathbf{p}^0, m)$ yields the *Konüs-Laspeyres price index*, when $u = u^1 = v(\mathbf{p}^1, m)$ we can refer to the *Konüs-Paasche price index*.

Since the Konüs-Laspeyres index is based on the original prices, you shouldn't be surprised to hear that this index is closely related to the compensating variation. Indeed, we can express the Konüs-Laspeyres index in terms of the compensating variation:

$$\begin{aligned}
P_K(\mathbf{p}^0, \mathbf{p}^1; u^0) &= \frac{e(\mathbf{p}^1, u^0)}{e(\mathbf{p}^0, u^0)} \\
&= \frac{1}{e(\mathbf{p}^0, u^0)}\Big[e(\mathbf{p}^0, u^0) - \big(e(\mathbf{p}^0, u^0) - e(\mathbf{p}^0, u^1)\big)\Big] \\
&= 1 - \frac{CV(\mathbf{p}^0, \mathbf{p}^1; m)}{e(\mathbf{p}^0, u^0)} \\
&= 1 - \frac{CV(\mathbf{p}^0, \mathbf{p}^1; m)}{m} \\
&= \frac{m - CV(\mathbf{p}^0, \mathbf{p}^1; m)}{m}.
\end{aligned}$$

The Konüs-Laspeyres index is one minus the ratio of the compensating variation and consumer income. In the case of a welfare loss, the compensating variation is positive and less than m. It follows that the price index is less than one. If there is a welfare gain, the compensating variation is negative and the price index will be larger than one. Thus a price index less than one indicates a welfare loss, while a price index greater than one signifies a welfare gain. Moreover, the magnitude of the index informs us about the size of the gain or loss, as measured by the compensating variation.

10.3.7 Paasche and Laspeyres Style Konüs Index II

Similarly, the Konüs-Paasche index can be written in terms of the equivalent variation.

$$P_K(\mathbf{p}^0, \mathbf{p}^1; u^1) = \frac{m}{m + EV(\mathbf{p}^0, \mathbf{p}^1; m)}.$$

As with the Konüs-Laspeyres index, the Konüs-Paasche index can be used to measure welfare changes. However, it is based on the equivalent variation rather than the compensating variation.

10.3.8 Konüs Bounds SKIPPED

We can use the Laspeyres and Paasche indices to derive bounds on the Konüs indices.

Theorem 10.3.2. *Suppose* $u^i = u(\mathbf{x}^i)$ *for* $i = 0, 1$. *Then*

$$P_K(\mathbf{p}^0, \mathbf{p}^1; u^0) \le P_L(\mathbf{p}^0, \mathbf{p}^1; \mathbf{x}^0) \quad \textit{and} \quad P_P(\mathbf{p}^0, \mathbf{p}^1; \mathbf{x}^1) \le P_K(\mathbf{p}^0, \mathbf{p}^1; u^1).$$

Proof. We start with the Konüs-Laspeyres index. We use the fact that $u(\mathbf{x}^0) = u^0$ to find that

$$P_K(\mathbf{p}^0, \mathbf{p}^1; u^0) = \frac{e(\mathbf{p}^1, u^0)}{e(\mathbf{p}^0, u^0)} = \frac{e(\mathbf{p}^1, u^0)}{\mathbf{p}^0 \cdot \mathbf{x}^0} \le \frac{\mathbf{p}^1 \cdot \mathbf{x}^0}{\mathbf{p}^0 \cdot \mathbf{x}^0} = P_L(\mathbf{p}^0, \mathbf{p}^1; \mathbf{x}^0).$$

Similarly, for the Konüs-Paasche index

$$P_K(\mathbf{p}^0, \mathbf{p}^1; u^1) = \frac{e(\mathbf{p}^1, u^1)}{e(\mathbf{p}^0, u^1)} = \frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{e(\mathbf{p}^0, u^1)} \ge \frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{\mathbf{p}^0 \cdot \mathbf{x}^1} = P_P(\mathbf{p}^0, \mathbf{p}^1; \mathbf{x}^1)$$

because $e(\mathbf{p}^0, u^1) \le \mathbf{p}^0 \cdot \mathbf{x}^1$. □

When preferences are homothetic, we can combine the two bounds.

