15. Competitive Equilibrium

Outline February 23, 2023

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We now turn our attention to the economy as a whole. We start by focusing on the problem of general equilibrium. We consider all markets together. The simplest general equilibrium models include goods markets, service markets, and factor and resource markets.

The central problem of general equilibrium is to determine whether it is possible to find prices so that all markets simultaneously clear. After all, it could be that adjusting prices to clear one market will always force another market out of equilibrium.

If we can answer the central problem in the affirmative, we can then go on to analyze the properties of general equilibrium. How does the equilibrium change when parameters change? What principles govern equilibrium resource and goods allocations? How good a job does the equilibrium do at efficiently using resources and allocating goods and services?

We start by describing the basic characteristics of an economy in section one. Section two defines a type of general equilibrium, the Walrasian equilibrium, and investigates some of its basic properties. We then provide some examples of equilibrium in pure exchange economies—economies without production, in section three. Section four examines equilibrium in production economies. Production economies with constant returns to scale are often easier to analyze. Section five includes several general results for such economies, including the Non-substitution Theorem, factor price equalization, and the Rybczynski and Stolper-Samuelson effects.

15.1 What is an Economy?

An *economy* consists of consumers and producers engaging in economic activity–consuming, producing, and trading. We formalize this as follows.

Economy. An *economy* is a collection of I consumers, labeled $i = 1, \dots, I$, and F firms, $f = 1, \dots, F$, using m goods, $j = 1, \dots, m$. Each *consumer* is characterized by a consumption set $\mathfrak{X}_i \subset \mathbb{R}^m_+$, preferences $\succsim_i$ defined on the consumption set $\mathfrak{X}_i$, and an exogenous endowment of goods $\boldsymbol{\omega}^i \in \mathbb{R}^m_+$. If preferences are continuous, we represent them via a utility function u_i. Denote the aggregate endowment by $\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}^i$. Each *firm* f is characterized by a technology set Y_f. We can write the economy as $\mathcal{E} = \big((\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I, (Y_f)_{f=1}^F\big)$.[1]

When writing vectors belonging to consumers or firms, we will superscripts to denote ownership, which consumer or firm the vector pertains to, and subscripts to indicate the goods. For example, x^i_k is the amount of good k consumed by consumer i.

[1] When there is no risk of confusion, we may use the shorthand form $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y_f)$.

15.1.1 Allocations

We will be interested in what is produced, consumed, and traded in the economy. The totality of consumption and production choices are described by an *allocation*. An allocation is *feasible* if the economy has sufficient resources to support the allocation. That is, it is feasible if the sum of consumption vectors is the production possibilities set.

Allocations. An *allocation* is a $(I+F)$-tuple of vectors in $\mathbb{R}^m$, $\big((\mathbf{x}^i)_{i=1}^I, (\mathbf{y}^f)_{f=1}^F\big)$ such that $\mathbf{x}^i \in \mathfrak{X}_i$ for all i, $\mathbf{y}^f \in Y_f$ for all f. We say an allocation is *feasible* if it obeys the feasibility constraint

$$\sum_{i=1}^{I} \mathbf{x}^i \le \sum_{i=1}^{I} \boldsymbol{\omega}^i + \sum_{f=1}^{F} \mathbf{y}^f,$$

and that the allocation is *non-wasteful* if everything available is consumed, $\sum_i \mathbf{x}^i = \sum^i \boldsymbol{\omega}^i + \sum_f \mathbf{y}^f$. We will often use the shorthand $(\mathbf{x}^i, \mathbf{y}^f)$ to denote the allocation $\big((\mathbf{x}^i)_{i=1}^I, (\mathbf{y}^f)_{f=1}^F\big)$.

The economy starts with an endowment of goods, some of these goods become inputs into some production process, others may be directly consumed. The net output of the production sector, together with the remainder of the endowment, is then available for consumption. Firms may produce intermediate goods that are used as inputs to production processes within the firm or by other firms. Such intermediate goods wash out when summing net output vectors over all firms.

15.1.2 Pure Exchange Economies

One special case is the *pure exchange economy*. In a pure exchange economy, no production takes place. The only economic activity is the exchange of goods from consumer endowments. We can regard this as an economy with one firm that has production set $Y_1 = \mathbb{R}^m_-$, the negative orthant. The negative orthant is the smallest possible production set. It is not productive. It only allows disposal of goods or inaction.[2]

[2] Since feasibility requires that aggregate consumption be no more than the endowment plus net production, setting $Y_1 = \{0\}$ would yield the same feasible allocations. However, the non-wasteful allocations would differ.

15.2 Walrasian General Equilibrium Models

The first comprehensive theory of general equilibrium was created by Walras in 1874 (Walras, 1926). He built well, and his basic framework is still the one we use. In fact, if you read Walras, you will find that his full framework entails a much richer economic structure than used here, allowing for dynamics, uncertainty, and existence of multiple currencies. He even briefly considers the impact of taxation and monopoly.

Our notion of Walrasian equilibrium has been stripped to the basics. Consumers maximize utility, firms maximize profit, and markets clear.

Like Walras, we will later consider dynamics and uncertainty (Chapters 25 to 28). For these subjects, the types of models used still build on the basic Walrasian model, but in a rather different way from Walras' models that include dynamics and uncertainty. For now, we focus on the basics.

15.2.1 Circular Flow

Defining an equilibrium requires we account for all money flowing through the economy. The circular flow diagram in Figure 15.2.1 illustrates the situation.

Consumers obtain income by selling all or part of their endowments, including labor. They spend income by purchasing goods. Firms pay for their inputs and receive revenue from their outputs, possibly making a profit. Before defining an equilibrium, we must specify where the profit goes, or who pays if there is a loss. We will follow standard practice and attribute it to the owners of the firms—the consumers. We do this via ownership shares.

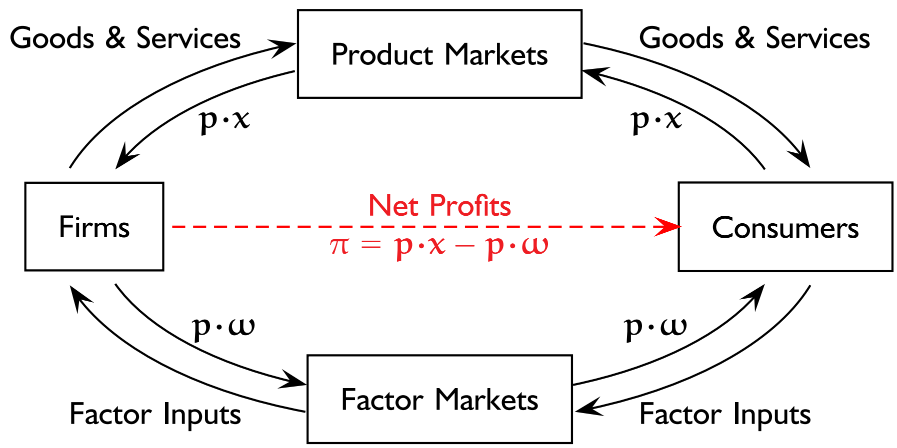

Figure 15.2.1: The basic Walrasian equilibrium model has a simple circular flow diagram. Trades within the firm and consumer sectors are not shown. All that is left is that consumers buy goods and services from the firms and sell labor and other resources to the firms. However, this leaves firms with a profit that is not spent. In the Walrasian model this is returned to the consumers according to their endowments of firm shares. The shares are endowments, not factors, so there is only a money flow without the corresponding capital goods flow often seen in circular flow diagrams.

15.2.2 Firm Shares

Let θ_f^i denote consumer i's ownership share of firm f. Here $\theta_f^i \geq 0$ for all i and f and $\sum_i \theta_f^i = 1$ for all f. We can think of θ_f^i as being the percentage of firm f owned by consumer i. The numbers θ_f^i indicate the disposition of firm f's profits and losses. The ownership shares are not traded.[3]

The *economy* is written[4]

$$\mathcal{E} = \big((\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I, (Y_f)_{f=1}^F, (\theta_f^i)_{i=1,f=1}^{I,F}\big).$$

If $\mathcal{E}$ is an exchange economy, firm 1 has $Y_1 = \mathbb{R}_-^m$ and its maximum profit is zero. In that case, we do not have to worry about the ownership shares as there is no income from the firm to distribute. We then write $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)$.

[3] Nothing is gained by allowing trade in the ownership shares. If a share of the firm earns \$1 profit, then that share would sell for \$1. Buying or selling it would not affect the consumer's budget constraint. This argument only holds in a world of certainty. If profits were uncertain, the price of a share with expected profit of \$1 may no longer be \$1.

[4] When there is no risk of confusion, we may use the shorthand notation $\mathcal{E} = (X_i, \succsim_i, \boldsymbol{\omega}^i, Y_f, \theta_f^i)$.

15.2.3 Walrasian Equilibrium

We are now ready to define general equilibrium. The definition we use is basically that of Walras. In a Walrasian equilibrium, both firms and consumers are treated as price-takers. Firms maximize profit and consumers maximize utility. Equilibrium prices are price vectors where all markets clear.[5].

