16. Existence of Equilibrium

Feb. 16, 2023

The centerpiece of general equilibrium theory is the demonstration that economies have Walrasian equilibria. If economies do not have equilibria, there's no point to studying equilibrium. If small changes in the specification of an economy can eliminate all the equilibria, it will make equilibrium models practically useless as we could always be very slightly in error concerning the model specification and end up with a model without equilibria.

It is by no means obvious that an equilibrium must exist. It's easy enough to find equilibrium prices in a single market, but changing the price in one market changes supply and demand in all of them. This forces other markets out of equilibrium, and their adjustment to a new equilibrium may in turn force the original market back out of equilibrium.

16.0.1 Chapter Outline

The first section discusses the history of the existence problem. We demonstrate existence of equilibrium in exchange economies in section two. To do this, we construct a tâtonnement map that adjusts prices if they are out of equilibrium. We then find a fixed point via the Brouwer Fixed Point Theorem, and show the fixed point is an equilibrium.

Section three examines the role of convexity in the existence proof. An example shows how continuum economies can convexify the economy and yield an equilibrium even when preferences are not convex. The concept of irreducibility and its relation to survival is examined in section four.

We prove the existence of equilibrium in constant returns to scale production economies in section five. This involves a more complex tâtonnement map and the Kakutani Fixed Point Theorem. Section six examines existence under weaker survival and preference assumptions. Finally, section seven shows that any economy with diminishing returns production may be converted to a constant returns model by introducing "entrepreneurial factors" à la McKenzie. This allows us to apply our equilibrium existence theorems to production economies under diminishing returns to scale.

16.1 The Existence Question

Walras already raised the existence question in the 1870s, when he formalized the concept of general equilibrium in the 1874 (first) edition of *Éléments d'économie politique pure, ou théorie de la richesse sociale*, the Elements of Pure Economics. Walras observed that the number of equations matched the number of unknowns. This made it at least plausible that there might be an equilibrium. Of course, this argument is inadequate. Walras seemed to recognize that, and gave a further description of how an equilibrium might be found via his tâtonnement process.[1]

The tâtonnement was inspired by the operation of the Paris Bourse (stock exchange), which operated as an open outcry exchange between the *agents de change*, who took orders from buyers and sellers. Walras imagined a single auctioneer calling out prices, and then taking orders from buyers and sellers. If markets cleared, the transactions would be made, otherwise, the Walrasian auctioneer would cry out a new set of prices. The auctioneer would raise prices for goods in excess demand, and lower them for goods in excess supply. Walras suggested that this process would lead to a set of equilibrium prices.

There is a literature on attempts to model price adjustment in this fashion, with differential equations playing the role of auctioneer.[2] For this to converge to an equilibrium price vector, the system must be stable. Scarf (1960) demonstrated that some economies were globally unstable under such a price adjustment process. Unless you started with the equilibrium prices in such an economy, you would never approach equilibrium.

[1] Tâtonnement means "trial and error".

[2] See Arrow and Hahn (1971) for a summary.

16.1.1 Fixed Point Theorems and Existence of Equilibrium

Fortunately, convergence of a price adjustment process was not needed to prove that an equilibrium exists. During the 1930's, the existence problem was investigated by Karl Schlesinger, Abraham Wald, and John von Neumann.

This research culminated in von Neumann's use of a fixed point theorem to show the existence of equilibrium in a model of economic growth (von Neuman, 1937) and Wald's unpublished (and presumed lost) paper using a fixed point argument to show existence of equilibrium in a Walras-Cassel model.[3]

Along the way, two previously unrecognized problems had been noticed. By the 1920's, Cassel's (1918) presentation of Walras' model had become popular. Remak (1929) observed that it was not enough to solve Cassel's equations for a Walrasian equilibrium. Prices must also be nonnegative. Schlesinger (1935) and Zeuthen (1933) then pointed out that the supply of resources could exceed demand. This would prevent any possibility of equilibrium requiring equality of supply and demand. Schlesinger's solution was to require that demand be no larger than supply in equilibrium—that there be no excess demand.

The first fully general proofs of the existence of equilibrium were published simultaneously in *Econometrica* by Arrow and Debreu (1954) and McKenzie (1954). Like von Neumann and Wald, these existence proofs relied on a fixed point argument to find an equilibrium, rather than trying to mimic the Walrasian tâtonnement process. However, the fixed point argument merely sidesteps the convergence process. It still includes something similar to Walras' tâtonnement when defining the map that determines the fixed point.[4] Similar arguments have also been used in game theory to show the existence of Nash Equilibrium.

[3] See Düppe and Weintraub (2016) for a discussion of Wald's lost proof.

[4] Methods such as fixed point theorems were not available to Walras. The Brouwer Fixed Point Theorem dates to 1910, the year Walras died. It was not published until 1911 (Brouwer, 1911). Most other fixed point theorems came later. The Kakutani Fixed Point Theorem was published in 1941 (Kakutani, 1941). The Eilenberg and Montgomery theorem used by Debreu (1952b) and so by Arrow and Debreu (1954) dates to 1946 (Eilenberg and Montgomery, 1946) while the KKM Lemma used by Gale was proved in 1929 (Knaster, Kuratowski, and Mazurkiewicz, 1929). See Weintraub (1985) and Düppe and Weintraub (2014) for a history of general equilibrium.

16.2 Equilibrium Existence in Exchange Economies

An *exchange economy* is an economy without production (or one where the only production sets are $-\mathbb{R}^m_+$). The only goods supplied in an exchange economy are those present in consumer endowments. The excess demand becomes

$$z(\mathbf{p}) = \sum_i \mathbf{x}^i(\mathbf{p}, \mathbf{p}\cdot\boldsymbol{\omega}^i) - \sum_i \boldsymbol{\omega}^i.$$

As always, it is the difference between demand and supply.

We have an equilibrium if $\mathbf{z}(\mathbf{p}) \le \mathbf{0}$. We will focus on the case where consumers are locally non-satiated, in which case Walras' Law tells us that $\mathbf{p}\cdot\mathbf{x}^i(\mathbf{p}, \mathbf{p}\cdot\boldsymbol{\omega}^i) = \mathbf{p}\cdot\boldsymbol{\omega}^i$. Summing over all consumers i, we obtain $\mathbf{p}\cdot\mathbf{z}(\mathbf{p}) = 0$. If $\mathbf{p}$ is a equilibrium price vector, $\mathbf{z}(\mathbf{p}) \le \mathbf{0}$. Since $\mathbf{p} \ge \mathbf{0}$, the only goods that can have positive prices are those where the market clears. Goods in excess supply ($z_\ell(\mathbf{p}) < 0$) must have $p_\ell = 0$.

16.2.1 Walras' Existence Proof

The basic structure of the existence proof was laid out by Walras. He suggested the tâtonnement process could be used to obtain an equilibrium. In the tâtonnement process, we start with a price vector. If a good is in excess demand, we raise the price. If there is excess supply, and the price is positive, we lower it. The equilibrium price vector will be a fixed point of this adjustment process. At such prices, supply equals demand for all goods that have positive prices, and supply (the endowment) is greater than or equal to demand for goods with price zero.

16.2.2 Some Preliminaries

We must take care of a couple of preliminaries before constructing the tâtonnement map. First, we must choose a normalization for prices. Here modern practice breaks with Walras. Rather than using a numéraire, we normalize prices so $\mathbf{p}\cdot\boldsymbol{e} = 1$. Recall the price simplex is the set $\Delta = \{\mathbf{p} \geq \mathbf{0} : \mathbf{p}\cdot\boldsymbol{e} = 1\}$. Prices are restricted to the price simplex.

Second, we require the Brouwer Fixed Point Theorem, which was not available until 37 years after Walras' original existence argument.

Brouwer Fixed Point Theorem. *Let* $\varphi : D \to D$ *be a continuous function where* $D \subset \mathbb{R}^m$ *is a compact and convex set. There is a* $\mathbf{p}^* \in D$ *with* $\varphi(\mathbf{p}^*) = \mathbf{p}^*$.

We will not attempt to prove this theorem here. The original version is Brouwer (1911). A nice proof of the Brouwer Theorem based on approximation by differentiable functions may be found in Milnor (1965). An alternative proof based on simplicial subdivisions (Sperner's Lemma) is in Border (1985). A proof can also be found in Smart (1974). Another option is to use the Knaster-Kuratowski-Mazurkiewicz Lemma (KKM Lemma) or its generalization by Shapley (the KKMS Lemma).[5]

[5] A proof may be found in Border (1985). The original papers are Knaster, Kuratowski, and Mazurkiewicz (1929) and Shapley (1973).

16.2.3 Equilibrium Existence in an Exchange Economy

The possibility that some prices may be zero causes problems for the existence argument. There are two problems here. Some consumer's income may drop to zero. If prices were positive, that wouldn't create a problem. But when one of the prices is zero, it can cause a discontinuity in demand and lead to non-existence of equilibrium (see Example 16.4.3).

The other problem is that demand might not be well-defined on the price simplex. This happens because the budget set is not compact if any price is zero. Walras finessed both of these problems by positing satiation points for consumers. If the price of a good is zero, they just consume the satiation level. This makes his demands continuous for all non-negative prices.

We will handle this boundary problem in a somewhat different fashion. We require that all consumers have strictly positive endowments. This avoids the zero income problem.

The existence proof then proceeds in four main parts. First, we truncate the budget set so that it will be compact even when some prices are zero. We use the truncated budget set to construct the continuous "truncated" demand functions of Lemma 16.2.2. We choose the truncation level high enough that it cannot possibly affect any equilibrium. In part two of the proof, we use these continuous "truncated" demands to construct the tâtonnement map that adjusts prices. The Brouwer Theorem then yields a fixed point. The third step is to show that we have in equilibrium in terms of the "truncated" demands. We finish the proof by showing that the equilibrium for the "truncated" demands is also an equilibrium for the true demands.

16.2.4 Budget Correspondences and Continuity of Demand

To show continuity of demand, we need Berge's Maximum Theorem from section 33.6. We use that to show that the maximers of a continuous utility function are continuous in prices and income. Keeping in mind that budget sets are correspondences, set-valued functions of prices and incomes, we need some sort of continuity for correspondences to make this work.

We do this by dividing continuity into two parts: upper and lower continuity by analogy with upper and lower semicontinuity of functions. For functions, semicontinuity rules out sudden jumps, either upward or downward. For correspondences, semicontiunity rules out the sudden growth of the value-sets (explosions) and the sudden contraction of the value-sets (collapses).

In other words, if we let $\mathbf{p}^n \to \mathbf{p}$, if the budget set is at least some $\varepsilon > 0$ larger than at every $\mathbf{p}^n$, we have an explosion, if it is at least some $\varepsilon > 0$ smaller we have a collapse. If computed using a metric, as the ε above suggests, we speak of Hausdorff upper and lower continuity. Together, they make Hausdorff continuity.[6] If this is translated into topological terms first, we speak of upper and lower Vietoris continuity, which combine into Vietoris continuity.[7]

[6] Felix Hausdorff (1868–1942) was a German mathematician. Among other things, he was one of the main founders of modern set topology. He was Jewish and was ordered to a concentration camp in 1942. He committed suicide rather than allowing himself to be degraded and murdered by the Nazis.

[7] Leopold Vietoris (1891–2002) was a Austrian mathematican, born near the Slovenian border. He lived to be nearly 111, and is the oldest known Austrian man.

16.2.5 Equilibrium Existence in Exchange Economies

Equilibrium Existence Theorem: Exchange Economies. *Consider an exchange economy* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)$ *with* $\mathfrak{X}_i = \mathbb{R}^m_+$ *for each* i *and* $\boldsymbol{\omega}^i \gg \mathbf{0}$ *for all* $i = 1, \dots, I$. *Suppose each agent's preferences are continuous, monotonic, and strictly convex. Then a Walrasian equilibrium exists.*

Before proceeding to the proof, note that preferences are not only locally nonsatiated, but are also strongly monotonic due to the combination of strict convexity and monotonicity. It follows that if we have an equilibrium with excess supply, the same prices also yield an equilibrium with no excess supply (the consumers are no worse off if they eat the excess). Strict convexity will imply the demands are functions.

Now let $K > \max_\ell\{\omega_\ell\}$ and define the truncated budget set for i by $T_i(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geq \mathbf{0}, \mathbf{x} \leq K\boldsymbol{e}, \mathbf{p}\cdot\mathbf{x} \leq \mathbf{p}\cdot\boldsymbol{\omega}^i\}$. The truncated budget set is a closed convex subset of the ordinary budget set $B_i(\mathbf{p}, \mathbf{p}\cdot\boldsymbol{\omega}^i)$.

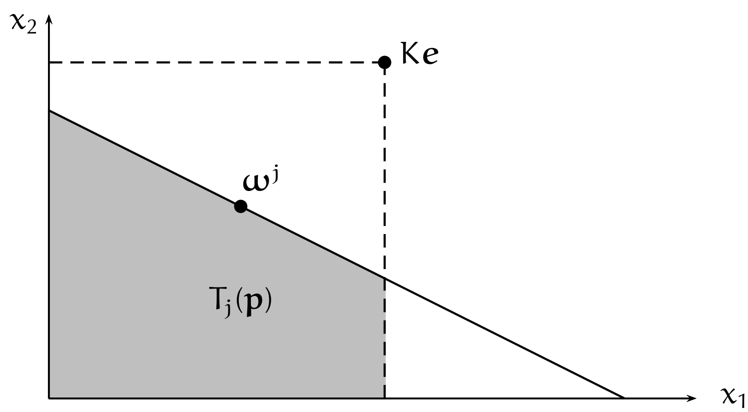

Figure 16.2.1: The budget line for $\mathbf{p}^*$ is shown on the diagram. The shaded area is the truncated budget set $T_j(\mathbf{p})$. The truncation will not affect the equilibrium because $K\boldsymbol{e} \gg \boldsymbol{\omega}$ and all equilibrium demand functions are bounded by $\boldsymbol{\omega}$.