Theorem 10.3.3. *Suppose preferences are homothetic. Then*

$$P_P(\mathbf{p}^0, \mathbf{p}^1; u^1) \le P_K(\mathbf{p}^0, \mathbf{p}^1; u) \le P_L(\mathbf{p}^0, \mathbf{p}^1; u^0) \tag{10.3.2}$$

for any utility level u.

Proof. Since preferences are homothetic, Theorem 10.3.1 shows that the Konüs index is independent of utility, in which case we can combine the bounds, obtaining equation 10.3.7. □

The same type of calculation applies to the cost function. One reason that the Laspeyres and Paasche indices are still used is that they can help us estimate the true costs of living and production.

10.3.9 Cobb-Douglas Price Indices **SKIPPED**

Let's work through all this for Cobb-Douglas utility. Here we have formulas for the demand functions and can reduce everything to prices.

Example 10.3.4: Suppose utility is $u(\mathbf{x}) = \prod_i x_i^{\gamma_i}$ where $0 < \gamma_i < 1$ for each $i = 1, \ldots, L$ and $\sum_i \gamma_i = 1$.

We start with the Konüs true cost of living index. For that, we need the expenditure function, which is

$$e(\mathbf{p}, \bar{u}) = \bar{u} \prod_i \left(\frac{p_i}{\gamma_i}\right)^{\gamma_i}.$$

Then the Konüs true cost of living index is

$$P_K(\mathbf{p}^0, \mathbf{p}^1; u) = \frac{e(\mathbf{p}^1, u)}{e(\mathbf{p}^0, u)} = \prod_i \left(\frac{p_i^1}{p_i^0}\right)^{\gamma_i}.$$

Notice that utility does not appear in the above equation. The fact that Cobb-Douglas preferences are homothetic has led to its elimination.

For the Laspeyres and Paasche indices, we can allow consumer income to change as well as prices.

Given income m^i and prices $\mathbf{p}^i$ where $i = 0, 1$, demand is $\mathbf{x}_i^i = m^i(\gamma_i/p_i^i)$. Then $\mathbf{p}^1 {\cdot} \mathbf{x}^0 = m^0 \sum_i \gamma_i (p_i^1/p_i^0)$ and $\mathbf{p}^0 {\cdot} \mathbf{x}^0 = m^0$. Similarly, $\mathbf{p}^0 {\cdot} \mathbf{x}^1 = m^1 \sum_i \gamma_i (p_i^0/p_i^1)$ and $\mathbf{p}^1 {\cdot} \mathbf{x}^1 = m^1$. Since the consumption bundle is held constant (different, but constant) for both the Laspeyres and Paasche indices, the income terms cancel out.

The Laspeyres price index is

$$P_L(\mathbf{p}^0, \mathbf{p}^1) = \frac{\mathbf{p}^1 {\cdot} \mathbf{x}^0}{\mathbf{p}^0 {\cdot} \mathbf{x}^0} = \sum_i \gamma_i \left(\frac{p_i^1}{p_i^0}\right)$$

and the Paasche price index is

$$P_P(\mathbf{p}^0, \mathbf{p}^1) = \frac{\mathbf{p}^1 {\cdot} \mathbf{x}^1}{\mathbf{p}^0 {\cdot} \mathbf{x}^1} = \left[\sum_i \gamma_i \left(\frac{p_i^0}{p_i^1}\right)\right]^{-1}$$

◀

10.3.10 Konüs Bounds for Cobb-Douglas Utility SKIPPED

We can now examine the Konüs bounds for Cobb-Douglas utility from another perspective.