Walrasian Equilibrium. A price vector $\hat{\mathbf{p}} > 0$ and allocation $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ form a *Walrasian equilibrium* for the economy $\mathcal{E}$ if:

1. Firms maximize profit: For each $f = 1, \dots, F$, $\hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f \geq \hat{\mathbf{p}}\cdot\mathbf{y}$ for all $\mathbf{y} \in Y_f$.
2. Consumers maximize utility: For each $i = 1, \dots, I$, $\hat{\mathbf{x}}^i \succsim_i \mathbf{x}$ for all $\mathbf{x} \in B(\hat{\mathbf{p}}, m^i)$ where $m^i = \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i + \sum_f \theta^i_f \hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f$.
3. Markets clear: $\sum_i \hat{\mathbf{x}}^i \leq \sum_i \boldsymbol{\omega}^i + \sum_f \hat{\mathbf{y}}^f$.[6]

Walras' original definition of equilibrium used a stricter definition of market clearing, requiring the equality of supply and demand. Schlesinger (1935) and Zeuthen (1933) showed that supply of resources could exceed demand for unproduced resources, precluding equilibrium. Schlesinger suggested solving this problem by requiring that demand be no more than supply, the now standard definition of market clearing.

Many fancier versions of general equilibrium exist, allowing for taxation and government spending, public goods and externalities, intertemporal choice, uncertainty, financial markets, imperfect competition, and other complications.[7] Including additional types of economic behavior requires some minor adjustment to the model, and sometimes a revised accounting for income flows, but all are similar to the basis Walrasian model. They contain the core ideas: consumers maximize utility, firms maximize profits, and markets clear.

[5] Other names for Walrasian equilibrium are competitive equilibrium, Arrow-Debreu equilibrium, and Arrow-Debreu-McKenzie equilibrium. In this book, the term Arrow-Debreu equilibrium is reserved for the contingent commodity equilibrium of section 27.2

[6] This means the allocation $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ is feasible.

[7] Intertemporal choice is considered in Chapter 25, while financial markets are the subject of Chapters 27 and 28.

15.2.4 Excess Demand and the Equilibrium Set

Let $\pi_f(\mathbf{p}) = \mathbf{p}\cdot\mathbf{y}^f(\mathbf{p})$ be the profit of firm f and $\mathbf{x}^i(\mathbf{p}, m^i)$ be the Marshallian demand of consumer i. Keep in mind that we may have to use correspondences for the Marshallian demands or for the firms' supply. Given an economy $\mathcal{E}$, we define the *excess demand*, the difference between demand and supply, by

$$\boldsymbol{z}(\mathbf{p}) = \sum_i \mathbf{x}\big(\mathbf{p}, m^i(\mathbf{p})\big) - \sum_i \boldsymbol{\omega}^i - \sum_f \mathbf{y}^f(\mathbf{p})$$

where

$$m^i(\mathbf{p}) = \mathbf{p}\cdot\boldsymbol{\omega}^i + \sum_f \theta^i_f\,\pi_f(\mathbf{p})$$

is consumer i's income from sale of the endowment and shares of firm profits.

Excess demand can be either a function or correspondence. The *equilibrium set*, the set of Walrasian equilibrium prices, is defined by

$$\boldsymbol{W}(\mathcal{E}) = \big\{\mathbf{p} : \boldsymbol{z}(\mathbf{p}) \le 0\big\}.$$

We will sometimes consider economies characterized by excess demands $\boldsymbol{z}(\mathbf{p})$. In that case we will abuse notation slightly and write $\boldsymbol{W}(\boldsymbol{z})$ for the corresponding equilibrium set.

15.2.5 The Price Level is Not Determined

One basic result is that the price level doesn't matter for equilibrium. Absolute prices don't matter. For Walrasian equilibrium, only relative prices matter. The absolute price level is irrelevant in general equilibrium, we can set it as we please.

Proposition 15.2.2. *If* $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ *is a Walrasian equilibrium for* $\mathcal{E}$, *so is* $(t\hat{\mathbf{p}}, \hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ *whenever* $t > 0$. *In other words,* $\boldsymbol{W}(\mathcal{E})$ *is a cone.*

Proof. The point is that both firm supplies and consumer demands are homogeneous of degree zero in $\mathbf{p}$, so the same allocation maximizes profit and utility under both $\mathbf{p}$ and $t\mathbf{p}$ for any $t > 0$. Since markets clear at $\mathbf{p}$ and the allocation is unchanged, they also clear at $t\mathbf{p}$. $\square$

15.2.6 Choosing a Price Level

Since the actual absolute level of prices is irrelevant, we can normalize prices in any fashion we wish. It is purely a matter of convenience. Three methods are often used. One is to choose a good as the unit of account, a *numéraire good*. We then measure all prices relative to the price of the numéraire good.

The widespread use of fiat currency now hides it, but the numéraire method was long used. Metals such as bronze, silver, and gold were all used as numéraire goods, with coins denominated in weights of that metal. The Roman *denarius* was originally 1/72 of a Roman pound of silver, the British pound sterling was originally a pound of silver.

When the United States moved to a gold standard in the 19^{th} century, the $20 Double Eagle was a troy ounce of gold. The term "dollar" itself indicates a weight of metal. It derives from *thaler*, a one ounce silver coin produced by various Counts von Schlick from the mines near Joachimsthal (now part of the Czech Republic). The first US dollars were based on the Spanish *reales de a ocho* or *pesos de ocho* (pieces of eight), themselves derived from the *thaler*.[8]

Although metal currency was progressively replaced by fiat currency during the 20^{th} century, the US government still used gold as money internationally until President Nixon closed the gold window on August 15, 1971.

[8] The eight pieces were often referred to as "shillings" or "bits". Thus the term "two bits" for a quarter. These and other Spanish coins such as the doubloon (four dollars) were long considered legal tender in the US. They were about a quarter of all circulating circulating coins as late as 1830. This was ended by the Coinage Act of 1857, which removed the legal tender status of all foreign coins, giving the US Mint a monopoly on legal coinage in the US.

15.2.7 Three Ways to Choose a Numéraire

We can treat any commodity as the unit of account for purposes of setting the price level.

To use good k as the numéraire, we need only multiply $\mathbf{p}$ by $1/p_k$ to obtain the normalized prices. Of course, this only works if $p_k > 0$, which can't always be guaranteed.

A second method of setting the price level is based on the fact that some price must be non-zero. This normalization sets $\sum_k p_k = 1$, insuring the price vector is in the $(m-1)$-dimensional simplex

$$\Delta_m = \left\{ \mathbf{p} \geq 0 : \sum_k p_k = 1 \right\}.$$

To normalize prices in this fashion, we multiply by $1/\sum_k p_k$. It doesn't matter if some prices are zero. This normalization works unless they're all zero, and they can't all be zero.

Finally, normalizing prices so the aggregate endowment has value 1 is sometimes convenient. In that case we set $\mathbf{p}\cdot\boldsymbol{\omega} = 1$.

15.2.8 Walras' Law

Now that we have a price system, we can ask about the value of consumption and the value of production. The fundamental relation between them is given by Walras' Law. It establishes that everything has been properly accounted for.

Walras' Law. *Suppose consumer preferences are locally non-satiated, that for each firm* f*, the net output* $\mathbf{y}^f(\mathbf{p})$ *maximizes profit for firm* f*, and that for each consumer* i*, the consumption bundle* $\mathbf{x}^i(\mathbf{p})$ *solves* i*'s consumer's problem with income* $\mathbf{p}\cdot\boldsymbol{\omega}^i + \sum_f \theta^i_f\,\mathbf{p}\cdot\mathbf{y}^f(\mathbf{p})$*. Then*

$$\sum_i \mathbf{p}\cdot\mathbf{x}^i(\mathbf{p}) - \sum_i \mathbf{p}\cdot\boldsymbol{\omega}^i - \sum_f \mathbf{p}\cdot\mathbf{y}^f(\mathbf{p}) = 0. \qquad (\,)$$

Proof. As in Lemma 4.3.3, local non-satiation implies each consumer spends all available income. Thus

$$\mathbf{p}\cdot\mathbf{x}^i(\mathbf{p}) = \mathbf{p}\cdot\boldsymbol{\omega}^i + \sum_f \theta^i_f\,\mathbf{p}\cdot\mathbf{y}^f(\mathbf{p}).$$

Summing over $i = 1, \dots, I$, yields

$$\begin{aligned}
\sum_i \mathbf{p}\cdot\mathbf{x}^i(\mathbf{p}) &= \sum_i \mathbf{p}\cdot\boldsymbol{\omega}^i + \sum_i\sum_f \theta^i_f\,\mathbf{p}\cdot\mathbf{y}^f(\mathbf{p}) \\
&= \sum_i \mathbf{p}\cdot\boldsymbol{\omega}^i + \sum_f\left(\sum_i \theta^i_f\right)\mathbf{p}\cdot\mathbf{y}^f(\mathbf{p}) \\
&= \sum_i \mathbf{p}\cdot\boldsymbol{\omega}^i + \sum_f \mathbf{p}\cdot\mathbf{y}^f(\mathbf{p})
\end{aligned}$$

which is equivalent to equation . $\square$

15.2.9 Walras' Law and GDP

As we saw in the proof, Walras' Law can also be written:

$$\sum_i \mathbf{p} \cdot x^i(\mathbf{p}) = \sum_i \mathbf{p} \cdot \omega^i + \sum_f \mathbf{p} \cdot y^f(\mathbf{p}).$$

In other words, the aggregate value of consumption (the left-hand side) must equal the value of the economy's resources together with the value of production (the right-hand side).

This form of Walras' Law is basic to our notions of GDP accounting, and hints at the way that general equilibrium can tie together microeconomic and macroeconomic issues. One can approach GDP accounting by considering either the value of consumption or the value of resources plus production. By Walras' Law, you get the same GDP either way.