16.2.6 Truncated Demand Lemma I

The following lemma proves the basic properties of the "truncated" demands.

Lemma 16.2.2. *Suppose the hypotheses of the Equilibrium Existence Theorem: Exchange Economies hold, that $\hat{\mathbf{x}}^i(\mathbf{p})$ maximizes utility over $T_i(\mathbf{p})$, and that $\hat{\mathbf{z}}^i(\mathbf{p}) = \hat{\mathbf{x}}^i(\mathbf{p}) - \boldsymbol{\omega}^i$. Then for all $\mathbf{p} \geq \mathbf{0}$:*

1. $\hat{\mathbf{z}}^i(\mathbf{p})$ *exists and is continuous in* $\mathbf{p}$.
2. $\hat{\mathbf{z}}^i(\mathbf{p})$ *is homogeneous of degree 0 in* $\mathbf{p}$.
3. $\mathbf{p} \cdot \hat{\mathbf{z}}^i(\mathbf{p}) = 0$ *(Walras' Law).*
4. $\hat{z}^i_\ell(\mathbf{p}) \geq -K$ *for each good* ℓ.

Proof of Lemma. Most of the proof is dedicated to showing that $\hat{\mathbf{z}}(\mathbf{p})$ is continuous in $\mathbf{p}$. The truncation ensures that the budget set is still bounded even if some goods have price zero. We can apply Lemma 33.2.3 to the function $\mathbf{p} \cdot (\boldsymbol{\omega} - \mathbf{x})$ to show that T_i is a closed correspondence. Since $T_i(\mathbf{p})$ is contained in a compact set, T_i is an upper Hausdorff continuous correspondence by Corollary 33.3.27, and is upper Vietoris continuous by Proposition 33.5.3.

The truncation itself would not be enough to make T_i a lower Vietoris continuous correspondence, as the possibility of a zero price could cause a sudden collapse of the budget set if some component of $\boldsymbol{\omega}^i$ were zero. The fact that $\boldsymbol{\omega}^i \gg \mathbf{0}$ rules out this possibility as $\mathbf{p} > \mathbf{0}$.

Proof continues...

16.2.7 Truncated Demand Lemma II

Rest of Proof. To see that T_i is lower semicontinuous, consider $\mathbf{x} \in T_i(\mathbf{p})$. If $\mathbf{p}\cdot\mathbf{x} < \mathbf{p}\cdot\boldsymbol{\omega}^i$, set $\mathbf{x}' = \mathbf{x}$. Then $\mathbf{p}\cdot\mathbf{x}' < \mathbf{p}\cdot\boldsymbol{\omega}^i$. Otherwise take $\varepsilon > 0$ with $x_\ell > \varepsilon$ for all ℓ with $x_\ell > 0$. Set $x'_\ell = x_\ell - \varepsilon/2$ when $x_\ell > 0$ and $x'_\ell = 0$ when $x_\ell = 0$. Note that since $\mathbf{p}\cdot\mathbf{x} = \mathbf{p}\cdot\boldsymbol{\omega}^i > 0$, there is a $p_k > 0$ with $x_k > 0$. It follows that $\mathbf{p}\cdot\mathbf{x}' < \mathbf{p}\cdot\boldsymbol{\omega}^i$.

Now that we have $\mathbf{x}' \le \mathbf{x}$ with $\mathbf{p}\cdot\mathbf{x}' < \mathbf{p}\cdot\boldsymbol{\omega}^i$, we can choose a $\delta > 0$ with $\mathbf{p}'\cdot\mathbf{x}' < \mathbf{p}'\cdot\boldsymbol{\omega}^i$ for $\mathbf{p}' \in B_\delta(\mathbf{p})$. Example 33.2.10 can be easily adapted to show that the interior of T_i is an open correspondence. Then Corollary 33.5.6 shows that T_i itself is lower Vietoris continuous.

Since T_i is both lower and upper Vietoris continuous, it is Vietoris continuous. The Maximum Theorem yields continuity of $\hat{\mathbf{z}}^i$.

Homogeneity follows as in the standard case. As usual, Walras' Law follows from local non-satiation. Notice that if the truncation binds, but the budget constraint does not, strong monotonicity shows that there is a better point where the budget constraint binds. Finally, since $\hat{\mathbf{x}}^i \ge \mathbf{0}$,

$$\hat{z}^i_\ell = \hat{x}^i_\ell - \omega^i_\ell \ge -\omega^i_\ell \ge -\omega_\ell > -K.$$

□

16.2.8 Proof of Equilibrium Existence I: Tâtonnement

Proof of Theorem. Let $\hat{z}(\mathbf{p}) = \sum_i \hat{z}^i(\mathbf{p})$ and define the tâtonnement map φ by

$$\varphi_\ell(\mathbf{p}) = \frac{p_\ell + \max\{0, \hat{z}_\ell(\mathbf{p})\}}{1 + \sum_k \max\{0, \hat{z}_k(\mathbf{p})\}} \geq 0.$$

This function is continuous since $\hat{z}$ is continuous. If $\mathbf{p} \in \Delta$,

$$\sum_\ell \varphi_\ell(\mathbf{p}) = \frac{\sum_\ell p_\ell + \sum_\ell \max\{0, \hat{z}_\ell(\mathbf{p})\}}{1 + \sum_k \max\{0, \hat{z}_k(\mathbf{p})\}} = 1$$

because $\sum_\ell p_\ell = 1$. Thus $\varphi\colon \Delta \to \Delta$. By the Brouwer Theorem, φ has a fixed point $\mathbf{p}^*$.

This is in the spirit of Walras' tâtonnement's effect on prices. Consider the new price ratio

$$\frac{\varphi_k(\mathbf{p})}{\varphi_\ell(\mathbf{p})} = \frac{p_k + \max\{0, \hat{z}_k(\mathbf{p})\}}{p_\ell + \max\{0, \hat{z}_\ell(\mathbf{p})\}}.$$

If good k is in in excess demand and good ℓ in excess supply, the relative price of k rises. If both goods are in excess demand, the effect on relative price depends on how much demand increases relative to price.

16.2.9 Proof of Equilibrium Existence II: Truncated Equilibrium

Since we have a fixed point, $p_\ell^* = \varphi_\ell(\mathbf{p}^*)$, meaning

$$p_\ell^* = \frac{p_\ell^* + \max\{0, \hat{z}_\ell(\mathbf{p}^*)\}}{1 + \sum_k \max\{0, \hat{z}_k(\mathbf{p}^*)\}}.$$

Clearing the denominator, we obtain

$$p_\ell^*\Big(1 + \sum_k \max\{0, \hat{z}_k(\mathbf{p}^*)\}\Big) = \Big(p_\ell^* + \max\{0, \hat{z}_\ell(\mathbf{p}^*)\}\Big).$$

Cancelling the first term (p_ℓ^*) yields

$$p_\ell^*\Big(\sum_k \max\{0, \hat{z}_k(\mathbf{p}^*)\}\Big) = \max\{0, \hat{z}_\ell(\mathbf{p}^*)\}$$

We proceed by contradiction. Suppose there is a good m with $\hat{z}_m(\mathbf{p}^*) > 0$. We can then divide by $\sum_k \max\{0, \hat{z}_k(\mathbf{p}^*)\} > 0$ to obtain

$$p_\ell^* = \frac{\max\{0, \hat{z}_\ell(\mathbf{p}^*)\}}{\sum_k \max\{0, \hat{z}_k(\mathbf{p}^*)\}}$$

for every ℓ. Thus $p_\ell^* = 0$ whenever $\hat{z}_\ell(\mathbf{p}^*) \leq 0$ and $p_\ell^* > 0$ when $\hat{z}_\ell(\mathbf{p}^*) > 0$ (in particular, for $\ell = m$). But then $\mathbf{p}^* \cdot \hat{\mathbf{z}}(\mathbf{p}^*) > 0$, violating Walras' Law. This contradiction implies $\hat{\mathbf{z}}(\mathbf{p}^*) \leq 0$.

16.2.10 Proof of Equilibrium Existence III: De-Truncation

Proof continues. The last step is to show that $\hat{\mathbf{z}}(\mathbf{p}^*) = \mathbf{z}(\mathbf{p}^*)$. Suppose there is a consumer j with $\hat{\mathbf{z}}^j(\mathbf{p}^*) \neq \mathbf{z}^j(\mathbf{p}^*)$. In other words, $\hat{\mathbf{x}}^j(\mathbf{p}^*) \neq \mathbf{x}^j(\mathbf{p}^*)$.

Because $\hat{\mathbf{z}}(\mathbf{p}^*) \leq \mathbf{0}$, we have $\sum_k \hat{\mathbf{x}}^k(\mathbf{p}^*) \leq \boldsymbol{\omega}$. As each $\hat{\mathbf{x}}^k$ is non-negative, each $\hat{\mathbf{x}}^k(\mathbf{p}^*) \leq \boldsymbol{\omega} \ll K\mathbf{e}$. In particular, $\hat{\mathbf{x}}^j(\mathbf{p}^*) \ll K\mathbf{e}$. This is illustrated in Figure 16.2.3.

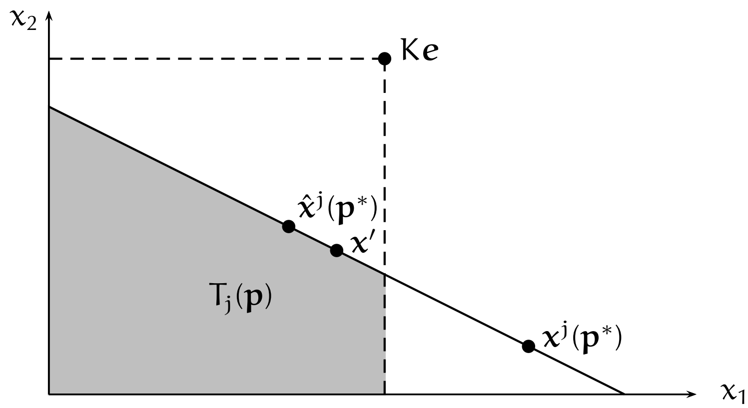

Figure 16.2.3: The budget line for $\mathbf{p}^*$ is shown on the diagram. The shaded area is the truncated budget set, and $\hat{\mathbf{x}}^j(\mathbf{p}^*)$ is j's truncated demand. If j's actual demand $\mathbf{x}^j(\mathbf{p}^*)$ differs from his truncated demand, it must be in the unshaded portion of the budget set. We take a convex combination of $\hat{\mathbf{x}}^j(\mathbf{p}^*)$ and $\mathbf{x}^j(\mathbf{p}^*)$ that is in the budget set (shown here at $\mathbf{x}'$). The strict convexity of preferences combined with the fact that $\mathbf{x}^j(\mathbf{p}^*) \succ_j \hat{\mathbf{x}}^j(\mathbf{p}^*)$ implies $\mathbf{x}' \succ_j \hat{\mathbf{x}}^j(\mathbf{p}^*)$, contradicting the fact that $\hat{\mathbf{x}}^j(\mathbf{p}^*)$ is optimal in $T_j(\mathbf{p}^*)$.

Since $T_j(\mathbf{p}^*) \subset B_j(\mathbf{p}^*, \mathbf{p}^* \cdot \omega^j)$, $\mathbf{x}^j(\mathbf{p}^*) \succ_i \hat{\mathbf{x}}^j(\mathbf{p}^*)$. Take $\varepsilon > 0$ small enough that $\mathbf{x}' = (1-\varepsilon)\hat{\mathbf{x}}^j(\mathbf{p}^*) + \varepsilon \mathbf{x}^j(\mathbf{p}^*)$ is in the truncated budget set $T_j(\mathbf{p}^*)$. This is possible because $\hat{\mathbf{x}}^j(\mathbf{p}^*) \ll K\mathbf{e}$.

Due to strict convexity and the fact that $\mathbf{x}^j(\mathbf{p}^*) \succ_j \hat{\mathbf{x}}^j(\mathbf{p}^*)$, $\mathbf{x}'$ is also better than $\hat{\mathbf{x}}^j(\mathbf{p}^*)$. This is impossible since $\hat{\mathbf{z}}(\mathbf{p}^*)$ is optimal in the truncated budget set, contradicting $\hat{\mathbf{z}}(\mathbf{p}^*) \neq \mathbf{z}(\mathbf{p}^*)$. Therefore $\mathbf{z}(\mathbf{p}^*) = \hat{\mathbf{z}}(\mathbf{p}^*) \leq 0$, showing that $\mathbf{p}^*$ is an equilibrium price vector. □

16.2.11 Generalization of Existence Theorem

The assumption of strict convexity made the proof technically simpler. It ensured that demand was a function and allowed us to use the Brouwer Fixed Point Theorem. In fact, the result (and more) is also true for convex preferences (quasiconcave utility). Proving it leads to some technical complications due to the fact that demand may not a function.

One way to deal with this is to make it a function. Given a convex demand correspondence $\mathbf{x}(\mathbf{p})$, as would be generated by continuous convex preferences, we can find a continuous function $\phi(\mathbf{p}) \in \mathbf{x}(\mathbf{p})$. Such a function is called a *continuous selection* from $\mathbf{x}$, and its existence follows from the fact that $\mathbf{x}(\mathbf{p})$ is a convex set for every $\mathbf{p}$.[8] Using ϕ in place of the demand function, we simply follow the same steps as before to obtain an equilibrium.

Another method, which we follow for the production case, is to abandon the Brouwer Fixed Point Theorem for something stronger, the Kakutani Fixed Point Theorem. Then we can work directly with demand correspondences, and also allow the possibility of zero income. We will employ this method when showing equilibrium existence in production economies, but the theorem applies equally to exchange economies if we set the production set equal to the negative orthant.

[8] This is the Michael Selection Theorem of section 33.7. See also Klein and Thompson (1984, sec. 8.1). Gale and Mas-Colell (1975) used it to show the existence of equilibrium with non-ordered preferences.

16.3 The Role of Convexity

When preferences are not convex, there may not be an equilibrium.[9] The easiest way to see this is by example.