Example 10.3.4: We rewrite equation (10.3.7) using our various expressions for the Cobb-Douglas price indices. Thus

$$\sum_i \gamma_i \left(\frac{p_i^0}{p_i^1}\right) \le \prod_i \left(\frac{p_i^1}{p_i^0}\right)^{\gamma_i} \le \left[\sum_i \gamma_i \left(\frac{p_i^1}{p_i^0}\right)\right]^{-1}. \tag{10.3.7}$$

Notice that the Konüs index is concave in $\mathbf{p}^1$ and convex in $\mathbf{p}^0$ (due to the division).

We will evaluate the $\mathbf{p}^1$-derivative at $\mathbf{p}^0$ and use the Support Property Theorem to verify that the lower Konüs bound holds.

Since the Konüs index for Cobb-Douglas utility is independent of the utility level, we write it using the abbreviated notation $P_K(\mathbf{p}^1, \mathbf{p}^0)$. We exploit the fact that P_K is concave in the first argument and use the Support Property to write

$$P_K(\mathbf{p}^1, \mathbf{p}^0) \le P_K(\mathbf{p}^0, \mathbf{p}^0) + [D_1 P_K(\mathbf{p}^0, \mathbf{p}^0)]\cdot(\mathbf{p}^1 - \mathbf{p}^0).$$

Now $\partial P_K / \partial p_i^1 = \gamma_i (p_i^1)^{\gamma_i - 1}/(p_i^0)^{\gamma_i}$. When evaluated at p_i^0, this becomes γ_i / p_i^0. Substituting in our inequality, we find that

$$P_K(\mathbf{p}^1, \mathbf{p}^0) \le 1 + \sum_i \gamma_i \left[\left(\frac{p_i^1}{p_i^0}\right) - 1\right] = \sum_i \gamma_i \left(\frac{p_i^1}{p_i^0}\right) = P_L(\mathbf{p}^1, \mathbf{p}^0; \mathbf{x}^0)$$

because $\sum_i \gamma_i = 1$. This verifies the upper Konüs bound of equation (10.3.7). A similar technique applies to the lower Konüs bound, where convexity is important. ◀

10.4 Quantity Indices

Now that we have a set of price indices, our next problem is to construct quantity indices. One simple way to do this is to use the same method, but reverse the role of prices and quantities.

Rather than defining a standard commodity bundle and computing its value under two different price vectors, we can define a standard price vector, and compute how much two different commodity vectors would cost. The ratio is then our quantity index.

Let's try this with the Laspeyres and Paasche quantity indices.

While the Laspeyres price index use the original consumption or input vector and varies the prices, the Laspeyres quantity index keeps the original prices and varies the consumption or input vector. The *Laspeyres quantity index* is defined by

$$Q_L(\mathbf{x}^0, \mathbf{x}^1; \mathbf{p}^0) = \frac{\mathbf{p}^0 \cdot \mathbf{x}^1}{\mathbf{p}^0 \cdot \mathbf{x}^0}.$$

The Paasche quantity index is based on the new consumption or input vector, and varies the prices. There is also a *Paasche quantity index* is defined by

$$Q_P(\mathbf{x}^0, \mathbf{x}^1; \mathbf{p}^1) = \frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{\mathbf{p}^1 \cdot \mathbf{x}^0}.$$

The Laspeyres and Paasche quantity indices are similar to the price indices in that multiplying all quantities by the factor $t > 0$ multiplies the quantity index by t. These indices are also homogeneous of degree -1 in the initial quantities, homogeneous of degree $+1$ in the final quantities, and have the property that reversing the order of the quantity changes inverts the index.

10.4.1 Compatible Price and Quantity Indices

If we have both price and quantity indices, are they compatible? Do they work together? How do we even determine such a thing? One obvious method is to multiply the price and quantity indices together and ask whether the result is the spending ratio

$$\frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{\mathbf{p}^0 \cdot \mathbf{x}^0}.$$

If P and Q are price and quantity indices, we say that the pair (P, Q) is *compatible* if

$$\mathrm{P} \cdot \mathrm{Q} = \frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{\mathbf{p}^0 \cdot \mathbf{x}^0} \tag{10.4.3}$$

for all pairs $(\mathbf{p}^i, \mathbf{x}^i)$ that solve either solve an expenditure minimization problem for the same utility level, or a cost minimization problem for the same output level.