15.2.10 Walras' Law: Market Clearing

For general equilibrium models, Walras' Law has two particularly important consequences: 1) If all markets but one clear, then all markets clear. 2) Any good that is in excess supply must be free—its price must be zero.

The first consequence of Walras' Law follows immediately.

Corollary 15.2.3. *Suppose consumer preferences are locally non-satiated and let prices obey* $\mathbf{p} \gg 0$*. Suppose further that for each firm* f*, the net output* $\mathbf{y}^f$ *maximizes firm* f*'s profit, and that for each consumer* i*, the consumption bundle* $\mathbf{x}^i$ *solves* i*'s utility maximization problem with income* $m^i = \mathbf{p}\cdot\boldsymbol{\omega}^i + \sum_f \theta^i_f \mathbf{p}\cdot\mathbf{y}^f$*. If every market but one clears, then the remaining market must also clear. That is, if* $\sum_i x^i_k = \sum_i \omega^i_k + \sum_f y^f_k$ *for* $k = 1, \dots, m$*, with* $k \neq j$*, then market* j *obeys* $\sum_i x^i_j = \sum_i \omega^i_j + \sum_f y^f_j$*. Moreover,* $(\mathbf{x}^i, \mathbf{y}^f, \mathbf{p})$ *is a Walrasian equilibrium.*

Proof. We appeal to Walras' Law as stated in equation .

$$
\begin{aligned}
0 &= \sum_i \mathbf{p}\cdot\mathbf{x}^i(\mathbf{p}) - \sum_i \mathbf{p}\cdot\boldsymbol{\omega}^i - \sum_f \mathbf{p}\cdot\mathbf{y}^f(\mathbf{p}) \\
&= \sum_k \left(\sum_i p_k x^i_k(\mathbf{p}) - \sum_i p_k \omega^i_k - \sum_f p_k y^f_k(\mathbf{p}) \right) \\
&= p_j \left(\sum_i x^i_j - \sum_i \omega^i_j - \sum_f y^f_j \right)
\end{aligned}
$$

where the last line follows because all of the sums for goods $k \neq j$ are zero. Since $p_j > 0$, the market for good j also clears with equality. Since firms are maximizing profits, consumers are maximizing utility, and markets clear, $(\mathbf{x}^i, \mathbf{y}^f, \mathbf{p})$ is a Walrasian equilibrium. $\square$

15.2.11 Walras' Law: Complementary Slackness

The heart of the second result is a complementary slackness condition. For each good, either the equilibrium constraint binds, or the corresponding price is zero.

Lemma 15.2.4. *Suppose preferences are locally non-satiated and* $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ *is a Walrasian equilibrium with* $\hat{\mathbf{p}} \geq \mathbf{0}$. *Then*

$$\hat{p}_k \left(\sum_i \hat{x}_k^i - \sum_i \omega_k^i - \sum_f \hat{y}_k^f \right) = 0$$

for every $k = 1, \ldots, m$.

Proof. By market clearing,

$$\sum_i \hat{x}_k^i \leq \sum_i \omega_k^i + \sum_f \hat{y}_k^f$$

for every $k = 1, \ldots, m$. As $\hat{p}_k \geq 0$,

$$\hat{p}_k \left(\sum_i \hat{x}_k^i - \sum_i \omega_k^i - \sum_f \hat{y}_k^f \right) \leq 0$$

for every $k = 1, \ldots, m$. That is, each term in the sum in Walras' Law is non-positive. Since they sum to zero, each term must be zero, which is the result we were trying to prove. □

15.2.12 Walras' Law: Excess Supply implies Zero Price

Lemma 15.2.4 allows us to make short work of the second consequence of Walras' Law. Any good in excess supply must be a free good—its price must be zero.

Corollary 15.2.5. *Suppose preferences are locally non-satiated and that free disposal is satisfied for at least one firm. Let* $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ *be a Walrasian equilibrium. If* $\sum_i \hat{x}^i_k < \sum_i \omega^i_k + \sum_f \hat{y}^f_k$*, then* $\hat{p}_k = 0$

Proof. Free disposal implies that prices must be non-negative as profit maximization would be impossible otherwise.

By Lemma 15.2.4,

$$\hat{p}_k \left(\sum_i \hat{x}^i_k - \sum_i \omega^i_k - \sum_f \hat{y}^f_k \right) = 0$$

for every $k = 1, \dots, m$. If $\sum_i \hat{x}^i_k < \sum_i \omega^i_k + \sum_f \hat{y}^f_k$, it must be that $\hat{p}_k = 0$. □

Together with normalization of prices, these results simplify calculation of equilibria by reducing the number of equations and variables to consider, and by handling cases involving excess supply.

15.3 Pure Exchange: The Edgeworth Box

We now specialize to the case of a pure exchange economy (no production) with two goods and two consumers. The consumption set is $\mathfrak{X}_i = \mathbb{R}^2_+$. Then $(\mathbf{x}^1, \mathbf{x}^2)$ is a feasible allocation if $\mathbf{x}^i \geq \mathbf{0}$, $x^1_1 + x^2_1 \leq \omega_1$ and $x^1_2 + x^2_2 \leq \omega_2$.[9] We presume preferences are monotonic and focus our attention on the non-wasteful allocations where $x^1_1 + x^2_1 = \omega_1$ and $x^1_2 + x^2_2 = \omega_2$.

When an allocation is non-wasteful, it is enough to know how much person 1 gets to identify the allocation. Then person 2 receives the rest, $\mathbf{x}^2 = \boldsymbol{\omega} - \mathbf{x}^1$. Equally, if we know what person 2 gets, we also know what person 1 gets. This is the key to constructing the Edgeworth box.[10] We can describe the non-wasteful allocation $(\mathbf{x}^1, \mathbf{x}^2)$ by the single point $\mathbf{x}^1$. We put a second origin at ω, and rotate the axes there by 180 degrees. If we use the original origin to label the point, we obtain person 1's allocation. If we use the new origin at ω, we obtain person 2's allocation.

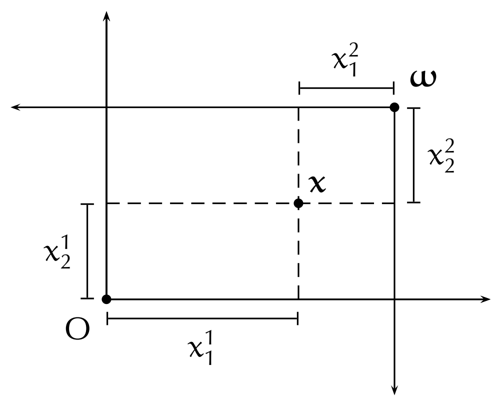

Figure 15.3.1: The point $\mathbf{x}$ has co-ordinates (x^1_1, x^1_2) when measured from the origin (O) and co-ordinates (x^2_1, x^2_2) when measured from $\boldsymbol{\omega}$. This allows us to represent the allocation $(\mathbf{x}^1, \mathbf{x}^2)$ as a single point. Note that $x^1_1 + x^2_1 = \omega_1$ and $x^1_2 + x^2_2 = \omega_2$.

[9] Recall that we use superscripts to denote consumers or producers and subscripts to indicate which good. Thus x^i_k is the amount of good k consumed by consumer i.

[10] Although named after Frances Edgeworth, and sometimes Arthur Bowley, this diagram is actually due to Vilfredo Pareto (inspired by Edgeworth). Bowley got his name on it by popularizing it.

15.3.1 Budget Lines in the Edgeworth Box

Budget lines also conform to this convention, with the same budget line serving for both person 1 and person 2. There are two key points. First, slopes are unaffected by the 180° rotation that creates the other coordinate system. And second, if $\mathbf{p}\cdot\mathbf{x}^1 = \mathbf{p}\cdot\boldsymbol{\omega}^1$, then $\mathbf{p}\cdot\mathbf{x}^2 = \mathbf{p}\cdot\boldsymbol{\omega}^2$. To see the latter, recall that non-wasteful allocations obey $\mathbf{x}^1 + \mathbf{x}^2 = \boldsymbol{\omega} = \boldsymbol{\omega}^1 + \boldsymbol{\omega}^2$. Then price the equation, finding that $\mathbf{p}\cdot\mathbf{x}^1 + \mathbf{p}\cdot\mathbf{x}^2 = \mathbf{p}\cdot\boldsymbol{\omega} = \mathbf{p}\cdot\boldsymbol{\omega}^1 + \mathbf{p}\cdot\boldsymbol{\omega}^2$. It follows that $\mathbf{p}\cdot\mathbf{x}^1 = \mathbf{p}\cdot\boldsymbol{\omega}^1$ if and only if $\mathbf{p}\cdot\mathbf{x}^2 = \mathbf{p}\cdot\boldsymbol{\omega}^2$.