Example 16.3.1: Elliptical Utility: No Equilibrium Suppose there are two consumers with endowments $(1,2)$ and elliptical utility $u_i(\mathbf{x}^i) = (x_1^i)^2 + 4(x_2^i)^2$. The aggregate endowment is then $(2,4)$ Normalize prices so $p_1 + p_2 = 1$ and let $p = p_1$. Each consumer has income $(2-p)$. The consumers choose corner solutions if $[(2-p)/p]^2 \neq 4[(2-p)/(1-p)]^2$, which translates to $p \neq 1/3$. Individual demand is

$$\mathbf{x}(p) = \begin{cases} \left(\left(\frac{2-p}{p}\right), 0\right) & \text{if } p < 1/3 \\ \left(\left(\frac{2-p}{p}\right), 0\right) \text{ or } \left(0, \left(\frac{2-p}{1-p}\right)\right) & \text{if } p = 1/3 \\ \left(0, \left(\frac{2-p}{1-p}\right)\right) & \text{if } p > 1/3. \end{cases}$$

If $p < 1/3$, only good 1 is purchased. Market demand for it is $2(2-p)/p$. Since supply of good 1 is 2, market clearing for good 1 requires $p = 1$, which is inconsistent with $p < 1/3$. If $p > 1/3$, only good 2 is purchased. Market demand is $2(2-p)/(1-p) = 4$, so $p = 0$, which is also impossible.

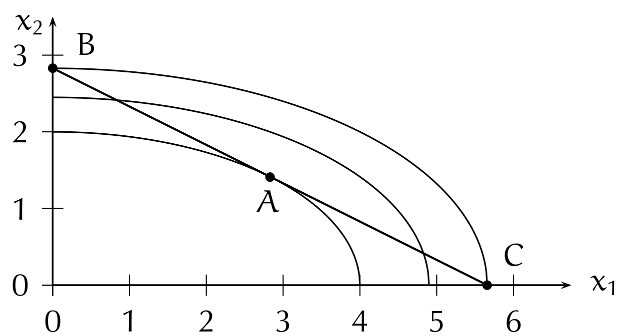

Figure 16.3.2: A sample budget line of slope $-\frac{1}{2}$ is shown on the diagram. Notice that the tangency at A ctually minimizes utility as higher indifference curves cross the budget line to both left and right of the tangency. In this case, where the budget line differs a bit from the problem, both endpoints of the budget line (B, C) maximize utility. In the problem, this happens at slope $-/\mathtt{frac}13$.

[9] The selection argument mentioned above fails as there may not be a continuous selection from the demand correspondence.

16.3.1 Elliptical Utility Continued

It follows that $p = 1/3$ is the only possible equilibrium price. Individual demand is $(5, 0)$ or $(0, 5/2)$, so market demand must be one of $(10, 0)$, $(5, 5/2)$, or $(0, 5)$. Since the aggregate endowment is $(2, 4)$, the first two result in excess demand for good 1 while the third has excess demand for good 2. It follows that there can be no equilibrium for this economy.

The lack of equilibrium here is not a fluke. With this utility and identical endowments, equilibrium requires that the aggregate endowment be a multiple of $(1, 2)$. If we allow endowments to differ, the prices are still $(1/3, 2/3)$ and the aggregate endowment must be either $(3w_1, 6w_2)$ or $(3w_2, 6w_1)$ where w_i are the incomes in order to obtain an equilibrium. In the other cases with strictly positive aggregate endowment, equilibrium does not exist. ◀

16.3.2 Continuum Economies

If we could continuously vary the proportions of consumers demanding $(5, 0)$ and $(0, 5/2)$, we would not have problems finding an equilibrium. It is possible to do that by taking competition one step farther. Instead of having a large finite number of consumers, our economy will have infinitely many consumers.

We will consider economies that not only have infinitely many consumers, but have a continuum of consumers. Each consumer can be labeled by a point in the interval $[0, 1]$. One way to describe such an economy is a mapping from space of consumers, $[0, 1]$, to the space of consumer characteristics. Since we are presuming all consumers have $\mathbb{R}^m_+$ as consumption set, the characteristics are the preferences (or utility) and endowment. The simplest case is to work with utility. Then consider a mapping Φ defined by $\Phi(s) = \big(u(s, \cdot), \boldsymbol{\omega}(s)\big)$ where $u(s, \cdot)$ is a continuous utility function defined on $\mathbb{R}^m_+$ and $\boldsymbol{\omega}(s) \in \mathbb{R}^m_+$ is the endowment. The mapping Φ describes a continuum economy.[10]

[10] For more on continuum economies, see Aumann (1964) and Hildenbrand (1974).

16.3.3 Demand, Supply, and Equilibrium in Continuum Economies

Given a price vector $\mathbf{p} \gg 0$, we can consider consumer s's demand. When preferences are strictly convex, this is a continuous bounded function $\mathbf{x}(s, \mathbf{p})$, so there is no problem integrating (adding) it to get market demand

$$\mathbf{x}(\mathbf{p}) = \int_0^1 \mathbf{x}(s, \mathbf{p})\, ds.$$

If preferences are not strictly convex, there may be multiple demand points. In that case we pick a function $\mathbf{x}(s, \mathbf{p})$ so that $\mathbf{x}(s, \mathbf{p})$ solves the consumer's problem for each s.[11] Since this is an exchange economy, the only supply is from endowments. Thus market supply is

$$\boldsymbol{\omega} = \int_0^1 \boldsymbol{\omega}(s)\, ds.$$

Equilibrium is defined by a combination of consumer utility maximization and market clearing. In this case, market clearing means that aggregate demand equals aggregate supply, $\mathbf{x}(\mathbf{p}) = \boldsymbol{\omega}$ or

$$\mathbf{x}(\mathbf{p}) = \int_0^1 \mathbf{x}(s, \mathbf{p})\, ds = \int_0^1 \boldsymbol{\omega}(s)\, ds.$$

[11] This is a selection from the demand correspondence. Showing it is always integrable involves some technical complications.

16.3.4 Elliptic Utility in a Continuum Economy

Example 16.3.3: Continuum Economy: Elliptical Utility Suppose consumers have endowments $\boldsymbol{\omega}(s) = (1, 2)$ and elliptical utility $u(s, \mathbf{x}) = (x_1)^2 + 4(x_2)^2$.

Demand is then

$$\mathbf{x}(s, \mathbf{p}) = \begin{cases} \left(\left(\frac{2-p}{p}\right), 0\right) & \text{if } p < 1/3 \\ \left(\left(\frac{2-p}{p}\right), 0\right) \text{ or } \left(0, \left(\frac{2-p}{1-p}\right)\right) & \text{if } p = 1/3 \\ \left(0, \left(\frac{2-p}{1-p}\right)\right) & \text{if } p > 1/3. \end{cases}$$

As before, since both goods are valuable, the only possible equilibrium price is $p = 1/3$. At that price, individual consumer demand is either $(5, 0)$ or $(0, 5/2)$.

One difference between the continuum economy and the finite agent economy is that we can arrange to have demand be any convex combination of $(5, 0)$ and $(0, 5/2)$ by adjusting the proportion of consumers choosing one or the other.

For the equilibrium, we need 4 times as many choosing $(0, 5/2)$ as choose $(5, 0)$. In other words, $1/5$ of the consumers pick $(5, 0)$, the other $4/5$ choose $(0, 5/2)$. To that end, consider the demand function defined by

$$\mathbf{x}(s, 1/3) = \begin{cases} (5, 0) & \text{for } 0 \le s \le 1/5 \\ (0, 5/2) & \text{for } 1/5 < s \le 1. \end{cases}$$

Then $\int_0^1 \mathbf{x}(s)\, ds = (1, 2) = \int_0^1 (1, 2)\, ds$, yielding an equilibrium. ◀

16.3.5 Remarks on Continuum Example

In fact, there is also an equilibrium if the number of consumers is a multiple of 5. The number 5 arises when each have endowment $(1, 2)$. If the endowment were different, the required number of consumers would be different. For example, if the endowment were $(1, 1)$, in equilibrium, we would need twice as many consumers choosing $(0, 5/2)$ as choose $(5, 0)$, so that demand for goods one and two would be equal. That means that the number of consumers must be a multiple of 3.

Since individual endowments are $(1, 2)$ and even a single agent choosing $(5, 0)$ demands 5 units of good 1. If there were 5 units of good one, there would be twice that, 10 units of good two available. Since consumers that demand good two only use $5/2$ units, we would need $10/(5/2) = 4$ consumers to use up supply. The total number of consumers must also be 5. In general, it must be a multiple of 5. Thus there would be an equilibrium if the number of consumers where a multiple of 15. A continuum economy where consumer have these preferences will have an equilibrium regardless of the average endowment.

16.4 Zero Income, Survival, and Irreducibility

Remainder of Chapter Skipped

The possibility of zero income causes problems for the existence of equilibrium. The problem is not just that the proof doesn't work. There may not be an equilibrium at all. It can also call into question whether the equilibrium is sensible. Can consumers survive and supply labor services if they have no income? We sidestepped these issues in the Equilibrium Existence Theorem by assuming that individual endowments were strictly positive, guaranteeing positive income, and by taking the consumption set to be the positive orthant. The later means, rather unrealistically, that survival is possible with zero consumption. If the consumption set incorporates a survival constraint, we might require the explicit survival assumption that $\boldsymbol{\omega}^i \in \mathfrak{X}_i$. If this is violated there may not be an equilibrium.

16.4.1 Death in Equilibrium

Example 16.4.1: Death in Equilibrium Consider an exchange economy with two goods and three consumers. Each consumer has equal-weighted Cobb-Douglas utility $u_i(\mathbf{x}^i) = (x_1^i)^{1/2}(x_2^i)^{1/2}$ and consumption set $\mathfrak{X}_i = \{\mathbf{x} \in \mathbb{R}^2_+ : x_1 + x_2 \geq 2\}$. Here at least two units of consumption goods of any type must be consumed in order to survive. The endowments are $\boldsymbol{\omega}^1 = \boldsymbol{\omega}^2 = (2, 2)$ and $\boldsymbol{\omega}^3 = (0, 1)$, yielding total endowment $(4, 5)$. Then $\omega_1 + \omega_2 = 9$, meaning that there are sufficient consumption goods available for everyone to survive. However, that is not what happens.

For the moment we ignore the survival constraint. In that case the equilibrium prices must obey $p_1/p_2 = 5/4$. We set $p_2 = 1$ and $p_1 = 5/4$. Demand is then $\mathbf{x}^1 = \mathbf{x}^2 = (9/5, 9/4)$ and $\mathbf{x}^3 = (2/5, 1/2)$. However, this cannot be the actual equilibrium as consumer three **does not survive** because $\mathbf{x}^3 \notin \mathfrak{X}_3$.

If we recalculate demand by consumer three when prices are $\mathbf{p} = (p, 1)$, we find that $\mathbf{x}^3(p) = (1/2p, 1/2)$ when $p \leq 1/3$ and $\mathbf{x}^3(p)$ is undefined otherwise. In that case demand is $\mathbf{x}^1(p) = \mathbf{x}^2(p) = (p + 1)(1/p, 1)$, so market demand is

$$\mathbf{x}(p) = \begin{cases} (2p + \frac{5}{2})\left(\frac{1}{p}, 1\right) & \text{when } p \leq 1/3 \\ \text{undefined} & \text{when } p > 1/3. \end{cases}$$

There is no equilibrium since the demand for good 1 is $2 + \frac{5}{2p} \geq 19/2$, which is larger than the endowment of good one.

How then do we handle the death of consumer three? If good 2 is labor, consumer three's labor will not be supplied when he is dead. In that case the relevant equilibrium involves only the first two consumers. It is easily calculated that $\mathbf{x}^1 = \mathbf{x}^2 = (2, 2)$ and $\mathbf{p} = (1, 1)$ is an equilibrium between the first two consumers.

Another possibility is that the endowment of three is a consumption good and survives his death. Then we would need a rule for dividing it up. But if it is a consumption good, consumer three may well consume it all in an attempt to survive as long as possible. ◀

16.4.2 Survival Through Trade

Of course, requiring that each consumer be able to survive without trade ($\omega^i \in \mathfrak{X}_i$) is rather strong. In fact, it is only specialization and the division of labor that allows large human populations to exist. Requiring survival in complete autarky is unreasonable.

In fact, the idea that we should think not in terms of autarkic survival but in terms of equilibrium survival has a long history in economics. Adam Smith wrote:

> "When the division of labour has been once thoroughly established, it is but a very small part of a man's wants which the produce on his own labour can supply. He supplies the far greater part of them by exchanging that surplus part of the produce of his own labour, which is over and above his own consumption, for such parts of the produce of other men's labour as he has occasion for. Every man thus lives by exchanging, or becomes in some measure a merchant, and the society itself grows to be what is properly called a commercial society.[12]"

Humans live by exchange. We don't presume any individual has the skills and resources to survive in autarky. Rather, they have skills or resources valuable to others, and their survival (and more) is based on this fact. This idea is the essence of McKenzie's concept of irreducibility. Irreducibility assumptions are but Smith's words translated into the language of modern general equilibrium models.

[12] Smith (1789), Bk.1, Chap.4

16.4.3 Irreducible Economies

McKenzie (1959, 1961, 1981, 1987) and Boyd and McKenzie (1993) introduced various notions of irreducibility for general production economies. Roughly speaking, an economy is irreducible if no matter how you divide consumers into two groups, each group is able to supply useful goods to the other group. One consequence of irreducibility is that all consumers will have income in equilibrium.[13] We will see later that this can be used to ensure survival in equilibrium even if some consumers cannot survive without trade.

Let $J \subset \{1, \dots, I\}$ be a group of consumers. Given $\boldsymbol{x}^i$ for $i \in J$, define the aggregate vector

$$\boldsymbol{x}^J = \sum_{i \in J} \boldsymbol{x}^i.$$

The definition applies to production economies as well as exchange economies. We let Y denote the aggregate production set.