We can go a step further, and call P and Q *totally compatible* if equation (10.4.8) holds for every initial and final price-quantity pair $(\mathbf{p}^0, \mathbf{x}^0)$ and $(\mathbf{p}^1, \mathbf{x}^1)$.

10.4.2 Laspeyres and Paasche are Compatible

Let's try this with the Laspeyres and Paasche indices. Are the two Laspeyres indices compatible? What about the Paasche indices? The sad fact is that two easy computations show that both pairs of indices fail the test. They are not compatible.

However, the pairs (P_L, Q_P) and (P_P, Q_L) are both totally compatible.

$$P_L\left(\mathbf{p}^0, \mathbf{p}^1; \mathbf{x}^0\right) \cdot Q_P\left(\mathbf{x}^0, \mathbf{x}^1; \mathbf{p}^1\right) = \frac{\mathbf{p}^1 \cdot \mathbf{x}^0}{\mathbf{p}^0 \cdot \mathbf{x}^0} \cdot \frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{\mathbf{p}^1 \cdot \mathbf{x}^0} = \frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{\mathbf{p}^0 \cdot \mathbf{x}^0}.$$

and

$$P_P\left(\mathbf{p}^0, \mathbf{p}^1; \mathbf{x}^1\right) \cdot Q_L\left(\mathbf{x}^0, \mathbf{x}^1; \mathbf{p}^0\right) = \frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{\mathbf{p}^0 \cdot \mathbf{x}^1} \cdot \frac{\mathbf{p}^0 \mathbf{x}^1}{\mathbf{p}^0 \cdot \mathbf{x}^0} = \frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{\mathbf{p}^0 \cdot \mathbf{x}^0}.$$

This means that the Laspeyres price index is totally compatible with the Paasche quantity index. The same is true of the Paasche price index and the Laspeyres quantity index.

10.4.3 Fisher Ideal Index

Fisher (1922) preferred to use a geometric mean of Laspeyres and Paasche indices. This is the *Fisher ideal index*. The *Fisher ideal indices* are defined by

$$
\begin{aligned}
P_F\left(\mathbf{p}^0,\mathbf{p}^1;\mathbf{x}^0,\mathbf{x}^1\right) &= \left[P_L(\mathbf{p}^0,\mathbf{p}^1;\mathbf{x}^0)\,P_P(\mathbf{p}^0,\mathbf{p}^1;\mathbf{x}^1)\right]^{1/2}\\
&= \left[\frac{\mathbf{p}^1\cdot\mathbf{x}^0}{\mathbf{p}^0\cdot\mathbf{x}^0}\cdot\frac{\mathbf{p}^1\cdot\mathbf{x}^1}{\mathbf{p}^0\cdot\mathbf{x}^1}\right]^{1/2} \quad \text{and}\\
Q_F\left(\mathbf{x}^0,\mathbf{x}^1;\mathbf{p}^0,\mathbf{p}^1\right) &= \left[Q_L(\mathbf{x}^0,\mathbf{x}^1;\mathbf{p}^0)\,Q_P(\mathbf{x}^0,\mathbf{x}^1;\mathbf{p}^1)\right]^{1/2}\\
&= \left[\frac{\mathbf{p}^0\cdot\mathbf{x}^1}{\mathbf{p}^0\cdot\mathbf{x}^0}\cdot\frac{\mathbf{p}^1\cdot\mathbf{x}^1}{\mathbf{p}^1\cdot\mathbf{x}^0}\right]^{1/2}.
\end{aligned}
$$

Multiplying the expressions for P_F and Q_F shows that

$$P_F Q_F = \frac{\mathbf{p}^1\cdot\mathbf{x}^1}{\mathbf{p}^0\cdot\mathbf{x}^0}.$$

10.4.4 The Malmquist Quantity Index

We still need to find a quantity index that is compatible with the Konüs price index. One quantity index that is closely related to Konüs style indices is Malmquist's (1953) quantity index. The Malmquist index is derived from the distance function introduced in Section 8.3. The distance function can be applied to either indifference curves or isoquants, as needed.