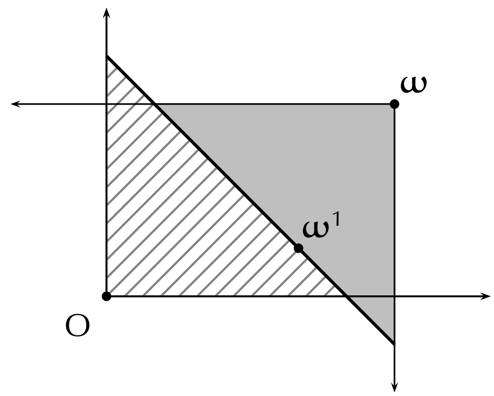

Figure 15.3.2: The same line serves as budget line for consumer one (read from the origin as $\mathbf{p}\cdot\mathbf{x}^1 = \mathbf{p}\cdot\boldsymbol{\omega}^1$) and for consumer two (read from $\boldsymbol{\omega}$ as $\mathbf{p}\cdot\mathbf{x}^2 = \mathbf{p}\cdot\boldsymbol{\omega}^2$). Consumer one's budget set is hatched, consumer two's budget set is shaded. Notice that one or both budget sets may contain points that are not feasible allocations.

15.3.2 Edgeworth's Ideas

For a moment, we will think like Edgeworth (1881), not Walras, and consider the possibilities for trade without reference to a price system. Consumers trade because it makes them better off. As a result, neither consumer will be willing to trade unless they end up with a consumption bundle that they value at least as highly as their endowment point. This means that for each consumer, the equilibrium point (if any) must yield at least as much utility as the endowment point. This property of being at least as good as the endowment is referred to as *individual rationality*.

The equilibrium point must lie within the lenticular region of mutually beneficial trades, as shown in Figure 15.3.3.a. To find the equilibrium, we can use offer curves.

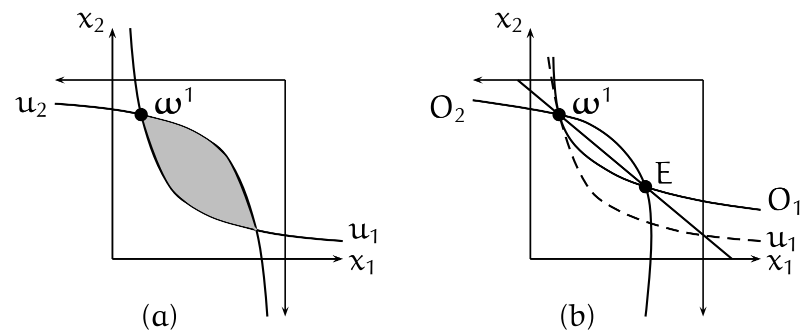

Figure 15.3.3: Both consumers have equal-weighted Cobb-Douglas preferences. The endowments are $\boldsymbol{\omega}^1 = (1,2)$ and $\boldsymbol{\omega}^2 = (2,1)$.

In figure (a), the indifference curves through the endowment point are u_1 and u_2. The shaded lenticular region between them denotes the region of potential trades, the region where both consumers are at least as well off as at their respective endowments. Given the initial endowment $\boldsymbol{\omega}$, any equilibrium point must lie within the lenticular area.

Figure (b) shows the offer curves O_1 and O_2 which trace demand for each consumer at various prices. Notice how O_1 lies above the indifference curve u_1. The equilibrium is at E. The budget line connects $\boldsymbol{\omega}$ and E. It determines the equilibrium price vector, up to scalar multiplication. Here $\mathbf{p}^* = (2.5, 3)$.

The *offer curve* is the locus traced-out by the demand points as prices change. Since the demand point is individually rational, the offer curve for consumer 1 lies above 1's indifference curve. Any intersection of the offer curves (other than the endowment point) is an equilibrium allocation, as in Figure 15.3.3.b where there is a unique such intersection. The equilibrium price vector is then perpendicular to the line connecting the endowment point and the equilibrium allocation.

15.3.3 Equilibrium with Two Cobb-Douglas Consumers

Example 15.3.4: Suppose utility is identical for both people and given by $u(\mathbf{x}^i) = (x_1^i)^\gamma (x_2^i)^{(1-\gamma)}$ where $0 < \gamma < 1$ and endowments are $\boldsymbol{\omega}^1 = (1, 2)$ and $\boldsymbol{\omega}^2 = (2, 3)$. Incomes are $m^1 = p_1 + 2p_2$ and $m^2 = 2p_1 + 3p_2$, yielding demands

$$\mathbf{x}^1(\mathbf{p}) = (p_1 + 2p_2)\begin{pmatrix} \gamma/p_1 \\ (1-\gamma)/p_2 \end{pmatrix} \quad \text{and} \quad \mathbf{x}^2(\mathbf{p}) = (2p_1 + 3p_2)\begin{pmatrix} \gamma/p_1 \\ (1-\gamma)/p_2 \end{pmatrix}.$$

.

When drawn in an Edgeworth box, these demand curves are called *offer curves*. Market demand is then

$$\mathbf{x}(\mathbf{p}) = (3p_1 + 5p_2)\begin{pmatrix} \gamma/p_1 \\ (1-\gamma)/p_2 \end{pmatrix}.$$

We find all the equilibria by setting $\mathbf{x}(\mathbf{p}) = \boldsymbol{\omega} = (3, 5)$. Clearly either $p_1 = 0$ or $p_2 = 0$ would lead to infinite demand, so we are free to pick either as numéraire. Set $p_1 = 1$. We now have two equations to determine one unknown: $3 = \gamma(3 + 5p_2)$ and $5 = (1-\gamma)(3 + 5p_2)/p_2$. In fact, one of these equations is redundant due to Walras' Law. We solve the first equation to find $p_2 = 3(1-\gamma)/5\gamma$.[11] This yields the equilibrium allocation

$$\left(\begin{pmatrix} (6-\gamma)/5 \\ (6-\gamma)/3 \end{pmatrix}, \begin{pmatrix} (9+\gamma)/5 \\ (9+\gamma)/3 \end{pmatrix}\right).$$

Any prices with $p_2/p_1 = 3(1-\gamma)/5\gamma$ are equilibrium prices. ◀

[11] In equilibrium problems, you can check your work by verifying that it solves the unused equation.

15.3.4 Equilibrium with Many Cobb-Douglas Consumers SKIPPED

Example 15.3.5: Suppose there are m goods and I consumers with identical Cobb-Douglas utility functions $u(x) = x_1^{\gamma_1} \cdots x_m^{\gamma_m}$ where each $\gamma_j > 0$ and $\sum_{j=1}^m \gamma_j = 1$. Let $\boldsymbol{\omega}^i$ be consumer i's endowment and $\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}^i$ the social endowment.

Now solve for the equilibrium. Consumer i has demand

$$\mathbf{x}^i(\mathbf{p}) = (\mathbf{p}\cdot\boldsymbol{\omega}^i)\begin{pmatrix}\gamma_1/p_1\\ \vdots\\ \gamma_m/p_m\end{pmatrix}.$$

Aggregate demand is

$$\mathbf{x}(\mathbf{p}) = \sum_{i=1}^{I}\mathbf{x}^i(\mathbf{p}) = \left(\sum_{i=1}^{I}\mathbf{p}\cdot\boldsymbol{\omega}^i\right)\begin{pmatrix}\gamma_1/p_1\\ \vdots\\ \gamma_m/p_m\end{pmatrix} = (\mathbf{p}\cdot\boldsymbol{\omega})\begin{pmatrix}\gamma_1/p_1\\ \vdots\\ \gamma_m/p_m\end{pmatrix}.$$

We normalize prices so $\mathbf{p}\cdot\boldsymbol{\omega} = 1$. The market clearing condition becomes

$$\boldsymbol{\omega} = \begin{pmatrix}\omega_1\\ \vdots\\ \omega_m\end{pmatrix} = \begin{pmatrix}\gamma_1/p_1\\ \vdots\\ \gamma_m/p_m\end{pmatrix}.$$

It follows that $\omega_j = \gamma_j/p_j$, or equivalently, $p_j = \gamma_j/\omega_j$ for $j = 1, \ldots, m$. This price vector and its positive multiples are the only equilibrium prices. The corresponding equilibrium allocation is

$$\hat{\mathbf{x}}^i = (\mathbf{p}\cdot\boldsymbol{\omega}^i)\,\boldsymbol{\omega} = \left(\sum_{j=1}^{m}\gamma_j\frac{\omega_j^i}{\omega_j}\right)\boldsymbol{\omega}.$$

◀

15.3.5 Equilibrium with Two Leontief Consumers

SKIPPED

Example 15.3.6: Let $\omega^1 = (2,1)$ and $\omega^2 = (1,1)$, so $\omega = (3,2)$. Suppose utility has the Leontief form $u_i(\mathbf{x}^i) = \min\{x_1^i, x_2^i\}$.

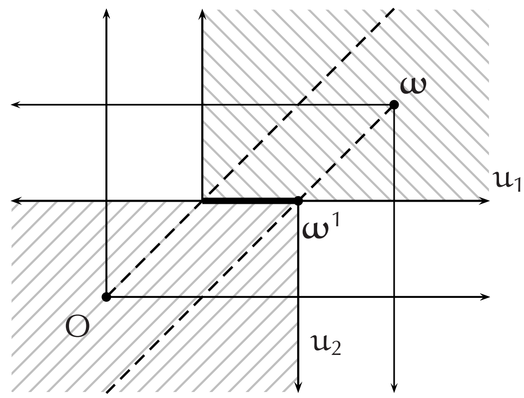

Figure 15.3.7: Two Leontief indifference curves are shown above. The dashed lines indicate the corners of the Leontief indifference curves. The region of mutually beneficial trades is the heavy line to the left of the endowment point ω^1. It is clear that the budget line must be horizontal to move to such points. That means the price vector is any $(0, p)$ for $p > 0$. With such a price vector, any point on the heavy line is an equilibrium allocation.