Irreducible Economy. An economy $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)$ is *irreducible* if for every non-trivial partition of consumers into two groups I_1 and I_2 and every feasible allocation $(\hat{\boldsymbol{x}}^i, \hat{\boldsymbol{y}}^f)$, there exists $\boldsymbol{x}^h \succ_h \hat{\boldsymbol{x}}^h$ for all $h \in I_1$, a $\boldsymbol{v} \in \sum_{i \in I_2}(\boldsymbol{\omega}^i - \mathfrak{X}_i)$, and $\boldsymbol{y} \in Y$ such that $\boldsymbol{x}^{I_1} + \hat{\boldsymbol{x}}^{I_2} = \boldsymbol{y} + \boldsymbol{\omega} + \boldsymbol{v}$.

I have slightly modified the statement of irreducibility from McKenzie's version to improve clarity. In particular, I have flipped the sign of $\boldsymbol{v}$ to better reflect that it is being given to the individuals in I_1. By replacing $\hat{\boldsymbol{y}}$ with $\boldsymbol{y} + \boldsymbol{v}$, the individuals in I_1 are able to obtain better consumption vectors. When $\mathcal{E}$ is an exchange economy, $Y = -\mathbb{R}^m_+$ and we can normally replace both $\hat{\boldsymbol{y}}$ and $\boldsymbol{y}$ by $\boldsymbol{0}$, thus $\boldsymbol{x}^{I_1} + \hat{\boldsymbol{x}}^{I_2} = \boldsymbol{\omega} + \boldsymbol{v}$.

[13] Gale (1957) introduced a related notion of irreducibility for linear economies. Hammond (1993) contains a discussion of irreducibility and its relation to survival and Arrow and Hahn's (1971) resource-relatedness.

16.4.4 Criteria for Irreducibility

Many economies are irreducible. The following result implies that all of the economies covered by the Equilibrium Existence Theorem: Exchange Economies are irreducible.

Proposition 16.4.2. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, -\mathbb{R}^m_+)$ be an exchange economy. Suppose for each $i = 1, \dots, I$, $\mathfrak{X}_i = \mathbb{R}^m_+$ and $\boldsymbol{\omega}^i > 0$. Then $\mathcal{E}$ is irreducible if for every consumer i, either (1) $\succsim_i$ is strongly monotonic or (2) $\succsim_i$ is monotonic and $\boldsymbol{\omega}^i \gg 0$.*

Proof. Let $(\hat{\mathbf{x}}^i)$ be a feasible allocation. We may set $\hat{\mathbf{y}} = \sum_i(\hat{\mathbf{x}}^i - \boldsymbol{\omega}^i) \in -\mathbb{R}^m_+$. Let I_1, I_2 be a partition of the consumers. Set $\boldsymbol{\nu} = \sum_{i\in I_2} \boldsymbol{\omega}^i > 0$. Define $\mathbf{x}^h = \hat{\mathbf{x}}^h + (1/\#I_1)\boldsymbol{\nu} > \hat{\mathbf{x}}^h$ for all $h \in I_1$. Then $\mathbf{x}^{I_1} + \hat{\mathbf{x}}^{I_2} = \hat{\mathbf{y}} + \boldsymbol{\omega} + \boldsymbol{\nu}$.

If all $h \in I_1$ obey (1), we have $\mathbf{x}^h \succ_h \hat{\mathbf{x}}^h$ for all $h \in I_1$ by strong monotonicity. Otherwise, some $h \in I_1$ obeys (2) and $\boldsymbol{\nu} \gg 0$. Then monotonicity is enough to insure that $\mathbf{x}^h \succ_h \hat{\mathbf{x}}^h$ for all $h \in I_1$. This establishes irreducibility. □

In Proposition 16.4.2, each consumer is able to supply their full endowment. Since the endowment is strictly positive and preferences are monotonic, that means that even an isolated consumer can provide resources that are valuable to anyone else in the economy. That doesn't happen in Example 16.4.1 because consumer three requires two units of consumption before he can supply anything.

16.4.5 A Reducible Economy without Equilibrium

An economy that is not irreducible will be called *reducible*. Reducible economies need not have an equilibrium, even in the pure exchange case. There is also the possibility that they may break up into two or more autarkic economies.

Example 16.4.3: A Simple Reducible Economy without Equilibrium Consider an economy with 2 consumers and 2 goods. Consumer one has endowment $\boldsymbol{\omega}^1 = (1,0)$. Utility is $u_1(\mathbf{x}^1) = x_1^1$. Consumer two has endowment $\boldsymbol{\omega}^2 = (0,1)$ and utility $u_2(\mathbf{x}^2) = \sqrt{x_1^2} + \sqrt{x_2^2}$. The consumption sets are $\mathfrak{X}_i = \mathbb{R}^2_+$.

This economy is reducible. Consider $I_1 = \{1\}$, $I_2 = \{2\}$, and the allocation $\big((1,0),(0,1)\big)$. To increase the utility of the consumer $1 \in I_1$ requires giving them more of good 1. Thus v_1 must be positive. No such vectors exist in $\{(0,1) - \mathbb{R}^2_+\}$. This means the economy is reducible.

We now turn to equilibrium. Demand by consumer one is $\mathbf{x}^1(\mathbf{p}) = (1,0)$. Demand by consumer two is

$$\mathbf{x}^2(\mathbf{p}) = \left(\frac{p_2}{p_1(p_1+p_2)}, \frac{p_1}{p_2(p_1+p_2)}\right).$$

In equilibrium, $\mathbf{x}(\mathbf{p}) \le \boldsymbol{\omega} = (1,1)$. If we normalize prices so that $p_1 + p_2 = 1$, $\mathbf{x}(\mathbf{p}) = (1 + p_2/p_1, p_1/p_2)$. Equilibrium in market 1 requires $p_2 = 0$, which means demand for good 2 is infinite. Equilibrium in market 2 requires $p_1/p_2 = 1$, in which case there is excess demand for good 1. There is no equilibrium and there is no trade. There is the possibility of autarky, where all that the consumers can do is to consume their own endowments. ◀

16.4.6 Another Reducible Economy without Equilibrium

Example 16.4.4: Example 16.4.1 Revisited The economy of Example 16.4.1 is also reducible. Consider the allocation $\hat{x}^1 = \hat{x}^2 = (1/2, 2)$, $\hat{x}^3 = (3, 1)$. Note that each $\hat{x}^i \in \mathfrak{X}_i$. Now partition the consumers into $I_1 = \{1, 2\}$ and $I_3 = \{3\}$. Then $\boldsymbol{v} \in (\boldsymbol{\omega}^3 - \mathfrak{X}_3) = \big((0, 1) - \mathfrak{X}_3\big) = \{\boldsymbol{x} : x_1 \leq 0, x_2 \leq 1, x_1 + x_2 \leq -1\}$. Consumer three can only give good 2 in exchange for more than 1 unit of good 1. But the consumers in I_1 only have 1 unit of good two, no more. So there is no $\boldsymbol{v} \in (\boldsymbol{\omega}^3 - \mathfrak{X}_3)$ that can make them better off. ◄

16.4.7 Wald's Counterexample I

Long before the concept of irreducibility was invented, Wald (1936) provided an example of a reducible exchange economy without an equilibrium. Wald's example is particularly interesting from an irreducibility standpoint since the allocations where the economy fails to be irreducible are allocations that are not reasonable to consider as potential equilibrium allocations—they are not individually rational.

Example 16.4.5: Wald's Counterexample Define

$$\varphi_2(x) = \begin{cases} -\frac{b}{x} - \ln x & \text{when } x \leq b \\ -1 - \ln b & \text{when } x > b. \end{cases}$$

and

$$\varphi_3(x) = \begin{cases} -\frac{2c}{x} - 2\ln x & \text{when } x \leq c \\ -2 - 2\ln c & \text{when } x > c. \end{cases}$$

There are three consumers with utility $u_1(\mathbf{x}^1) = \ln x_1^1 + \varphi_2(x_2^1) + \varphi_3(x_3^2)$, $u_2(\mathbf{x}^2) = -\frac{1}{x_1^2} + \ln x_2^2$, and $u_3(\mathbf{x}^3) = -\frac{1}{x_1^3} + \ln x_3^3$. The endowments are $\boldsymbol{\omega}^1 = (a, 0, 0)$, $\boldsymbol{\omega}^2 = (0, b, 0)$ and $\boldsymbol{\omega}^3 = (0, 0, c)$.

The fact that utility is not defined for some $\mathbf{x} \geq \mathbf{0}$ means that the consumption sets are not the positive orthant. They are $\mathfrak{X}_1 = \{\mathbf{x} \in \mathbb{R}^3_+ : x_1 > 0,\ x_2 < b,\ x_3 < c\}$, $\mathfrak{X}_2 = \{\mathbf{x} \in \mathbb{R}^3_+ : x_1, x_2 > 0\}$, and $\mathfrak{X}_3 = \{\mathbf{x} \in \mathbb{R}^3_+ : x_1, x_3 > 0\}$

16.4.8 Wald's Counterexample II

Each good is always valuable to some consumer, so equilibrium prices have to be strictly positive. We choose good 1 as numéraire, so $\mathbf{p} = (1, p_2, p_3)$. Since consumer one has infinite marginal utility at zero consumption of any good, consumer one must consume all three goods in equilibrium. This means that consumers two and three will both consume some of good 1. They will not give up all of their endowments because they too have infinite marginal utility at zero.

Solving for the equilibrium can be rather messy, so we follow Wald's simpler route. The only trades are of good two from consumer two to consumer one, of good three from consumer three to consumer one, and good one from consumer one to consumers two and three. We can write the resulting consumption vectors as follows.

$$\begin{aligned}
\mathbf{x}^1 &= \big(a - p_2\Delta x_2 - p_3\Delta x_3, p_2\Delta x_2, p_3\Delta x_3\big),\\
\mathbf{x}^2 &= \big(p_2\Delta x_2, b - \Delta x_2, 0\big),\\
\mathbf{x}^3 &= \big(p_3\Delta x_3, 0, c - \Delta x_3\big),
\end{aligned}$$

Where $\Delta x_2 = b - x_2^2 > 0$ and $\Delta x_3 = c - x_3^3 > 0$.

16.4.9 Wald's Counterexample III

We now turn our attention to the first-order conditions. In the case of consumer two, $\mathrm{MRS}^2_{12} = 1/p_2$, so

$$\mathrm{MRS}^2_{12} = \frac{1}{p_2} = \frac{b - \Delta x_2}{p_2^2 \Delta x_2^2}.$$

Similarly, for consumer three,

$$\mathrm{MRS}^3_{13} = \frac{1}{p_3} = \frac{c - \Delta x_c}{p_3^2 \Delta x_3^2}.$$

Thus

$$p_2 = \frac{b - \Delta x_2}{\Delta x_2^2} \quad \text{and} \quad p_3 = \frac{c - \Delta x_3}{\Delta x_3^2}.$$

Taking the ratio, we find

$$\frac{p_2}{p_3} = \frac{(b - \Delta x_2)\Delta x_3^2}{(c - \Delta x_3)\Delta x_2^2}$$

But for consumer one,

$$\frac{p_2}{p_3} = \mathrm{MRS}^1_{23} = \frac{1}{2}\frac{(b - \Delta x_2)\Delta x_3^2}{(c - \Delta x_3)\Delta x_2^2}.$$

Since the prices are the same for all consumers, this is impossible. There is no equilibrium. ◀

16.4.10 Remarks on Wald's Counterexample

The economy in Wald's example is not irreducible. Consider the allocation $\hat{x}^1 = (a, b, c)$, $\hat{x}^2 = \hat{x}^3 = 0$ and the partition $I_1 = \{1\}$, $I_2 = \{2, 3\}$. In this exchange case we must find $v \in \left((0, b, c) - \mathbb{R}^3_+\right)$ with $x^1 = \omega - v \succ_1 \omega = \hat{x}^1$. This is impossible because consumer one is satiated in both goods 2 and 3 at x^1 and $v_1 \leq 0$.

In Wald's example, it is not the reducibility that prevents the economy from having an equilibrium. Rather, it is the fact that the consumption sets are not closed. As McKenzie (1959) pointed out, closing them can give the economy an equilibrium. In fact, if we merely close the consumption sets of consumers two and three by making the boundary the lowest indifference curve, the $(1, 0, 0)$ is an equilibrium price vector. Consumers two and three are then indifferent between consuming the endowment and nothing at all. The resulting allocation is $x^1 = (a, b, c)$, $x^2 = x^3 = 0$.

16.5 Equilibrium Existence in a Production Economy

We will show existence of equilibrium in a production economy with a single constant returns to scale firm. When the aggregate production set is irreversible, Theorem 14.5.6 and Proposition 14.5.2 show that we can treat the production side as a single firm. Constant returns to scale is important because it guarantees we can maximize profit whenever the profit function is finite. The use of constant returns to scale also saves us from having to worry about the distribution of profits, because profit is also zero in equilibrium. The economy can then be written as $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)$. The case of possibly decreasing returns will be considered later, in section 16.7.

16.5.1 Kakutani Fixed Point Theorem

The existence proof for production economies is similar in spirit to that for exchange economies, but at a higher level of generality. For one, we will not assume strict convexity of preferences. As a result, we have to allow for the possibility of demand correspondences rather than demand functions. The presence of constant returns to scale means that supply is also a correspondence. Because of this, the tâtonnement map must be modified to take these correspondences into account. The resulting map adjusts both prices and demands. We then appeal to a more powerful fixed point theorem to do the heavy lifting—Kakutani's Fixed Point Theorem.[14][15] We use irreducibility to show the fixed point is an equilibrium, which lets us dispense with the assumption of strictly positive individual endowments.

Kakutani Fixed Point Theorem. *Let $\Delta \subset \mathbb{R}^m$ be compact and convex. If $F\colon \Delta \to \Delta$ is an upper Vietoris continuous correspondence that is convex-valued, then there is a $\boldsymbol{v}^* \in \Delta$ with $\boldsymbol{v}^* \in F(\boldsymbol{v}^*)$.*

See Chapter 33 for more on correspondences. One key fact is that any closed correspondence whose values are all contained in a compact set is upper Vietoris continuous (Continuity Theorem).