The Malmquist quantity index is based on either a utility level u or production level q. The *Malmquist quantity index* is given by

$$Q_M(\mathbf{x}^0, \mathbf{x}^1; u) = \frac{\eth(\mathbf{x}^1, u)}{\eth(\mathbf{x}^0, u)} \quad \text{or} \quad Q_M(\mathbf{z}^0, \mathbf{z}^1; q) = \frac{\eth(\mathbf{z}^1, q)}{\eth(\mathbf{z}^0, q)}$$

where $\mathbf{x}^i$ denotes consumption vectors and $\mathbf{z}^i$ denotes input vectors. Utility is given by a utility function u, and production is determined by a production function f.

This construction is similar to Konüs' true cost of living index

$$P_K(\mathbf{p}^0, \mathbf{p}^1; u) = \frac{e(\mathbf{p}^1, u)}{e(\mathbf{p}^0, u)},$$

but with prices replaced by quantities and the expenditure function replaced by the distance function.

10.4.5 Malmquist Index with Homothetic Utility

*** SKIPPED REST OF CHAPTER ***

Let's take a look at the Malmquist quantity index when utility or production is homothetic.

Theorem 10.4.1. *Suppose u (f) is a continuous homothetic and increasing utility (production) function on $\mathbb{R}^m_+$. Then there is a homogeneous of degree one function v and increasing function F such that $u = F \circ v$ ($f = F \circ v$) and*

$$M_Q(\mathbf{x}^0, \mathbf{x}^1; \bar{u}) = \frac{v(\mathbf{x}^1)}{v(\mathbf{x}^0)}.$$

Proof. By the Homothetic Representation Theorem, there is an increasing continuous function F and a continuous v that is homogeneous of degree one with $u(\mathbf{x}) = F \circ v(\mathbf{x})$.

By Theorem 8.3.2, the distance function $\eth = \eth(\mathbf{x}, \bar{u})$ is the unique solution to $u(\mathbf{x}/\eth) = \bar{u}$. Let $\bar{v} = F^{-1}(\bar{u})$. Then

$$v(\mathbf{x}/\eth) = (F^{-1} \circ u)(\mathbf{x}/\eth) = \bar{v}.$$

Since v is homogeneous of degree one, we can rewrite this as $v(\mathbf{x})/\eth = \bar{v}$. It follows that $\eth(\mathbf{x}, \bar{u}) = v(\mathbf{x})/\bar{v}$.

We can now compute the Malmquist index for any $\bar{u}$

$$M_Q(\mathbf{x}^0, \mathbf{x}^1; \bar{u}) = \frac{\eth(\mathbf{x}^1, \bar{u})}{\eth(\mathbf{x}^0, \bar{u})} = \frac{v(\mathbf{x}^1)}{\bar{v}} \cdot \frac{\bar{v}}{v(\mathbf{x}^0)} = \frac{v(\mathbf{x}^1)}{v(\mathbf{x}^0)}.$$

□

10.4.6 Malmquist Index with Homogeneous Production

Suppose the production function f is homogeneous of degree $\gamma > 0$. We can set $v(\boldsymbol{x}) = f(\boldsymbol{x})^{1/\gamma}$ to obtain a linear homogeneous function. Then Theorem 10.4.1 shows that the Malmquist index is

$$Q_M(\boldsymbol{z}^0, \boldsymbol{z}^1; q) = \left[\frac{f(\boldsymbol{z}^1)}{f(\boldsymbol{z}^0)}\right]^{1/\gamma}$$

Recall that the Konüs price index is the ratio of unit costs, $P_K(\boldsymbol{w}^0, \boldsymbol{w}^1; q) = b(\boldsymbol{w}^1)/b(\boldsymbol{w}^0)$. It follows that

$$P_K(\boldsymbol{w}^0, \boldsymbol{w}^1; q) \cdot Q_M(\boldsymbol{x}^0, \boldsymbol{z}^1; q) = \frac{b(\boldsymbol{w}^1)}{b(\boldsymbol{w}^0)} \left[\frac{f(\boldsymbol{z}^1)}{f(\boldsymbol{z}^0)}\right]^{1/\gamma} = \frac{c(\boldsymbol{w}^1, q)}{c(\boldsymbol{w}^0, q)} = \frac{\boldsymbol{w}^1 \cdot \boldsymbol{z}^1}{\boldsymbol{w}^0 \cdot \boldsymbol{z}^0}.$$

The Konüs cost of production index and the Malmquist quantity index are compatible because their product is the ratio of the conditional factor costs.