We first consider the case of both prices positive. In that case, market demand for both goods must be the same. However, if $x_1 < 3$, there will be excess supply of good 1, which is impossible with positive prices. But if $x_1 = 3$, $x_2 = 3$ also, and there is excess demand for good 2. The market fails to clear and this is not an equilibrium.

If $p_2 = 0$, $\mathbf{x}^1 = (2, x_2^1)$ with $x_2^1 \geq 2$ and $\mathbf{x}^2 = (1, x_2^2)$ with $x_2^2 \geq 1$. This means $x_2 \geq 3$. Demand is larger than the endowment. The market for good two does not clear and this cannot be an equilibrium.

That leaves $\mathbf{p} = (0, p)$. We use good two as numéraire and set $p = 1$. Then incomes are $m^1 = 1$ and $m^2 = 1$. Demands for good two are $x_2^1 = 1$ and $x_2^2 = 1$, clearing that market. For good one, any $x_1^1, x_1^2 \geq 1$ with $x_1^2 \geq 1$ and $x_1^1 + x_1^2 \leq 2$ is an equilibrium allocation, as shown by Figure 15.3.7.

Still larger consumption of good one is allowed by the budget constraint. Although the consumers are equally well off here, it is not an equilibrium because there will be excess demand for good one. ◄

15.3.6 Equilibrium with Linear Indifference Curves SKIPPED

Example 15.3.8: We again consider a two-person exchange economy with endowments $\boldsymbol{\omega}^1 = (2,1)$ and $\boldsymbol{\omega}^2 = (1,1)$. The utility functions are linear: $u_1(\mathbf{x}^1) = x_1^1 + 2x_2^1$ and $u_2(\mathbf{x}^2) = 2x_1^2 + x_2^2$.

As you can see in Figure 15.3.9, the region of mutual improvement lies above and to the left of the endowment point. This means that the relative price of good one must be between 1/2 and two, the slopes of the indifference curves through $\boldsymbol{\omega}^1$.

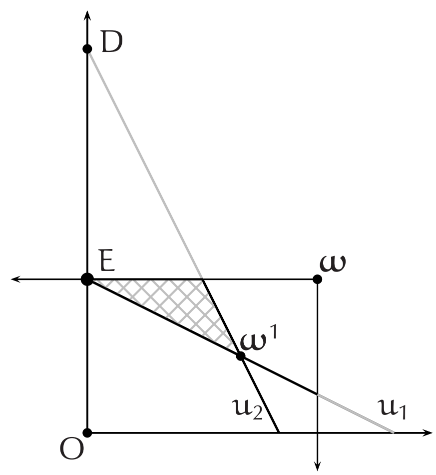

Figure 15.3.9: Again the endowments are $\boldsymbol{\omega}^1 = (2,1)$ and $\boldsymbol{\omega}^2 = (1,1)$. Both consumers have linear indifference curves. Consumer one's indifference curves have slope $-\frac{1}{2}$ while consumer two's have slope -2. The shaded area shows the potential mutual improvements.

Since the budget line must connect the equilibrium with the endowment, it must have a slope between $-\frac{1}{2}$ and -2. However, if the slope is not $-\frac{1}{2}$, consumer one's demand point will lie outside the Edgeworth box. This is illustrated for a slope of -2 when consumer one's demand point is D.

The equilibrium budget line must have slope $-\frac{1}{2}$ when consumer one is indifferent along the entire budget line, while consumer two consumes only good one. The point E is the only such point and is the equilibrium.

Let the price vector be $(p,1)$. If $p > \frac{1}{2}$, $MRS^1 = \frac{1}{2} < p$ and consumer one will only consume good two. Then $x_2^1 = (p,1)\cdot(2,1)/p = (2p+1)/p > 2$. This means that demand for good two exceeds its supply and we cannot have an equilibrium.

It follows that $p = \frac{1}{2}$ is the only possible price. Consumer one is then indifferent between all points on the budget line. Consumer two has $MRS^2 = 2 > p$ and will spend everything on good one, so $x_1^2 = (\frac{1}{2},1)\cdot(1,1)/\frac{1}{2} = 3$ and $\mathbf{x}^2 = (3,0)$. By market clearing, $\mathbf{x}^1 = (0,2)$, yielding the equilibrium point E in Figure 15.3.9. ◀

15.4 Equilibrium in Production Economies

It's a little more complex to find equilibria in economies with production. It's usually best to find the profit functions first, then substitute in the consumer's problem to find demands. Finally, use the market clearing conditions to find the equilibria. When production is constant returns to scale, the problem is somewhat simplified since profits are zero. There may also be strong restrictions on prices.

15.4.1 Equilibrium with CRS Production

Example 15.4.1: Suppose there are two goods, two consumers and one firm. The consumers have Cobb-Douglas utility $u_i(\mathbf{x}^i) = (x_1^i)^{1/2}(x_2^i)^{1/2}$ and endowments $\boldsymbol{\omega}^1 = (2,0)$ and $\boldsymbol{\omega}^2 = (1,0)$. Good two must be produced because none is available from the endowment. The production set is $Y = \{(y_1, y_2) : y_2 \leq -y_1, y_1 \leq 0\}$.

Profits will be maximized when $y_2 = -y_1$, so we must maximize

$$p_1 y_1 + p_2(-y_1) = (p_1 - p_2)y_1$$

under the constraint $y_1 \leq 0$. There is no maximum if $p_1 < p_2$ and so no equilibrium. If $p_1 > p_2$, the maximum is only at $y_1 = y_2 = 0$. Since nothing is produced, consumers can only consume good one. This requires that good two be unaffordable. Its relative price must be infinite, so $p_1 = 0$. Then $p_2 < 0$, so there is again no equilibrium. Finally, if $p_1 = p_2$, any $(y_1, -y_1)$ with $y_1 \leq 0$ will maximize profits, which are zero. In this case we may as well use good one as numéraire, normalizing prices so that $\mathbf{p} = (1,1)$.

Notice how the fact of production determines the relative prices in equilibrium. The only role demand played in determining prices was insuring they were strictly positive!

Consumer wealth is then $m^1 = 2$ and $m^2 = 1$, which yields demands $\mathbf{x}^1 = (1,1)$ and $\mathbf{x}^2 = (1/2, 1/2)$. Market demand is $(3/2, 3/2)$. The market clearing condition is

$$\mathbf{y} + \boldsymbol{\omega} = \mathbf{x}^1 + \mathbf{x}^2 \quad \text{or} \quad \mathbf{y} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix},$$

so $y_2 = 3/2$ and $y_1 = -3/2$. The equilibrium is $\mathbf{x}^1 = (1,1)$, $\mathbf{x}^2 = (1/2, 1/2)$, $\mathbf{y} = (-3/2, 3/2)$ and $\mathbf{p} = (1,1)$. ◀

15.4.2 Economy where Production is Possible, but Unused SKIPPED

It is possible that some, or even all firms do not produce in equilibrium. Here's a simple example with one CRS firm that is not used in equilibrium.

Example 15.4.2: Suppose there are two goods, two consumers and one firm. The consumers have Cobb-Douglas utility $u_i(\mathbf{x}^i) = (x_1^i)^{1/2}(x_2^i)^{1/2}$ and endowments $\boldsymbol{\omega}^1 = (3,1)$ and $\boldsymbol{\omega}^2 = (1,3)$, so the aggregate endowment is $\boldsymbol{\omega} = (4,4)$. Good two can be produced from good one. The production set is $Y = \{(y_1, y_2) : y_2 \le -y_1/2, y_1 \le 0\}$.

Profits will be maximized when $y_2 = -y_1/2$, so we must maximize

$$p_1 y_1 + p_2\left(-\frac{1}{2}y_1\right) = \left(p_1 - \frac{1}{2}p_2\right) y_1$$

under the constraint $y_1 \le 0$. There is no maximum if $p_1 < p_2/2$ and so no equilibrium. If $p_1 > p_2/2$, the maximum is only at $y_1 = y_2 = 0$. If nothing is produced, the consumers have only the aggregate endowment to consume. It becomes an exchange economy. If we take good one as numéraire, the equilibrium is unique, $\hat{\mathbf{p}} = (1,1)$. The corresponding allocation is $\hat{\mathbf{x}}^1 = \hat{\mathbf{x}}^2 = (2,2)$. Since $1 = p_1 > p_2/2 = 1/2$, production must be $\hat{\mathbf{y}} = (0,0)$. We have found an equilibrium, but it doesn't use the technology at all.

Are there any equilibria that do use the technology? For the technology to be productive, we must have $p_2 = 2p_1$. Again taking good one as numéraire, $\mathbf{p} = (1,2)$. Incomes are $m^1 = 5$ and $m^2 = 7$. Each consumer spends half of their income on each good, so demands are $\mathbf{x}^1 = (5/2, 5/4)$ and $\mathbf{x}^2 = (7/2, 7/4)$. Market demand is $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2 = (6,3)$. By market clearing,

$$\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \boldsymbol{\omega} + \mathbf{y} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} + \mathbf{y},$$

so $\mathbf{y} = (2,-1)$. But $\mathbf{y} \notin Y$ as $y_1 > 0$. It follows that the only equilibrium has $\hat{\mathbf{p}} = (1,1)$, $\hat{\mathbf{x}}^1 = \hat{\mathbf{x}}^2 = (2,2)$, and $\hat{\mathbf{y}} = \mathbf{0}$. It does not use the production technology at all! ◀

15.4.3 Equilibrium with DRS Production SKIPPED

Example 15.4.3: A production economy has 2 goods, 2 consumers, and 1 firm. The consumers have identical Cobb-Douglas utility functions $u_i(\mathbf{x}) = \sqrt{x_1^i x_2^i}$. Endowments are $\boldsymbol{\omega}^1 = (1,1)$ and $\boldsymbol{\omega}^2 = (1,0)$. Consumer one receives $\theta_1^1 = 2/3$ of the profits, while consumer two gets $\theta_1^2 = 1/3$ of the profits. The firm produces good two. Its technology is described by the production function $f(z) = 2\sqrt{z}$ where z is the input of good one.