Upper Vietoris Continuous Correspondence. A correspondence $\varphi\colon A \twoheadrightarrow B$ is *upper Vietoris continuous* at $x \in A$ if for all open sets U with $\varphi(x) \subset U$ and every sequence $x^n \to x$, there is an $N > 0$ with $\varphi(x^n) \subset U$ for $n \geq N$. A correspondence on A is *upper Vietoris continuous* if it is upper Vietoris continuous at every point $x \in A$.

We will also recall a definition from Chapter 2 before stating the existence theorem.

Semi-strictly Convex Preferences. A preference order $\succsim$ defined on a convex set $\mathfrak{X}$ is *semi-strictly convex* if whenever $\boldsymbol{x} \succ \boldsymbol{y}$, then $(1-t)\boldsymbol{x} + t\boldsymbol{y} \succ \boldsymbol{y}$ for all $t \in (0,1)$.

Semi-strict convexity does not imply that there is a unique demand point. Indifference surfaces containing flats are allowed. Even linear utility is semi-strictly convex, and that can have the entire budget line as a utility maximum.

[14] See Border (1985), Klein and Thompson (1984), or Smart (1974) for a proof.

[15] It is possible to use other fixed point theorems. E.g., Gale (1955) used the KKM Lemma (Knaster, Kuratowski, and Mazurkiewicz, 1929), while Arrow and Debreu (1954) based their proof on Debreu's social equilibrium existence theorem (Debreu, 1952b), which itself used a theorem of Eilenberg and Montgomery (1946).

16.5.2 Equilibrium Existence in a Production Economy

Recall that consumer i's net trading set is defined by $X_i = \mathfrak{X}_i - \boldsymbol{\omega}^i$ where $\mathfrak{X}_i$ is i's consumption set. The aggregate net trading set is $X = \sum_i X_i = \left(\sum_i \mathfrak{X}_i\right) - \boldsymbol{\omega}$. If the consumption sets are closed, X will also be closed by Proposition 14.4.8. We can think of preferences as being defined on the net trading sets rather than on the consumption sets.

We are now ready to state the main existence theorem for production economies.

Equilibrium Existence Theorem: Production Economies. *Consider a CRS production economy $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)$. Suppose that*

1. *Each $\mathfrak{X}_i \subset \mathbb{R}^m_+$ is closed and convex with $\boldsymbol{\omega}^i \in \mathfrak{X}_i$.*
2. *For each $i = 1, \dots, I$, $\succsim_i$ is an semi-strictly convex and continuous preference order on $\mathfrak{X}_i$.*
3. *The aggregate production set Y is a non-empty closed convex cone obeying inaction, no-free lunch, and free disposal.*
4. $-\boldsymbol{\omega} \in \operatorname{int} Y$.
5. *$\mathcal{E}$ is irreducible.*

Then a Walrasian equilibrium $(\hat{\mathbf{p}}, \hat{\mathbf{z}}, \hat{\mathbf{y}})$ exists. Moreover, $\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i > 0$ for all $i = 1, \dots, I$.

Concerning (4), if the endowment is strictly positive, $-\boldsymbol{\omega}$ will be in interior of the production due to inaction and free disposal. Since endowments are non-negative, the only cases of interest here are when some $\omega_\ell = 0$, in which case (4) is slightly more general than assuming $\boldsymbol{\omega} \gg \mathbf{0}$.

We have some preliminary points to cover before starting the proof of the existence theorem.

16.5.3 Tâtonement and Double Truncation

The general plan is the same as in the exchange economy case. Use demands to construct a tâtonnement map, find a fixed point, and show the fixed point is an equilibrium. The tâtonnement map for production economies is more complex than for exchange economies. Rather than map points in the price simplex to points in the price simplex, the tâtonnement map takes both a price vector and consumption/production vector as arguments, mapping them to a new price vector and new consumption/production vector.

The consumption/production vector will be both an aggregate excess demand vector and a production vector. That is, it must lie in both the sum of the net trading sets and in the production set. This restricts it to the net trade version of the production possibilities set. This restriction will enable us to focus on a compact subset of the production set, something that is necessary for use of the Kakutani Fixed Point Theorem.

This restriction to the production possibilities set is one of two truncations we will use. The other truncation affects the consumption sets in order to maintain compactness of the budget set even when one or more prices is zero. This truncation is essentially the same at that used to prove the Equilibrium Existence Theorem: Exchange Economies. It will be defined once we have our set of consumption/production vectors.

16.5.4 Truncation via the Production Possibilities Set

The truncation will be accomplished by focusing on the set $Y \cap T$ where $T = X + \boldsymbol{\omega} + \bar{\mathbf{y}}$ for some $\bar{\mathbf{y}} \ll -\boldsymbol{\omega}$. Here $\bar{\mathbf{y}} \in \operatorname{int} Y$.

To get a better feel how this works, we will convert the net trades to consumption vectors and show the truncation we used for exchange economies can be written in this form.

Market clearing requires that equilibria are both in the production possibilities set $(Y + \boldsymbol{\omega})$ and in the aggregate consumption set $\mathfrak{X} = \sum_i \mathfrak{X}_i$. Equilibria must be in the intersection $(Y + \boldsymbol{\omega}) \cap \mathfrak{X}$. In the net trade picture, this corresponds to $Y \cap X$.

Since $\mathfrak{X} \subset \mathbb{R}^m_+$, $\mathbf{x} \in \mathfrak{X}$ implies $\mathbf{0} \le \mathbf{x}$. Recall that we can write exchange economies as production economies with $Y = -\mathbb{R}^m_+$. Then $\mathbf{x} = \mathbf{y} + \boldsymbol{\omega} \le \boldsymbol{\omega}$ since $\mathbf{y} \le \mathbf{0}$. In net trade terms, where the corresponding net trade is $\mathbf{z} = \mathbf{x} - \boldsymbol{\omega}$, our inequality becomes

$$-\boldsymbol{\omega} \le \mathbf{z} = \mathbf{x} - \boldsymbol{\omega} = \mathbf{y} \in Y.$$

The truncation we used in the proof of the Equilibrium Existence Theorem for Exchange Economies relaxes the limit on demand from $\boldsymbol{\omega}$, requiring only that $\mathbf{x} \le K\mathbf{e}$ where $K\mathbf{e}$ is strictly greater than $\boldsymbol{\omega}$. For the tâtonnement to work, this must still be in the exchange economy production set, $Y = -\mathbb{R}^m_+$. Denote $-K\mathbf{e}$ by $\bar{\mathbf{y}}$ and notice that $\bar{\mathbf{y}} \in Y$. Of course $\bar{\mathbf{y}} \in \operatorname{int} Y$, which we will need to retain in the production case for technical reasons.

Now define $\mathbf{y}' = \mathbf{y} + (\boldsymbol{\omega} + \bar{\mathbf{y}})$, so that $\mathbf{y} + \boldsymbol{\omega} = \mathbf{y}' - \bar{\mathbf{y}}$. Then

$$\mathbf{y}' = \mathbf{y} + (\boldsymbol{\omega} + \bar{\mathbf{y}}) \le \boldsymbol{\omega} + \bar{\mathbf{y}} \ll \mathbf{0},$$

so $\mathbf{y}' \in Y$ by free disposal. We can now write the our truncation equation in consumption terms as

$$\mathbf{0} \le \mathbf{x} = \mathbf{y}' + K\mathbf{e} = \mathbf{y}' - \bar{\mathbf{y}} \le -\bar{\mathbf{y}} = K\mathbf{e}$$

with $\mathbf{y}' \in Y$. In net trades $\mathbf{z} = \mathbf{x} - \boldsymbol{\omega}$, so this becomes

$$-\boldsymbol{\omega} \le \mathbf{z} = \mathbf{x} - \boldsymbol{\omega} = \mathbf{y}' - (\boldsymbol{\omega} + \bar{\mathbf{y}}).$$

Then $\mathbf{z} = \mathbf{x} - \boldsymbol{\omega} \in X$ and $\mathbf{z} = \mathbf{y}' - (\boldsymbol{\omega} + \bar{\mathbf{y}})$, implying $\mathbf{z} + \boldsymbol{\omega} + \bar{\mathbf{y}} = \mathbf{y}' \in Y$ It follows that $\mathbf{z} \in X \cap \big(Y - (\boldsymbol{\omega} + \bar{\mathbf{y}})\big)$ or $\mathbf{z} + \omega + \bar{\mathbf{y}} \in (X + \boldsymbol{\omega} + \bar{\mathbf{y}}) \cap Y = T$.

The novel thing here is that when proving existence in production economies, we must make it explicit that we restrict production to be in $T = Y \cap (X + \boldsymbol{\omega} + \bar{\mathbf{y}})$. Our tâtonnement map will act on T as well as the price simplex Δ.

16.5.5 Truncation Lemma

We prove a lemma that will allow us to simultaneously truncate both demand and supply as discussed above. This lemma defines the truncation point $\bar{\mathbf{y}}$ and shows that $Y \cap T$ is non-empty and compact. It is also convex, making it suitable for use with fixed point theorems.

Lemma 16.5.1. *Suppose a CRS production economy $\mathcal{E}$ obeys conditions (1), (3), and (4) of the Equilibrium Existence Theorem: Production Economies. Then there is $\bar{\mathbf{y}} \ll -\boldsymbol{\omega}$ so that $T = Y \cap (X + \boldsymbol{\omega} + \bar{\mathbf{y}})$ is both non-empty and compact.*

Proof. By (4), $-\boldsymbol{\omega} \in \operatorname{int} Y$, so we can find $\varepsilon > 0$ with $B_\varepsilon(-\boldsymbol{\omega}) \subset Y$. Let

$$\bar{\mathbf{y}} = -\boldsymbol{\omega} - \frac{\varepsilon}{2}\boldsymbol{e} \in B_\varepsilon(-\boldsymbol{\omega}) \subset \operatorname{int} Y.$$

By (1), $\mathbf{0} \in X + \boldsymbol{\omega}$, so $\bar{\mathbf{y}} \in X + \boldsymbol{\omega} + \bar{\mathbf{y}}$. Since $\bar{\mathbf{y}} \in Y$, $T = Y \cap (X + \boldsymbol{\omega} + \bar{\mathbf{y}})$ is non-empty.

Let $T_0 = Y \cap (\mathbb{R}^m_+ + \bar{\mathbf{y}})$. As noted previously, X is closed and $X \subset \left(\sum_i X_i\right) - \boldsymbol{\omega}$. By (1), each $X_i \subset \mathbb{R}^m_+$, so $X \subset \mathbb{R}^m_+ - \boldsymbol{\omega}$. It follows that $(X + \boldsymbol{\omega} + \bar{\mathbf{y}}) \subset \mathbb{R}^m_+ + \bar{\mathbf{y}}$. Then T is a closed subset of $T_0 = Y \cap (\mathbb{R}^m_+ + \bar{\mathbf{y}})$. Showing that T_0 is compact will imply T is also compact.

By (3), Y is closed, so T_0 is closed. We complete the proof by showing T_0 is bounded, which we prove by contradiction. Suppose T_0 is not bounded. Then there are $\mathbf{y}_n \in T_0$ with $\|\mathbf{y}_n\| \to \infty$. Let $\mathbf{u}_n = \mathbf{y}_n/\|\mathbf{y}_n\|$. Since Y is a cone, $\mathbf{u}_n \in Y$. Now $\|\mathbf{u}_n\| = 1$, so $\mathbf{u}_n$ is a bounded sequence. As such it has a convergent subsequence, $\{\mathbf{u}_{n_j}\}$ with $\lim_j \mathbf{u}_{n_j} = \mathbf{u}^*$. Then $\|\mathbf{u}^*\| = 1$ and $\mathbf{u}^* \in Y$ since Y is closed. Moreover, since $\mathbf{y}_n \ge \bar{\mathbf{y}}$, $\mathbf{u}_n \ge \bar{\mathbf{y}}/\|\mathbf{y}_n\|$. Taking the limit, we find $\mathbf{u}^* \ge \mathbf{0}$. But $\mathbf{u}^* \in Y$, so $\mathbf{u}^* = \mathbf{0}$ by the no free lunch condition. This is impossible because $\|\mathbf{u}^*\| = 1$. This contradiction shows that T_0 is bounded.

Finally, since T_0 is closed and bounded, it is compact, and so is T, which is a closed subset of T_0. $\square$

16.5.6 Truncating the Budget Set

Now that we know T is compact, we can define the truncation for the budget and trading sets. In an exchange economy, no consumer can ever consume more than the endowment $\boldsymbol{\omega} \ll \bar{\mathbf{y}}$. In that case we can safely truncate the budget using $\bar{\mathbf{y}}$. But this is a production economy and consumption is limited by what can be produced with the endowment. To put an upper bound on consumption, we consider

$$\bar{t}_\ell = \max\{y_\ell : \mathbf{y} \in T\},$$

which exists for $\ell = 1, \dots, m$ since T is compact. This defines a vector $\bar{\mathbf{t}}$ that is an upper bound for T. The truncated net trading set is

$$\hat{X}_i = X_i \cap \{\mathbf{z} : \mathbf{z} \le \bar{\mathbf{t}}\},$$

and the corresponding consumption set is

$$\hat{\mathfrak{X}}_i = \boldsymbol{\omega}_i + X_i = \{\mathbf{x} \in \mathfrak{X}_i : \mathbf{x} \le \bar{\mathbf{t}} + \boldsymbol{\omega}_i\}.$$

For the aggregate sets, $\hat{\mathfrak{X}} = \sum_i \hat{\mathfrak{X}}_i$ is bounded above by $\boldsymbol{\omega} + I\bar{\mathbf{t}}$ and bounded below by $\mathbf{0}$. The aggregate net trading set $\hat{X} = \sum_i \hat{X}_i$ is bounded above by $I\bar{\mathbf{t}}$ and below by $\boldsymbol{\omega} \gg \bar{\mathbf{y}}$. The set T is bounded below by $\bar{\mathbf{y}}$ and above by $\bar{\mathbf{t}}$.