10.4.7 Compatibility of Konüs and Malmquist Indices

In the $\gamma = 1$ case, Diewert (1976) calls pairs of indices *exact* if their product is the ratio of the cost or expenditure functions.[5] The points are that the exact price and quantity indices are constructed in a natural way from cost and production, and that the resulting indices are compatible.

In fact, compatibility of the Konüs-Malmquist pair is guaranteed even outside the homogeneous case. We first prove a preliminary theorem and then obtain compatibility as a corollary.

Theorem 10.4.2. *Suppose $u\colon \mathbb{R}^m_+ \to \mathbb{R}$ is a continuous, locally non-satiated, and semi-strictly quasiconcave utility function. Then*

$$\frac{\eth(\mathbf{x}^1, u^1)}{\eth(\mathbf{x}^0, u^0)} \cdot \frac{e(\mathbf{p}^1, u^1)}{e(\mathbf{p}^0, u^0)} = \frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{\mathbf{p}^0 \cdot \mathbf{x}^0}$$

whenever $\mathbf{x}^i \in \mathbf{h}(\mathbf{p}^i, u^i)$ with each $\mathbf{x}^i \gg \mathbf{0}$.

Moreover, if f is a production function obeying the same conditions, then

$$\frac{\eth(\boldsymbol{z}^1, q^1)}{\eth(\boldsymbol{z}^0, q^0)} \cdot \frac{e(\boldsymbol{w}^1, q^1)}{e(\boldsymbol{w}^0, q^0)} = \frac{\boldsymbol{w}^1 \cdot \boldsymbol{z}^1}{\boldsymbol{w}^0 \cdot \boldsymbol{z}^0}$$

whenever $\boldsymbol{z}^i \in \boldsymbol{z}(\boldsymbol{w}^i, q^i)$ with each $\boldsymbol{z}^i \gg \mathbf{0}$.

Proof. This follows immediately from Proposition 8.3.11. □

Corollary 10.4.3. *Under the conditions of Theorem 10.4.2, the Konüs price index and Malmquist quantity index are compatible.*

Proof. We prove the production case. The consumer case is similar. Using the definitions of the two indices and Theorem 10.4.2, we find

$$Q_M(\mathbf{x}^0, \mathbf{x}^1; u)\, P_K(\boldsymbol{w}^0, \boldsymbol{w}^1; q) = \frac{\eth(\boldsymbol{z}^1, q)}{\eth(\boldsymbol{z}^0, q)} \cdot \frac{c(\boldsymbol{w}^1, q)}{c(\boldsymbol{w}^0, q)} = \frac{\boldsymbol{w}^1 \cdot \boldsymbol{z}^1}{\boldsymbol{w}^0 \cdot \boldsymbol{z}^0}$$

whenever $\boldsymbol{z}^i \in \boldsymbol{z}^i(\boldsymbol{w}^i, q)$ for $i = 0, 1$. This establishes compatibility. □

[5] The concept of an exact index was used by Diewert (1976) in the constant returns to scale case. I have modified it slightly to allow any uniform returns to scale. For prices, it can be considered a generalization of Konüs's (1926) true cost-of-living index.

10.4.8 Cobb-Douglas Quantity indices

Can the Malmquist quantity index be reduced to an expression involving quantities? In the Cobb-Douglas case, the answer is yes.

Example 10.4.4: We continue to use the utility function $u(\mathbf{x}) = \prod_i x_i^{\gamma_i}$ where $0 < \gamma_i < 1$ for each $i = 1, \ldots, L$ and $\sum_i \gamma_i = 1$.