We first consider the firm. The firm's profit is $2p_2\sqrt{z} - p_1 z$. The first-order condition for maximizing this concave profit function is $p_2/\sqrt{z} = p_1$. Thus optimal input is $z = (p_2/p_1)^2$. The net output vector is

$$\mathbf{y}(\mathbf{p}) = \left(-\left(\frac{p_2}{p_1}\right)^2, \frac{2p_2}{p_1}\right)$$

yielding profit $\pi(\mathbf{p}) = p_2^2/p_1$.

Preferences are equal-weighted Cobb-Douglas, so the consumers will spend half of their income on good one and half on good two. The form of the utility function ensures that neither good can have price zero, so it is okay to take good one as numéraire. Let p be the (relative) price of good two. The firm's profit is now p^2. Wealth is $m^1 = 1 + p + 2p^2/3$ and $m^2 = 1 + p^2/3$.

Consumer one has demand $\mathbf{x}^1(p) = \frac{3+3p+2p^2}{6}(1, 1/p)$ while consumer two's demand is $\mathbf{x}^2(p) = \frac{3+p^2}{6}(1, 1/p)$. Market demand is then $\mathbf{x}(p) = \frac{2+p+p^2}{2}(1, 1/p)$.

The market clearing condition is

$$(2 + p + p^2)(1/2, 1/2p) = (2, 1) + (-p^2, 2p) = (2 - p^2, 1 + 2p).$$

By Walras' Law, it is enough to clear the market for good one. Thus $2+p+p^2 = 4-2p^2$. Rewriting, $3p^2 + p - 2 = 0$. This has solutions $p = -1$ and $p = 2/3$. Thus $p = 2/3$ is the equilibrium price. Production is then $\mathbf{y} = (-4/9, 4/3)$, profits are $\pi = 4/9$, and the consumption vectors are $\mathbf{x}^1 = \frac{53}{54}(1, 3/2)$ and $\mathbf{x}^2 = \frac{31}{54}(1, 3/2)$. ◄

15.5 General Equilibrium in Action

In some cases we can get strong results from general equilibrium models. In other cases, a fair amount of structure is required to get clear answers to general equilibrium questions. In fact, an entire subfield, computable general equilibrium, has grown up to help model microeconomic problems where the whole economy must be examined.

In this section we present some basic results from general equilibrium theory: the Non-substitution Theorem, factor price equalization, and the Stolper-Samuelson and Rybczynski effects. All of these involve outcomes that are surprising or unobtainable from partial equilibrium models.

15.5.1 The Non-substitution Theorem

The usual view of price determination is that both supply and demand interact to determine prices. It is normally not possible to say that supply alone or demand alone determines prices. Alfred Marshall compared it asking which blade of a pair of scissors cuts the paper, even though it takes both to cut the paper.

Remarkably, in general equilibrium, there are conditions under which demand is irrelevant for price determination. One prominent case is that of the Non-substitution Theorem.

The Non-substitution Theorem applies when there is a single primary (unproduced) resource that is used by all firms. Labor is the canonical example. When the technology is constant returns to scale, and each firm produces exactly one good, the Non-substitution Theorem tells us that all prices are determined by the production technology. Demand does not affect the equilibrium prices.

This was illustrated in Example 15.4.1. There is one non-produced good, and one firm producing the single produced good. In the example, once we know that something is produced, the relative price is determined. It must be one. Demand for the produced good does not affect its price.

We say good j is an *essential input* for firm f if $\mathbf{y} \in Y_f$ with $\mathbf{y} \not\leq \mathbf{0}$ implies $y_j < 0$. In other words, if the firm produces a positive amount of anything, it must use good j as an input.

Non-substitution Theorem. *Let $\mathcal{E}$ by an economy with one non-produced good (good 0) where each produced good f can be produced by exactly one firm, firm f (thus $m = 1 + F$). Suppose further that the non-produced good is an essential input for every firm and that each firm has a CRS technology. If each firm produces a positive amount of their product in equilibrium, then prices must satisfy $p_f = b_f(\mathbf{p})$ for $f = 1, \ldots, F$, where b_f is the unit cost function for producing f. Moreover, once a price normalization has been chosen, this system has a unique solution.*

15.5.2 Proof of Non-substitution Theorem

Non-substitution Theorem. *Let $\mathcal{E}$ by an economy with one non-produced good (good 0) where each produced good f can be produced by exactly one firm, firm f (thus $m = 1 + F$). Suppose further that the non-produced good is an essential input for every firm and that each firm has a CRS technology. If each firm produces a positive amount of their product in equilibrium, then prices must satisfy $p_f = b_f(\mathbf{p})$ for $f = 1, \dots, F$, where b_f is the unit cost function for producing f. Moreover, once a price normalization has been chosen, this system has a unique solution.*

Proof. Due to constant returns to scale, the cost function of firm f can be written as $c_f(\mathbf{p}, q_f) = b_f(\mathbf{p})q_f$. The constant returns to scale also imply profits of each firm are zero, $p_f q_f - b_f(\mathbf{p})q_f = 0$. Since $q_f > 0$ in equilibrium, $p_f = b_f(\mathbf{p})$.

As good 0 is essential, demand for it must be positive. It follows that the price must be positive. We normalize prices by using it as numéraire. In fact, Shephard's Lemma implies demand for good 0 by firm f is

$$x_0^f = \frac{\partial c_f}{\partial p_0} = q_f \frac{\partial b_f}{\partial p_0} > 0.$$

Since the non-produced resource is essential, each firm must demand it, and $\partial b_f/\partial p_0 > 0$ for $f \neq 0$.

Now suppose we have two solutions to $\mathbf{p} = \mathbf{b}(\mathbf{p})$. Call them $\mathbf{p}$ and $\mathbf{p}'$. Let $\lambda = \max_{j\neq 0} p'_j/p_j$ (good 0 is excluded) and let k be a good where the maximum occurs.

Suppose, by way of contradiction, that $\lambda > 1$. Since $p_0 = p'_0 = 1$, $\lambda = \max_j p'_j/p_j$, with good 0 included. Then $\lambda\mathbf{p} \geq \mathbf{p}'$ and the previously chosen k with $p'_j = \lambda p_j$ is not good 0. Now

$$p'_k = \lambda p_k = \lambda b_k(\mathbf{p}) = b_k(\lambda\mathbf{p}) > b_k(\mathbf{p}') = p'_k, \tag{15.5.1}$$

where the strict inequality arises because b_k is strictly increasing in p_0 and the price of good 0 is higher at $\lambda\mathbf{p}$ ($\lambda p_0 = \lambda > 1 = p_0$).

But then $p'_k > p'_k$ by equation 15.5.1, a **contradiction**. It follows that we cannot have $\lambda > 1$, so $\lambda \leq 1$. This implies $\mathbf{p} \geq \mathbf{p}'$.

Reversing the role of $\mathbf{p}$ and $\mathbf{p}'$ yields $\mathbf{p} \leq \mathbf{p}'$, so the solution to $\mathbf{p} = \mathbf{b}(\mathbf{p})$ is unique. □

15.5.3 Factor Price Equalization

When there are multiple non-produced factors, the Non-substitution Theorem no longer applies. Nonetheless, the technology can still exert a very strong influence over prices. One related result is that when output prices are the same, so are factor prices, provided the factor endowments are not too different. This is known as factor price equalization.

To see how it works, let's start with an activity analysis model where three goods ($j = 1, 2, 3$) are produced using the two primary factors ($j = 4, 5$). Let

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \\ -4 & -2 & -1 \end{bmatrix}$$

be the matrix of activities. Now suppose the price of each type of output is \$1, so we have three different activities that produce \$1 of income at unit intensity.

We now construct the unit revenue curve in input space. Consider first the input required to produce one dollar of revenue with each activity. These are $\mathbf{b}_1 = (1, 4)$, $\mathbf{b}_2 = (2, 2)$, and $\mathbf{b}_3 = (4, 1)$.

15.5.4 Lerner Diagram

To obtain the unit revenue curve, first consider the isoquant for the production of each output good that yields unit revenue. Since the price of each output is one, these input vectors correspond to an output of one unit. For each activity, consider the factor endowments that can be used with that activity to produce at least \$1 of revenue. Keeping in mind that free disposal is possible, we find that the cone with vertex $\mathbf{b}_j$ are input vectors that produce at least \$1 of revenue. These cones have vertices $\mathbf{b}_1 = (1,4)^\mathsf{T}$, $\mathbf{b}_2 = (2,2)^\mathsf{T}$ and $\mathbf{b}_3 = (4,1)^\mathsf{T}$, respectively.