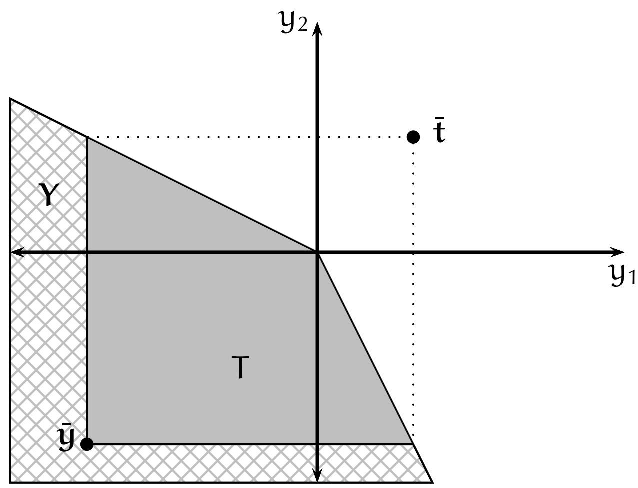

Figure 16.5.2: This diagram illustrates the set T, bounded below by $\bar{\mathbf{y}}$ and bounded above by $\bar{\mathbf{t}}$.

16.5.7 The Domain of the Tâtonnement

The biggest problem in showing existence of equilibrium is to construct the proper tâtonnement map. The key is to define a tâtonnement map that adjusts both prices and quantities. This map will compute demand based on prices, and a supply price based on demands. A fixed point will then be an equilibrium as the quantity demanded will be supplied at that price.

We will work with a price simplex defined by $\Delta = \{\mathbf{p} \in Y^* : \mathbf{p}\cdot\boldsymbol{e} = 1\}$. By free disposal, $\Delta \subset \mathbb{R}^m_+$. Of course, Δ is compact and convex.

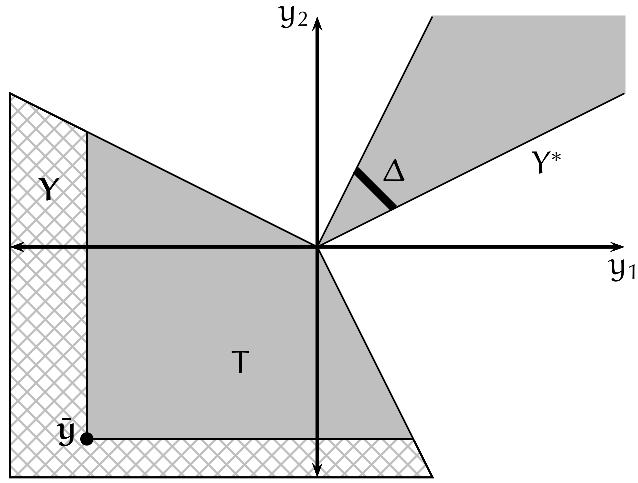

Figure 16.5.3: The price simplex is defined as all $\mathbf{p} \in Y^*$ obeying $\mathbf{p}\cdot\boldsymbol{e} = 1$. Notice that in this example $\operatorname{int} Y$ is not just $-\mathbb{R}^m_{++}$, but contains all of $-\mathbb{R}^m_+$ and more. Further, we have taken $\mathfrak{X} = \mathbb{R}^m_+$ rather than a closed convex subset, which accounts for the vertical and horizontal sides of T at $\bar{\mathbf{y}}$.

The tâtonnement map will act on $\Delta \times T$ where T has the form given above by Lemma 16.5.1. The set T is a compact convex subset of Y.

16.5.8 Equilibrium Existence Theorem: Production Economies

We restate the theorem before proceeding.

Equilibrium Existence Theorem: Production Economies. *Consider a CRS production economy* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)$. *Suppose that*

1. *Each* $\mathfrak{X}_i \subset \mathbb{R}^m_+$ *is closed and convex with* $\boldsymbol{\omega}^i \in \mathfrak{X}_i$.
2. *For each* $i = 1, \ldots, I$, $\succsim_i$ *is an semi-strictly convex and continuous preference order on* $\mathfrak{X}_i$.
3. *The aggregate production set* Y *is a non-empty closed convex cone obeying inaction, no-free lunch, and free disposal.*
4. $-\boldsymbol{\omega} \in \operatorname{int} Y$.
5. $\mathcal{E}$ *is irreducible.*

Then a Walrasian equilibrium $(\hat{\mathbf{p}}, \hat{\mathbf{z}}, \hat{\mathbf{y}})$ *exists. Moreover,* $\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i > 0$ *for all* $i = 1, \ldots, I$.

16.5.9 Existence Proof: Summary

Proof of Equilibrium Existence Theorem: Production Economies. This proof is based on McKenzie (1981). It is lengthy, so we start with a summary.

Summary: As in the exchange case, we must truncate demands. Unlike the exchange case, we also truncate supply. This was detailed previously in and following Lemma 16.5.1. Truncation does not solve all of our continuity problems. Unlike the exchange case, we have not ruled out the possibility of zero income. To handle this, we modify the consumer's excess demand using a method due to Debreu (1962) which yields an upper Vietoris continuous excess demand correspondence.

Then we construct the tâtonnement map. This map will be somewhat more complex than before, as we will have to map both prices and demand/supply vectors in order to accomodate production. More precisely, we work on $\Delta \times T$, mapping price-quantity pairs to price-quantity pairs. The mapping has two parts. One part maps prices to demands using the modified demand correspondence. The other part maps demand to prices. The latter happens in two steps, first projecting the quantity onto the boundary of the production set, then finding a price vector that supports the production set at that point. Essentially, we are finding the price at which the demand will be supplied by the firm.

The Kakutani Fixed Point Theorem yields our fixed point. We then use irreducibility to show that all consumers have positive income. This implies that the modified demand correspondence does not involve Debreu's modification. We also know that the truncation of the consumption or net trading sets will not bind, just as it did not bind in exchange economies. We then obtain the production vector and show that all markets clear.

16.5.10 Existence Proof: Truncation and Demand

Truncation: The set T is compact by Lemma 16.5.1. Since T is a subset of the net trade version of the production possibility set, any equilibrium must be in T. We can replace each consumption set $\mathfrak{X}_i$ by the compact set $\hat{\mathfrak{X}}_i = \mathfrak{X} \cap \{\mathbf{x} : \mathbf{x} \le \mathbf{t}\}$. Equivalently, we can replace the net trading sets X_i by $\hat{X}_i$. As we know from our use of truncation in the exchange case, it is sufficient to find an equilibrium for the demands using the truncated consumption sets $\hat{\mathfrak{X}}_i$ is bounded. As we saw, the upper bound $\bar{\mathbf{t}}$ is outside of $Y \cap (X + \boldsymbol{\omega} + \bar{\mathbf{y}})$. Then semi-strict convexity allows us to show that if there were a better demand point outside the bound, there is also a better demand point inside the bound.

Demand Correspondence: Let $\mathbf{x}^i(\mathbf{p})$ be the demand correspondence for consumer i. This will exist for all $\mathbf{p} \in \Delta$ because $\succsim_i$ is continuous and $\hat{\mathfrak{X}}_i$ is compact. The demand correspondence is upper Vietoris continuous when $\mathbf{p} \cdot \boldsymbol{\omega}^i > 0$, but upper Vietoris continuity may fail if the consumer has no income, something that was a problem in Example 16.4.3.

Define consumer i's modified excess demand by

$$z^i(\mathbf{p}) = \begin{cases} \mathbf{x}^i(\mathbf{p}) - \boldsymbol{\omega}^i & \text{if } \mathbf{p} \cdot \boldsymbol{\omega}^i > 0 \\ \{\mathbf{x} - \boldsymbol{\omega}^i : \mathbf{x} \in \hat{\mathfrak{X}}_i, \mathbf{p} \cdot \mathbf{x} = 0\} & \text{if } \mathbf{p} \cdot \boldsymbol{\omega}^i = 0. \end{cases}$$

Because $\boldsymbol{\omega}_i \in \mathfrak{X}_i$, $\mathbf{0} \in z^i(\mathbf{p})$ for every $\mathbf{p} \in \Delta$. The excess demand correspondence is always non-empty. Lemmas 1 and 2 in Debreu (1962) show that $z^i(\mathbf{p})$ is upper Vietoris continuous, as is $z(\mathbf{p}) = \sum_i z^i(\mathbf{p})$.[16]

[16] The terminology was somewhat less clear when Debreu wrote, and he actually shows that the correspondence is closed. However, since we have assumed that each $\mathfrak{X}_i$ is bounded and so compact, everything takes place in a compact set. Any closed correspondence mapping into a compact set is upper Vietoris continuous by Corollary 33.5.12.

16.5.11 Existence Proof: Tâttonnement

Tâtonnement Map: The tâtonnement map $\mathbf{F}$ is a correspondence on $\Delta \times T$. We write $\mathbf{F}(\mathbf{p}, \boldsymbol{z}) = \big(F_1(\boldsymbol{z}), F_2(\mathbf{p})\big)$. The second component is easy to define,

$$F_2(\mathbf{p}) = \boldsymbol{z}(\mathbf{p}) = \sum_i \boldsymbol{z}^i(\mathbf{p}),$$

which is upper Vietoris continuous.

For the demand to price mapping F_1, we first map demand onto the boundary of the production set Y, and then find the supporting prices at that point.

By Lemma 16.5.1, we find $\bar{\mathbf{y}} \in \operatorname{int} Y$, $\bar{\mathbf{y}} \ll -\boldsymbol{\omega}$. Given $\boldsymbol{z} \in X$, consider the line through $\bar{\mathbf{y}}$ and $\boldsymbol{z}$ and define

$$A(\boldsymbol{z}) = \{\alpha : \bar{\mathbf{y}} + \alpha(\boldsymbol{z} - \bar{\mathbf{y}}) \in Y\}.$$

The set $A(\boldsymbol{z})$ is closed because it is the inverse image of Y under a continuous mapping. Now $\boldsymbol{z} - \bar{\mathbf{y}} \geq -\boldsymbol{\omega} - \bar{\mathbf{y}} \gg \mathbf{0}$. So if $\alpha \in A(\boldsymbol{z})$ and $\alpha' < \alpha$, $\bar{\mathbf{y}} + \alpha'(\boldsymbol{z} - \bar{\mathbf{y}}) \ll \bar{\mathbf{y}} + \alpha(\boldsymbol{z} - \bar{\mathbf{y}})$. By free disposal, $\alpha' \in A(\boldsymbol{z})$. It follows that $A(\boldsymbol{z})$ is a closed interval. Also, for α large $\bar{\mathbf{y}} + \alpha(\boldsymbol{z} - \bar{\mathbf{y}}) \gg \mathbf{0}$ and cannot be in Y. This bounds $A(\boldsymbol{z})$ above. Finally, $0 \in A(\boldsymbol{z})$, so $A(\boldsymbol{z})$ is non-empty. We can then write

$$A(\boldsymbol{z}) = \big(-\infty, \alpha(\boldsymbol{z})\big].$$

This defines a function $\alpha(\boldsymbol{z})$.

We define

$$\mathbf{h}(\boldsymbol{z}) = \bar{\mathbf{y}} + \alpha(\boldsymbol{z})(\boldsymbol{z} - \bar{\mathbf{y}}).$$

For $\alpha > \alpha(\boldsymbol{z})$, $\bar{\mathbf{y}} + \alpha(\boldsymbol{z} - \bar{\mathbf{y}}) \notin Y$, so $\mathbf{h}(\boldsymbol{z})$ is on the boundary of Y. The point $\mathbf{h}(\boldsymbol{z})$ is where the ray from $\bar{\mathbf{y}}$ through $\boldsymbol{z}$ intersects the boundary of Y.

Now given $\mathbf{y}$ on the boundary of Y, let $G(\mathbf{y}) = \{\mathbf{p} \in \Delta : \mathbf{p} \cdot \mathbf{y}' \leq \mathbf{p} \cdot \mathbf{y} \text{ for all } \mathbf{y}' \in Y\}$. Finally, the first component of $\mathbf{F}$ is

$$F_1(\boldsymbol{z}) = G\big(\mathbf{h}(\boldsymbol{z})\big),$$

completing the tâtonnement map $\mathbf{F}$.

16.5.12 Tâtonnement Illustrated

We will illustrate the tâtonnement graphically, by considering the mapping $\mathbf{p} \to F_2(\mathbf{p})$ followed by the mapping $\boldsymbol{z}(\mathbf{p}) \to F_1(\boldsymbol{z}(\mathbf{p}))$.

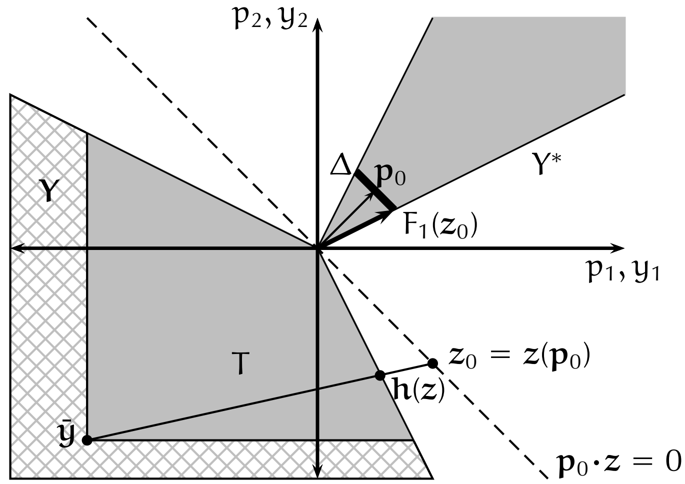

Figure 16.5.4: We start with a price vector $\mathbf{p}_0 \in \Delta$. This is mapped by F_2 to the excess demand correspondence $\boldsymbol{z}(\mathbf{p})$ which lies on the hyperplane $\mathbf{p}_0 \cdot \boldsymbol{z} = 0$.