We want to write everything in terms of quantities, so we will rewrite the relation $x_i^i = \gamma_i m^i / p_i^i$ as $p_i^i = \gamma_i m^i / x_i^i$. We substitute in the expressions for the quantity indices.

The Laspeyres quantity index is

$$Q_L(\mathbf{x}^0, \mathbf{x}^1) = \frac{\mathbf{p}^0 \cdot \mathbf{x}^1}{\mathbf{p}^0 \cdot \mathbf{x}^0} = \sum_i \gamma_i \left(\frac{x_i^1}{x_i^0} \right)$$

and the Paasche quantity index is

$$Q_P(\mathbf{x}^0, \mathbf{x}^1) = \frac{\mathbf{p}^1 \cdot \mathbf{x}^1}{\mathbf{p}^1 \cdot \mathbf{x}^0} = \frac{1}{\sum_i \gamma_i (x_i^0 / x_i^1)}.$$

◀

10.4.9 Producer and Consumer Indices

Both consumers and producers can be handled in the same way when constructing price and quantity indices.

We could equally define a Konüs producer price index by using the cost function:

$$\frac{c(\boldsymbol{w}^1, q)}{c(\boldsymbol{w}, q)}.$$

By analogy, we could also define Paasche and Laspeyres producer price and input quantity indices. In practice, we sometimes need price and quantity indices for consumer demand, and other times for producer demand. The methods are the same. Either way, we start with a *performance* or *aggregator* function.[6]

The performance function will typically be a real-valued function defined on the positive orthant. It tells us what we get from a given consumption or input vector. Whether we obtain utility or output, the performance function tells how much of this valuable item we receive. In other words, the performance function will generally be either a production or utility function, although these are not the only possibilities.

We take prices to be strictly positive, whether for consumer goods or productive inputs. We solve a cost (expenditure) minimization problem, The solutions to the cost/expenditure minimization problem are corresponding consumer (Hicksian) or producer (conditional factor) demands.

Duality allows us to connect the performance level with the demands and implied income level. The solutions apply to both the cost minimization problem and the maximizes indirect utility or production.

[6] *Aggregator function* is Diewert's (1976) terminology, but the term performance better describes its role.

10.4.10 Superlative Indices

Basing our indices on utility or production functions leaves open the question of what function to use. Diewert (1976) argued that we should *flexible functional forms*, functions that can provide a second order approximation of arbitrary $\mathcal{C}^2$ functions that are homogeneous of degree one. Among those functions, he recommended focusing on those where the index is exact. The term *superlative* applies to such indices.

The Törnqvist price index is one such example as it is exact for translog cost and expenditure functions of the form

$$\begin{aligned}\ln c(\boldsymbol{w}, q) = \alpha_0 &+ \sum_{i=1}^{m} \alpha_i \ln w_i + \frac{1}{2}\sum_{i,j=1}^{m} \gamma_{ij} \ln p_i \ln p_j \\ &+ \beta \ln q + \delta(\ln q)^2 + \sum_{i=1}^{m} \varepsilon_i \ln q \ln p_i\end{aligned}$$

where $\sum_i \alpha_i = 1$, $\gamma_{ij} = \gamma_{ji}$, $\sum_i \gamma_{ij} = 0$ and $\sum_i \varepsilon = 0$. The translog function, which includes CES functions as special cases, was introduced by Christensen, Jorgenson, and Lau (1973, 1975).

The Fisher ideal index is exact for unit cost functions that are quadratic means of order two with the form

$$b(\boldsymbol{w}) = \left(\sum_{i,j=1}^{m} b_{ij} w_i w_j\right)^{1/2}$$

with $b_{ij} = b_{ji}$. See Diewert (1976) for more.

Initially, it was hoped that these would always provide better approximations for price indices than the Paasche and Laspeyres indices, partly because the Fisher and Törnqvist indices are very close. That may often be true, but that is not always the case (Hill, 2006).

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