Now take the convex hull of these cones, as illustrated in Figure 15.5.1. The shaded region represents all factor combinations that generate at least \$1 of revenue. Its frontier is the unit revenue curve. Figure 15.5.1, which shows the inputs that can generate at least \$1 of revenue, is called a *Lerner diagram* or sometimes a *Lerner-Pearce diagram*.[12]

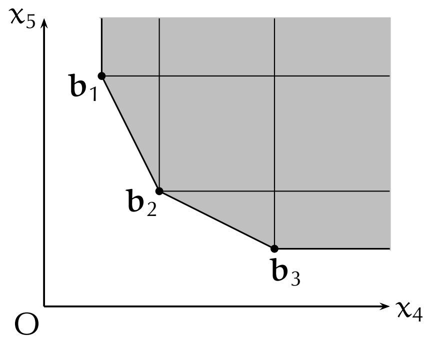

Figure 15.5.1: **Lerner Diagram.** On the left, the unit revenue isoquant is generated from activities $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ with corresponding unit revenue input vectors $(x_4, x_5) = \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$.

[12] The Lerner diagram using unit-value isoquants, was introduced by Lerner (1952). Pearce (1952) used a similar diagram using unit-quantity isoquants.

15.5.5 The Endowment Determines the Scale of Production

Now consider profit maximization. Linear activity models always have constant returns to scale. If profit can be maximized, the maximum profit is zero. Moreover, market clearing requires that the resource endowment be fully utilized. Demand for resources is given by Shephard's Lemma. Put together, this all means that the endowment must satisfy a cost minimization problem. In other words, the vector of input prices must support the revenue curve at the endowment. Moreover, the zero profit condition requires that factor prices are normalized so that cost equals revenue.

In practice, the endowment will rarely be on the unit revenue curve, so we have to scale it, obtaining an isorevenue curve that contains the factor endowment. The supporting prices give us the relative prices of the factors, and the absolute price of the factors is set so that profit is zero. This is illustrated in Figure 15.5.2.

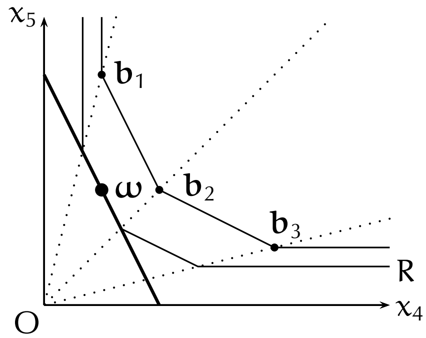

Figure 15.5.2: We choose the isorevenue curve R containing the endowment $\boldsymbol{\omega}$. As long as $\boldsymbol{\omega}$ is not proportional to any of the $\mathbf{b}_i$'s, the relative factor prices are uniquely determined by the isocost line that both supports the isorevenue curve R and contains the endowment $\boldsymbol{\omega}$.

In the Lerner diagram in Figure 15.5.2, an endowment at $\boldsymbol{\omega} = (1,2)^T$ leads us to scale production so that the isorevenue curve R that goes through ω. The slope of the isocost determines the relative factor prices ($p_4 = 2p_5$), while the zero profit condition determines the factor price level ($p_5 = 1/6$). Thus $\mathbf{p} = (1,1,1,1/3,1/6)$. Finally the corresponding intensity levels can be found by setting supply $\boldsymbol{\omega}$ equal to demand $z_1\mathbf{b}_1 + z_2\mathbf{b}_2$ and solving

$$\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1 & 2\\4 & 1\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}$$

to get $z_1 = 1/3$ and $z_2 = 1/3$.

15.5.6 Cones of Diversification

Now consider Figure 15.5.3, which is Figure 15.5.2 with slightly different labels. The dotted lines demarcate four cones, C_1 through C_4.

The same factor prices will obtain if the endowment is anywhere in the interior of the cone marked C_2. Moreover, if you have an endowment in C_2, it will always be cheaper to use activities $\mathbf{a}_1$ and $\mathbf{a}_2$ than $\mathbf{a}_3$. The cone C_2 is one of the four *cones of diversification* $C_1, \ldots, C_4$ determined by this activity model. If your factor endowment is in the interior of one of the cones, you use the activities that generate it and pay the factor prices defined by tangent to the isorevenue curve.

In fact, the cone the endowment belongs to determines how and whether you diversify production between the different activities. In cone C_2 you diversify between $\mathbf{a}_1$ and $\mathbf{a}_2$ (with inputs $\mathbf{b}_1$ and $\mathbf{b}_2$), while in cone C_3 you diversify between $\mathbf{a}_2$ and $\mathbf{a}_3$. Notice that if you are inside C_1, the price of factor 5 is zero. You throw away any excess and use only activity $\mathbf{a}_1$. Cone C_4 is similar to cone C_1, using only activity $\mathbf{a}_3$. If you are on one of the rays generated by the $\mathbf{b}_i$, you use only that activity and a range of factor prices are possible.

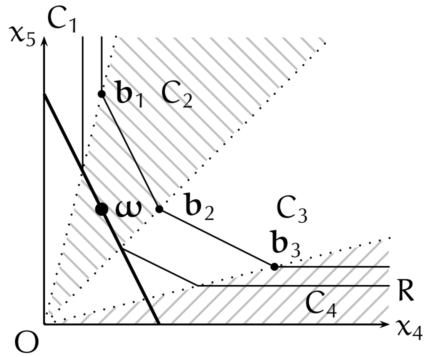

Figure 15.5.3: This diagram illustrates the cones of diversification, $C_1, \ldots, C_4$. The endowment $\boldsymbol{\omega}$ is in C_2, where the relative factor prices are determined by the slope of the dashed isocost line.

15.5.7 Output Prices and Endowments Determine Factor Prices

Now that we've seen how endowments determine factor prices, we can go even further. Let's hold output prices fixed and vary the endowment so that it remains within the same cone of diversification. Then factor prices remain unchanged.

This has implications for the theory of international trade. Consider a world of small open economies where final goods are freely traded and transport costs are zero. In such a world output prices must be the same in every country. Now suppose there are barriers to trade in factors, but all countries use the same technology. If all countries have factor endowments in the interior of the same cone of diversification, the factor prices must be the same in all countries![13] However, any country with a factor endowment in a different cone must have **different** factor prices.

[13] Samuelson (1958) and Diewert and Woodland (1977) discuss factor price equalization. Textbook presentations can be found in Dixit and Norman (1980), Feenstra (2016), or Woodland (1982).

15.5.8 Output Prices and Cones of Diversification

The cones of diversification depend on output prices, but generally do so in a continuous fashion. If your endowment is in the interior of a cone of diversification, small changes in output prices will keep you in the same cone of diversification. Such changes in output prices may change the level of factor prices, but they will not affect relative factor prices.

Sometimes changes in output prices can cause discontinuous changes in the cones of diversification. This happens when changes in output prices can make the production of some goods relatively unprofitable. Such products will not be produced in equilibrium, regardless of factor prices. In that case, the corresponding ray no longer determines the edge of a cone, and the cones can coalesce. This happens in our example if $p = (2, 1, 1)$. By running activities $\mathbf{a}_1$ and $\mathbf{a}_3$ at intensity $2/5$, we receive $6/5$ dollars while using two units of each factor. This is clearly better than running activity $\mathbf{a}_2$ at unit intensity, which produces only 1 dollar of revenue from the same input.

Similarly, cones of diversification can split in two when output prices vary (just reverse the order of the price change).

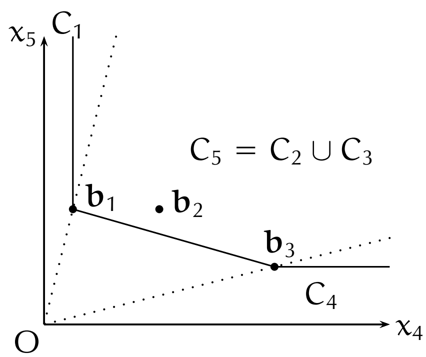

Figure 15.5.4: Compared with Figure 15.5.3, doubling the price of good one moves $\mathbf{b}_1$ down. This makes it more profitable to not produce good two. While cones C_1 and C_4 are unchanged, C_2 and C_3 are merged into a single cone of diversification C_5 using activities 1 and 3. Activity 2 is not used.

15.5.9 Factor Price Equalization with Smooth Cost Functions

Cones of diversification can also appear when factor proportions are variable. In this case diversification is required for factor prices to be unique. Factor prices are not unique if we are only producing one product, but may vary with the endowment.

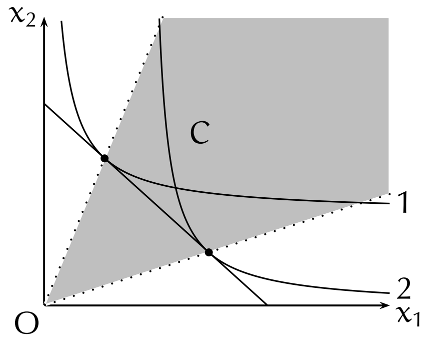

Figure 15.5.5: This Lerner diagram has been constructed from two variable proportions technologies, each with a smooth unit revenue curve. The line tangent to both of them determines the cone of diversification C. If the endowment is in C, the relative factor prices are given by the slope of the tangent line. If the endowment is above C, only technology one is used, while if the endowment is below C, only technology two is used.

Here's another take on the same type of problem.