Now form the line segment from $\bar{\mathbf{y}}$ to $\boldsymbol{z}(\mathbf{p}_0)$. The point where it hits the boundary of Y is $\mathbf{h}_0 = \mathbf{h}(\boldsymbol{z}(\mathbf{p}_0))$. The set of prices in Δ where profit is maximized at $\mathbf{h}_0$ constitutes the set $G(\mathbf{h}_0)$. In the diagram there is only one such price, marked at $F_1(\boldsymbol{z}_0)$ at the right end point of Δ.

Figure 16.5.4 is a little oversimplified in that only two prices are possible in this two dimensional case unless it degenerates into an exchange economy. Keeping Example 15.4.1 in mind, we know that production is only possible on the two endpoints of Δ, and that once we know what is produced, we have determined the prices. You have to use some imagination to see how it applies with more dimensions.

16.5.13 Existence Proof: Continuity of Tâttonnement

We still need to show that $F_1 = G \circ \mathbf{h}$ is upper Vietoris continuous. We begin by considering $\mathbf{h}$. Because $\alpha(\mathbf{z}) = \max A(\mathbf{z})$, we can employ the Maximum Theorem here if we show $\mathbf{z} \twoheadrightarrow A(\mathbf{z})$ is lower Vietoris continuous.

Let $\alpha \in A(\mathbf{z})$ and $\varepsilon > 0$. We need to show that $(\alpha - \varepsilon, \alpha + \varepsilon) \cap A(\mathbf{z}') \neq \emptyset$ for $\mathbf{z}'$ in some neighborhood of $\mathbf{z}$. If $\alpha - \varepsilon \leq 0$, we are done as $0 \in A(\mathbf{z})$. Suppose $\alpha - \varepsilon > 0$. Now $\mathbf{z}'' = \bar{\mathbf{y}} + (\alpha - \varepsilon/2)(\mathbf{z} - \bar{\mathbf{y}}) \in \operatorname{int} Y$ by free disposal and the fact that $\mathbf{z} - \bar{\mathbf{y}} \gg \mathbf{0}$. Choose ε' so that the ball of radius ε' about $\mathbf{z}''$ is contained in Y. Define $\delta = \varepsilon'/(\alpha - \varepsilon/2)$. If $\|\mathbf{z}' - \mathbf{z}\| < \delta$,

$$\left\|\bar{\mathbf{y}} + \left(\alpha - \frac{\varepsilon}{2}\right)(\mathbf{z}' - \bar{\mathbf{y}}) - \mathbf{z}''\right\| = \left(\alpha - \frac{\varepsilon}{2}\right)\|\mathbf{z}' - \mathbf{z}\| < \varepsilon',$$

so $\alpha - \varepsilon/2 \in A(\mathbf{z}')$. Thus $(\alpha - \varepsilon, \alpha + \varepsilon) \cap A(\mathbf{z}') \neq \emptyset$ for $\|\mathbf{z} - \mathbf{z}'\| < \delta$. The Maximum Theorem tells us that the function α is continous, and so is $\mathbf{h}$.

That leaves G, which is defined on the boundary of Y. Recall $G(\mathbf{y}) = \{\mathbf{p} \in \Delta : \mathbf{p}\cdot\mathbf{y}' \leq \mathbf{p}\cdot\mathbf{y} \text{ for all } \mathbf{y}' \in Y\}$. Clearly $G(\mathbf{y})$ is a closed correspondence. Since G's values are confined to the compact set Δ, G is upper Vietoris continuous, as is $F_1 = G \circ \mathbf{h}$.

The correspondence G has another important property. Because $G(\mathbf{y}) \subset \Delta \subset Y^*$, $\mathbf{p}\cdot\mathbf{y} \leq 0$ for every $\mathbf{p} \in G(\mathbf{y})$. If $\mathbf{p}\cdot\mathbf{y} < 0$, $\mathbf{p}\cdot(\mathbf{y}/2) > \mathbf{p}\cdot\mathbf{y}$, contradicting the definition of G. Thus $\mathbf{p}\cdot\mathbf{y} = 0$ for every $\mathbf{p} \in G(\mathbf{y})$.

16.5.14 Existence Proof: Fixed Points

Fixed Point: We now consider the tâtonnement map

$$F(\mathbf{p}, \mathbf{z}) = \Big(G\big(\mathbf{h}(\mathbf{z})\big), \mathbf{z}(\mathbf{p})\Big).$$

This is a closed correspondence on $\Delta \times T$. Since $\Delta \times T$ is compact, G is also upper Vietoris continuous by Corollary 33.5.12. Kakutani's Theorem then yields a fixed point $(\hat{\mathbf{p}}, \hat{\mathbf{z}})$. The fixed point obeys

$$\begin{aligned} \hat{\mathbf{p}} &\in G\big(\mathbf{h}(\hat{\mathbf{z}})\big) \\ \hat{\mathbf{z}} &\in \mathbf{z}(\hat{\mathbf{p}}) \end{aligned}$$

The pair $(\hat{\mathbf{p}}, \hat{\mathbf{z}})$ will be used to construct out candidate equilibrium. Define the production vector $\hat{\mathbf{y}}$ by $\hat{\mathbf{y}} = \mathbf{h}(\hat{\mathbf{z}}) \in Y$. By construction of $\hat{\mathbf{p}}$, $\hat{\mathbf{p}} \cdot \hat{\mathbf{y}} = 0$. Now $\hat{\mathbf{p}} \cdot (\hat{\mathbf{z}} - \hat{\mathbf{y}}) = 0$ since $\hat{\mathbf{p}} \cdot \hat{\mathbf{z}} = 0$. But $\hat{\mathbf{z}} - \hat{\mathbf{y}} = (\alpha(\hat{\mathbf{z}}) - 1)(\hat{\mathbf{z}} - \bar{\mathbf{y}})$. Since $\hat{\mathbf{z}} \gg \bar{\mathbf{y}}$ and $\hat{\mathbf{p}} > \mathbf{0}$, we have $\alpha(\hat{\mathbf{z}}) = 1$, implying $\hat{\mathbf{z}} = \hat{\mathbf{y}}$. Notice that profit is maximized at $\hat{\mathbf{y}}$ because $\hat{\mathbf{p}} \in G(\hat{\mathbf{y}})$.

16.5.15 Existence Proof: Positive Income in Equilibrium

Proof: Final Steps.

Positive Income: The next step is to replace the modified excess demand by ordinary excess demand. First, we strip away Debreu's modification. To do that, we must show that all consumers have positive income at $\hat{\mathbf{p}}$, that $\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i > 0$ for all i. At least some consumer must have positive income because $-\boldsymbol{\omega} \in \operatorname{int} Y$. We know that some component of $\hat{p}_\ell$ of $\hat{\mathbf{p}}$ is positive. If $\hat{\mathbf{p}}\cdot\boldsymbol{\omega} = 0$, we can find some $\mathbf{y} \in B_\varepsilon(-\boldsymbol{\omega}) \subset Y$ with $\hat{\mathbf{p}}\cdot\mathbf{y} > 0$. This is impossible since $\hat{\mathbf{p}}\cdot\mathbf{y} \le 0$ for all $\mathbf{y} \in Y$. It follows that $\hat{\mathbf{p}}\cdot\boldsymbol{\omega} > 0$. Some consumer must have positive income.

Now suppose there is a consumer with zero income and let $I_1 = \{i : \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i > 0\}$. Consider the allocation $\hat{\mathbf{x}}^i = \hat{\mathbf{z}}^i + \boldsymbol{\omega}^i$ obtained from the net trades $\hat{\mathbf{z}}^i$. The set $I_2 = \{i : \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i = 0\}$ is not empty by supposition. By irreducibility, there is $\mathbf{y}' \in Y$ and $\boldsymbol{\nu} \in -X_{I_2}$ with $\mathbf{x}^{I_1} + \hat{\mathbf{x}}^{I_2} = \mathbf{y}' + \boldsymbol{\omega} + \boldsymbol{\nu}$ and $\mathbf{x}^i \succ_i \hat{\mathbf{x}}^i$ for all $i \in I_1$. Then $\hat{\mathbf{p}}\cdot\mathbf{x}^i > \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i$ for each $i \in I_1$. Summing, we have

$$\begin{aligned}
\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^{I_1} &< \hat{\mathbf{p}}\cdot\mathbf{x}^{I_1} \\
&= \hat{\mathbf{p}}\cdot(\mathbf{y}' + \boldsymbol{\omega} - \hat{\mathbf{x}}^{I_2} + \boldsymbol{\nu}) \\
&\le \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^{I_1} + \hat{\mathbf{p}}\cdot\boldsymbol{\nu}
\end{aligned}$$

where we have used the fact that $\hat{\mathbf{p}}\cdot\mathbf{y}' \le 0$ and $\hat{\mathbf{p}}\cdot(\hat{\mathbf{x}}^i - \boldsymbol{\omega}^i) = 0$ for all $i \in I_2$. It follows that $\hat{\mathbf{p}}\cdot\boldsymbol{\nu} > 0$, so there is some $h \in I_2$ with $\hat{\mathbf{p}}\cdot\boldsymbol{\nu}^h > 0$. Now $\boldsymbol{\nu}^h = \boldsymbol{\omega}^h - \mathbf{x}^h$ for some $\mathbf{x}^h \in \mathfrak{X}_h$. Since $\mathbf{x}^h \in \mathbb{R}^m_+$, $0 \le \hat{\mathbf{p}}\cdot\mathbf{x}^h < \mathbf{p}\cdot\boldsymbol{\omega}^h = 0$. This contradiction shows that I_2 must be empty, that $\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i > 0$ for all $i = 1, \dots, I$.

Since every consumer has positive income, the definition of $\mathbf{z}^i(\mathbf{p})$ tells us that consumers maximize utility over their truncated budget sets at $\hat{\mathbf{z}}^i = \mathbf{z}^i(\hat{\mathbf{p}})$. The same method used in exchange economies can be used to show the truncation has no effect. This means that $\hat{\mathbf{z}}$ is the real excess demand. It follows that we have an equilibrium, which completes the proof. $\square$

16.5.16 A Simplified Equilibrium Existence Theorem

McKenzie's (1981) proof describes consumers via trading sets X_i that are closed, convex, and bounded below. These can be translated into our framework. Let $-\boldsymbol{\omega}^i$ be the greatest lower bound of X_i. Then $\mathfrak{X}_i = X_i + \boldsymbol{\omega}^i \subset \mathbb{R}^m_+$ and we can treat $\boldsymbol{\omega}^i$ as the endowment. The condition that $0 \in X_i$ translates directly to our condition that $\boldsymbol{\omega}^i \in \mathfrak{X}_i$.

Corollary 16.5.5. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, -\mathbb{R}^m_+)$ be an exchange economy. Suppose for each $i = 1, \dots, I$, $\mathfrak{X}_i = \mathbb{R}^m_+$; $\succsim_i$ is semi-strictly convex and continuous; $\boldsymbol{\omega}^i > 0$ and $\boldsymbol{\omega} \gg 0$. If either (1) $\succsim_i$ is strongly monotonic or (2) $\succsim_i$ is monotonic and $\boldsymbol{\omega}^i \gg 0$ for all i, then a Walrasian equilibrium $(\hat{\mathbf{p}}, \hat{\mathbf{x}}, \hat{\mathbf{y}})$ exists and $\hat{\mathbf{p}} \cdot \boldsymbol{\omega}^i > 0$ for all $i = 1, \dots, m$.*

Proof. By Proposition 16.4.2, $\mathcal{E}$ is irreducible. The hypotheses of the Equilibrium Existence Theorem: Production Economies are then satisfied, which proves existence of equilibrium.

16.6 Generalizations of the Existence Theorem

The existence theorems of Arrow-Debreu (1954) and McKenzie (1954) and their immediate successors did not end the search for equilibrium existence theorems. Rather, it started a long process of extending those theorems to encompass weaker assumptions concerning survival and preferences, and into new settings, involving externalities, public goods, nonconvexities, dynamic models, uncertainty, and models that could encompass product variety.

This section focus on the weakening of the survival and preference assumptions.

16.6.1 What about Survival?

As was the case for exchange economies, the Equilibrium Existence Theorem for Production Economies avoids grappling with the issue of survival. Each consumer has an endowment in their consumption set (equivalently, $0 \in X_i$). That means they can survive on their own resources. Of course, that is rather unrealistic in a modern economy. We previously quoted Adam Smith (1789) describing its implausibility in the 17^{th} century. He based his argument on the observed division of labor, and we know that at least some of the population already relied on the division of labor to thrive at least since the early Neolithic era.

If survival were in doubt, the meaning of the equilibrium would also be in doubt. If we try to relax the assumption that $\boldsymbol{\omega}^i \in \mathfrak{X}_i$, we run into the question of whether every consumer can afford to buy a bundle in his consumption set. If not, we have a problem. If consumers are not able to consume, are they really in the economy at all? How do they survive? If they don't survive, does it make sense to suppose they supply resources?

McKenzie (1981) proved another version of the existence theorem without the survival hypothesis that $\boldsymbol{\omega}^i \in \mathfrak{X}_i$ (i.e., $0 \in X_i$) under a strengthened irreducibility condition.[17] Irreducibility again guarantees positive income and ensures that each consumer can survive in equilibrium even if they cannot survive on their own. It ensures that they obtain a net trade in X_i (a consumption vector in $\mathfrak{X}_i$).[18]

In fact, the hypotheses of the theorem can still be considerably weakened. Not only is it possible to do without the free disposal assumption, but there is also no need for a complete preference order.[19] These possibilities were subsequently considered by McKenzie (1981).

[17] See also Moore (1975).

[18] See Hammond (1993) for more on the relation between irreducibility and survival.

[19] For more on the latter, see Mas-Colell (1974), Gale and Mas-Colell (1975), Shafer and Sonnenschein (1975), and Shafer (1976).