Example 15.5.6: Suppose endowments consist only of inputs to production and that each firm produces only a single consumption good. Let $\boldsymbol{\omega}$ denote the vector of factor inputs. Suppose further that the number of firms equals the number of consumption goods ($F = L$). We label firms according to the consumption good they produce. Production possibilities are described by a CRS production function. Let $\boldsymbol{w}$ be the vector of factor prices and p_j the price of output for firm j. If there is an equilibrium where all firms produce, $p_j = b_j(\boldsymbol{w})$ for each firm j, where b_j is firm j's unit cost function.

For given output prices $\mathbf{p}$, it will typically be the case that there is at most one $\boldsymbol{w}$ where all firms produce. All goods will have to be produced in equilibrium in cases where for every good, some consumer's marginal utility is infinite at zero consumption of that good (e.g. Cobb-Douglas utility).

By Shephard's Lemma, firm j's input demand is $\boldsymbol{z}^j = q_j D_{\boldsymbol{w}} b_j$. Notice that the factor demand $\boldsymbol{z}^j = q_j D_{\boldsymbol{w}} b_j$ is determined solely by output prices. Since the input market must clear, the output levels satisfy $\sum_j q_j \boldsymbol{z}^j = \boldsymbol{\omega}$. As long as $\boldsymbol{\omega}$ is in the cone generated by the $\boldsymbol{z}^j$, we can find a set of outputs corresponding to the output prices. ◄

15.5.10 Stolper-Samuelson and Rybczynski Effects

An alternative approach to the study of equilibrium is to use duality and work in factor price space rather than factor quantity space. For this we use the cost function, and confine our attention to the smooth case.

We will consider two major results relating factor intensity to changes in prices and quantities. The first result, due to Stolper and Samuelson, examines the effect of a change in product prices on factor prices. In the two-good, two-factor case, they found that that if factor 1 is more intensely used in production of good 1, an increase in the price of good 1 increases the price of factor 1, but decreases the price of factor 2. The second result, of Rybczynski (1955), shows that under the same intensity conditions, an increase in the endowment of factor 1 increases output of good 1 and decreases output of good 2.

15.5.11 The Basic Model

Let $c_j(w, y^j)$ denote the cost function for industry j. Due to constant returns, we may write this as $c_j(w, y^j) = y^j b_j(w)$ where b_j denotes the unit cost function for industry j, the cost of producing one unit of output. Shephard's Lemma tells us that factor demands by industry j for factor i at output level y^j are $x_j^i(w) = y^j\, \partial b_j/\partial w_i$.

Consider $\mathbf{b}(w)$ as a row vector of cost functions, so that

$$\sum_i y^j \frac{\partial b_j}{\partial w_i} = \left[D_w \mathbf{b}\right] \mathbf{y}$$

is the column vector whose i^{th}row is the factor demand for good i when the (column) vector y is produced.[14]

We focus on the case where all goods are produced in equilibrium, and all factors are scarce. Equilibrium in the factor market requires that factor supply ($\boldsymbol{\omega}$) equal factor demand, and that price equal cost. The equilibrium conditions are

$$\begin{aligned} \boldsymbol{\omega} &= \left[D_w \mathbf{b}\right] \mathbf{y}, \\ \mathbf{p} &= \mathbf{b}(w). \end{aligned}$$

Because the columns of $D_w \mathbf{b}$ represent factor demands by each industry at unit output and each $y^j \geq 0$, endowments must lie in the cone generated by the factor demands of the various industries.

[14] Be careful here! If b and w were (column) vectors, $\left[D_w \mathbf{b}\right]_{ij} = \partial b_j/\partial w_i$ would be the component of $D_w \mathbf{b}$ in the i^{th}row and j^{th}column. Since they are actually rows (covectors), $\left[D_w \mathbf{b}\right]_{ij} = \partial b_i/\partial w_j$, which is the transpose of the column vector case.

15.5.12 Stolper-Samuelson Effect I

Now consider the effect of a change in product price on factor prices. Differentiating $\mathbf{b}(\boldsymbol{w}) = \mathbf{p}$ with respect to $\mathbf{p}$, we obtain $D_{\boldsymbol{w}}\mathbf{b} \times D_{\mathbf{p}}\boldsymbol{w} = I$. When $D_{\boldsymbol{w}}\mathbf{b}$ is invertible, $D_{\mathbf{p}}\boldsymbol{w} = [D_{\boldsymbol{w}}\mathbf{b}]^{-1}$. Let's specialize to the case of two goods and two factors. Then

$$D_{\boldsymbol{w}}\mathbf{b} = \begin{pmatrix} \partial b_1/\partial w_1 & \partial b_2/\partial w_1 \\ \partial b_1/\partial w_2 & \partial b_2/\partial w_2 \end{pmatrix}$$

and

$$\left[D_{\boldsymbol{w}}\mathbf{b}\right]^{-1} = \frac{1}{\Delta}\begin{pmatrix} \partial b_2/\partial w_2 & -\partial b_2/\partial w_1 \\ -\partial b_1/\partial w_2 & \partial b_1/\partial w_1 \end{pmatrix}$$

where $\Delta = (\partial b_1/\partial w_1)(\partial b_2/\partial w_2) - (\partial b_1/\partial w_2)(\partial b_2/\partial w_1)$.[15] Let b_i^j denote $\partial b_j/\partial w_i = [D_{\boldsymbol{w}}\mathbf{b}]_{ij}$. In this notation,

$$D_{\mathbf{p}}\boldsymbol{w} = (D_{\boldsymbol{w}}\mathbf{b})^{-1} = \frac{1}{\Delta}\begin{pmatrix} b_2^2 & -b_1^2 \\ -b_2^1 & b_1^1 \end{pmatrix}$$

where $\Delta = b_1^1 b_2^2 - b_2^1 b_1^2$.

[15] Note the form $D_{\boldsymbol{w}}\mathbf{b}$ takes because $\mathbf{b}$ is a row vector.

15.5.13 Stolper-Samuelson Effect II

Suppose factor one is always used more intensively in industry one. In other words, $x_1^1/x_1^2 > x_2^1/x_2^2$. Since $x_i^j = y_i b_i^j$, this means $b_1^1/b_1^2 > b_2^1/b_2^2$ and it follows that $\Delta = b_1^1 b_2^2 - b_2^1 b_1^2 > 0$. Then $b_1^1 b_2^2 > \Delta > 0$. In that case, $p_1 = b_1^1 w_1 + b_1^2 w_2 > b_1^1 w_1$, so the elasticity of the factor price w_1 with respect to output price p_1 is

$$\frac{p_1}{w_1}\frac{\partial w_1}{\partial p_1} > b_1^1 \frac{b_2^2}{\Delta} > 1.$$

This shows the Stolper-Samuelson result that the price of factor one rises by a larger percentage than the price of good one when factor one is always used more intensively in industry one. Moreover, the price of factor two falls. If the factor intensity reverses, the Stolper-Samuelson result does not apply. Figure 15.5.7 shows such a case.

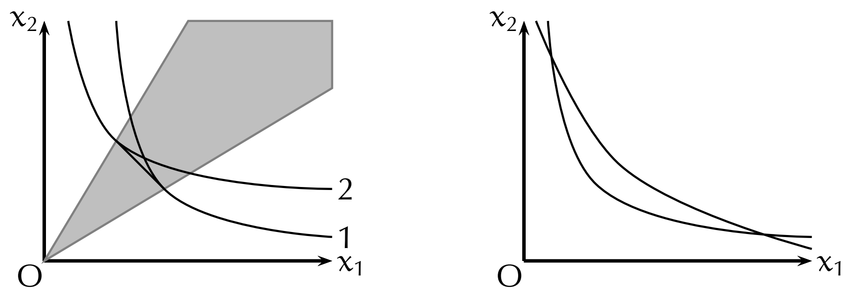

Figure 15.5.7: On the left, factor one is always used more intensively in the production of good one. We can see this by comparing the slopes of the isocost curves along each ray through the origin. Because factor demands are normal to the isocosts, the relative demand for good one is always higher in industry one.

The right panel shows a case involving a factor intensity reversal. The Stolper-Samuelson result does not apply in such a case.

15.5.14 Rybczynski Effect

There is a dual result, due to Rybczynski, concerning the effects of changes in endowments on outputs. Here we start with the other equilibrium equation, $\big[D_{\boldsymbol{w}}\mathbf{b}\big]\boldsymbol{y} = \boldsymbol{\omega}$ and differentiate with respect to $\boldsymbol{\omega}$, obtaining

$$D_{\boldsymbol{w}}\mathbf{b} \times D_{\boldsymbol{\omega}}\boldsymbol{y} + D_{\boldsymbol{w}}(D_{\boldsymbol{w}}\mathbf{b})\boldsymbol{y} \times D_{\boldsymbol{\omega}}\boldsymbol{w} = \mathbf{I}.$$

Provided we stay in the same cone of diversification, $D_{\boldsymbol{\omega}}\boldsymbol{w} = 0$. Thus $D_{\boldsymbol{\omega}}\boldsymbol{y} = [D_{\boldsymbol{w}}\mathbf{b}]^{-1}$. Since $b_1^1y^1 + b_2^1y^2 = \omega^1$, we conclude $b_1^1 < \omega^1/y^1$. Similarly,

$$\frac{\omega^1}{y^1}\frac{\partial y^1}{\partial \omega^1} > \frac{b_1^1 b_2^2}{\Delta} > 1,$$

so the output of good one rises by a larger percentage than the endowment of factor one (Rybczynski). The output of good two falls.

February 23, 2023