16.6.2 Survival and Modified Irreducibility

We will mostly work in terms of the net trading sets $X_i = \mathfrak{X}_i - \boldsymbol{\omega}^i$. If unaided survival is not possible for some consumer, $\boldsymbol{\omega}^i \notin \mathfrak{X}_i$ and so $0 \notin X_i$. Define the augmented trading set as the convex hull of 0 and X_i, $\overline{X}_i = \mathrm{co}(X_i \cup \{0\})$. If $0 \in X_i$, $\overline{X_i} = X_i$, but if $0 \notin X_i$, $\overline{X_i}$ is larger.

There is also a modified irreducibility assumption to go with the augmented trading sets.

Modified Irreducible Economy. An economy $\mathcal{E}$ obeys the *modified irreducibility* if for every non-trivial partition of consumers into two groups I_1 and I_2 and every $(\hat{\boldsymbol{x}}^i, \hat{\boldsymbol{y}}^f)$ with $\hat{\boldsymbol{x}}^{I_1} + \hat{\boldsymbol{x}}^{I_2} = \boldsymbol{\omega} + \hat{\boldsymbol{y}}$ with $\hat{\boldsymbol{x}}^{I_1} \in X_{I_1}$, $\hat{\boldsymbol{x}}^{I_2} \in \overline{X}_{I_2}$ and $\hat{\boldsymbol{y}} \in Y$, there exists a $\boldsymbol{v} \in -\overline{X}_{I_2}$ and $\boldsymbol{y} \in Y$ such that $\boldsymbol{x}^{I_1} + \hat{\boldsymbol{x}}^{I_2} = \boldsymbol{y} + \boldsymbol{\omega} + \boldsymbol{v}$ and $\boldsymbol{x}^h \in P_h(\hat{\boldsymbol{x}}^h)$ for all $h \in I_1$.

16.6.3 No Preference Orders!

Perhaps the most interesting modification is to dispense with preference orders! In section 2.3, we saw that completeness and transitivity are not needed for demand theory. We examined utility obeying the Shafer assumptions, which could be represented using a k where $k(\mathbf{x}, \mathbf{y}) > 0$ if and only if $\mathbf{x} \succ \mathbf{y}$. In section 4.4, we found that such preferences yield Marshallian demands which need not obey Slutsky symmetry.

All that is really needed for our purposes is the idea of strict preference together with continuity and convexity. Define the preference correspondence on the net trading set X_i by $P_i(\mathbf{x}^i) = \{\mathbf{z}^i \in X_i : \mathbf{z}^i \succ_i \mathbf{x}^i\}$. Since $P_i(\mathbf{x}^i)$ must be non-empty, the fact that P_i is a correspondence implies global non-satiation.

Instead of convexity, we will require that $\mathbf{x}^i \notin \operatorname{co} P_i(\mathbf{x}^i)$, which is implied by semi-strict convexity. As for continuity, we require that the upper contour set $P_i(\mathbf{x}^i)$ be open, and that the correspondence P_i be lower Vietoris continuous.[20] Both of these continuity conditions are automatically true for continuous preference orders.

Finally, we relax our assumption on Y to eliminate the free disposal requirement and replace the assumption that $-\boldsymbol{\omega} \in Y$ with $\operatorname{ri} X \cap \operatorname{ri} Y \neq \emptyset$ where $\operatorname{ri} A$ denotes the interior relative to the affine hull of A.

[20] Open-valued correspondences on connected spaces are not upper Vietoris continuous unless they are constant.

16.6.4 A General Equilibrium Existence Theorem

We can now state McKenzie's (1981) result.

Existence of Equilibrium in Production Economies (II). *Consider a CRS production economy* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)$ *and let* $X_i = \mathfrak{X}_i - \boldsymbol{\omega}^i$. *Suppose that*

1. *Each* X_i *is closed, convex, and bounded below.*
2. *For each* i, P *is a lower Vietoris continuous and open-valued correspondence on* X_i *with* $\mathbf{x}^i \notin \operatorname{co} P_i(\mathbf{x}^i)$.
3. *The aggregate production set* Y *is a non-empty closed convex cone obeying inaction, and no-free lunch.*
4. $\operatorname{ri} X \cap \operatorname{ri} Y \neq \emptyset$.
5. $\mathcal{E}$ *is modified irreducible.*

Then a Walrasian equilibrium $(\hat{\mathbf{p}}, \hat{\mathbf{z}}, \hat{\mathbf{y}})$ *exists. Moreover,* $\hat{\mathbf{p}} \cdot \boldsymbol{\omega}^i > 0$ *for all* $i = 1, \dots, I$.

16.7 Constant Returns is All You Need

There is one major weakness in the existence theorems of the previous section. They require constant returns to scale. In this section we show that any economy with diminishing returns production may be converted to a constant returns model by introducing "entrepreneurial factors" à la McKenzie (1959). Moreover, an equilibrium in either economy translates to an equilibrium in the other.

The use of constant returns to scale simplifies our problems in some ways. If profit is finite (i.e., zero), the supply correspondence exists. In the constant returns to scale world, there are no pesky cases where profits are bounded but cannot be maximized. There are also no profits to distribute and prices are restricted to the polar cone of the aggregate production set. There is also a cost as we must now deal with correspondences rather than functions. That is why we used the Kakutani Fixed Point Theorem in the previous section instead of something simpler such as the Brouwer Fixed Point Theorem.

16.7.1 Augmented Technology Sets

To carry out the conversion to a constant returns to scale technology, we start with the firms. If we have firms $Y_1, \dots, Y_f$. We introduce F entrepreneurial factors. The augmented technology set for firm f is $\hat{Y}_f = \mathrm{cl}\{(z\mathbf{y}, -z\mathbf{e}_f) : \mathbf{y} \in Y, z > 0\}$. Of course, $\hat{Y}_f \subset \mathbb{R}^{m+F}$. The aggregate technology set is $\hat{Y} = \sum_{f=1}^{F} \hat{Y}_f$. If $\hat{Y}$ is irreversible, it is closed, and we have translated the aggregate production set into a CRS form.

We augment the consumers by extending preferences to $\mathbb{R}_+^{L+F}$ via indifference and replacing their endowments by $\big(\boldsymbol{\omega}^i, (\theta_f^i)\big)$. This translates any economy with convex production sets into an economy with CRS convex production sets.

There is one potential problem with the new economy. We haven't verified that all of the appropriate assumptions are satisfied. As far as the consumers are concerned, everything is okay. They inherit the appropriate properties from the diminishing returns economy. The production set $\hat{Y}$ is a different matter. There are two properties that present a potential problem, closure and irreversibility.

16.7.2 Augmented Production and Irreversibility

Irreversibility is not too hard to handle. Any vector $(\mathbf{y}, -\mathbf{z}) \in T$ with $\mathbf{z} > \mathbf{0}$ is clearly irreversible since $\mathbf{z} < \mathbf{0}$ is not permitted. Vectors of the form $(\mathbf{y}, \mathbf{0}) \in \hat{Y}$ come from summing elements $(\mathbf{y}^f, \mathbf{0}) \in \hat{Y}_f$. By Lemma 14.3.6 the $\mathbf{y}^f$ are CRS production vectors. It now suffices to impose the restriction that $Y = \sum_f Y_f$ contains no straight lines.[21] As the following lemma demonstrates, $\hat{Y}$ is then irreversible.

Lemma 16.7.1. *Suppose $\hat{Y}_f$ is the augmented technology set corresponding to a DRS technology set Y_f. If $Y = \sum_f Y_f$ contains no straight lines through the origin, then $\hat{Y} = \sum_f \hat{Y}_f$ is irreversible. Moreover, $\hat{Y}$ is closed.*

Proof. By way of contradiction, suppose $\sum_f \hat{Y}_f$ contains a straight line. We know that it must have the form $(t\mathbf{y}, \mathbf{0})$ since the entrepreneurial factors must always be inputs. Thus $(\mathbf{y}, \mathbf{0}) \in \hat{Y}$ and $(-\mathbf{y}, \mathbf{0}) \in \hat{Y}$. There are then constant returns elements $\mathbf{y}^f, \hat{\mathbf{y}}^f \in Y_f$ with $\sum_f \mathbf{y}^f = \mathbf{y}$ and $\sum_f \hat{\mathbf{y}}^f = -\hat{\mathbf{y}}$. Both $\mathbf{y}, -\mathbf{y} \in Y$. Now $\mathbf{y}$ and $-\mathbf{y}$ are CRS elements of Y, which implies Y contains a straight line through the origin. This is impossible! Therefore $\sum_f \hat{Y}_f$ must be irreversible.

Proposition 14.4.8 now tells us that $\hat{Y}$ is closed. $\square$

[21] Debreu (1959) imposes this on page 40.

16.7.3 CRS Economies

Now we are ready to derive an equilibrium in an economy with convex production from an equilibrium in the associated constant returns economy, and vice-versa.

CRS Economy. Given an economy $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y_f, \theta^i_f)$, define the *CRS economy generated by* $\mathcal{E}$, $\mathrm{CRS}(\mathcal{E}) = \big(\mathfrak{X}'_i, \succsim'_i, (\boldsymbol{\omega}^i, \boldsymbol{\theta}^i), \hat{Y}\big)$, by:

1. $\mathfrak{X}'_i = \mathfrak{X}_i \times \mathbb{R}^F_+$. Write elements of $\mathfrak{X}'_i$ as $(\boldsymbol{x}, \boldsymbol{\nu})$ with $\boldsymbol{x} \in \mathfrak{X}_i$ and $\boldsymbol{\nu} \in \mathbb{R}^F_+$. Here $\boldsymbol{x}$ is a bundle of ordinary goods and $\boldsymbol{\nu}$ is a bundle of entrepreneurial factors.
2. Consumer i's endowment of entrepreneurial factors is $\boldsymbol{\theta}^i = (\theta^i_1, \dots, \theta^i_F)$.
3. Consumer i's preferences $\succsim'_i$ are given by $(\boldsymbol{x}', \boldsymbol{\nu}') \succsim'_i (\boldsymbol{x}, \boldsymbol{\nu})$ if and only if $\boldsymbol{x}' \succsim \boldsymbol{x}$.
4. Firm f has augmented technology set $\hat{Y}_f = \mathrm{cl}\{(z\boldsymbol{y}, -z\boldsymbol{e}^f) : \boldsymbol{y} \in Y_f, z > 0\}$. The aggregate technology is $\hat{Y} = \sum_f \hat{Y}_f$.

Condition (1) expands the consumption set to accommodate entrepreneurial factors while (2) endows each consumer with shares of the entrepreneurial factors that are equal to their firm shares in $\mathcal{E}$. By (3), only ordinary goods affect preferences. Each consumer i is indifferent concerning holdings of the entrepreneurial factors. This means that if the price of an entrepreneurial factor is strictly positive, consumers will supply all of it they have in order to earn maximum income. Condition (4) tells us that entrepreneurial factor f corresponds to firm f.

16.7.4 Equivalence of CRS and Walrasian Equilibria

The key result is that any equilibrium in CRS($\mathcal{E}$) is also an equilibrium in $\mathcal{E}$, and vice-versa.

Theorem 16.7.2. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y_f, \theta^i_f)$ be an economy where each Y_f is a convex production set and consumers are locally non-satiated. Suppose $Y = \sum_f Y_f$ contains no straight lines through the origin. If*

$$\big((\hat{\mathbf{x}}^i, \hat{\boldsymbol{\nu}}^i), (\hat{\mathbf{y}}^f, -\hat{\boldsymbol{z}}^f), (\hat{\mathbf{p}}, \hat{\mathbf{q}})\big)$$

is a Walrasian equilibrium in CRS($\mathcal{E}$). *Write $\hat{\mathbf{y}} = \sum_f \hat{\mathbf{y}}^f$. Then $q_f = \pi_f(\hat{\mathbf{p}})$ and $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f, \hat{\mathbf{p}})$ is a Walrasian equilibrium in $\mathcal{E}$.*

Conversely, if $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f, \hat{\mathbf{p}})$ is a Walrasian equilibrium in $\mathcal{E}$, then

$$\big((\hat{\mathbf{x}}^i, \mathbf{0}), (\hat{\mathbf{y}}^f, -\boldsymbol{e}), (\hat{\mathbf{p}}, \boldsymbol{\pi}(\hat{\mathbf{p}}))\big)$$

is a Walrasian equilibrium in CRS($\mathcal{E}$).

16.7.5 Proof of Equivalence Theorem I

Proof. The fact that Y contains no straight lines implies that $\hat{Y}$ is closed by Lemma 16.7.1. It is east to see that $\hat{Y}$ satisfies all of the properties of a production set except no free lunch.

If $\hat{\mathbf{y}} \in \hat{Y}$ with $\hat{\mathbf{y}} \geq \mathbf{0}$, then we may write $\hat{\mathbf{y}} = (\mathbf{y}, \mathbf{0})$ for some $\mathbf{y} \in Y$. Then $\hat{\mathbf{y}} = \sum_f (\mathbf{y}^f, \mathbf{0})$ for some $\mathbf{y}^f in Y_f$. By Lemma 14.3.6, each $\mathbf{y}^f$ is a CRS element of Y_f. Thus $\mathbf{y} \in Y$ for all $t > 0$. Free disposal then yields $t\mathbf{y} \in Y$ for all t. Since Y contains no straight lines through the origin, $\mathbf{y} = \mathbf{0}$, which shows that $\hat{Y}$ obeys the no free lunch condition.

Proposition 14.5.2 tells us that maximizing profit $(\hat{\mathbf{p}}, \hat{\mathbf{q}})\cdot(\mathbf{y}, -\mathbf{z})$ over $\hat{Y}$ entails maximizing $(\hat{\mathbf{p}}, \hat{\mathbf{q}})\cdot(\mathbf{y}, -\mathbf{z})$ over each $\hat{Y}_f$. This allows us to write $(\hat{\mathbf{y}}, -\hat{\mathbf{z}}) = \sum_f (\hat{\mathbf{y}}^f, -\hat{\mathbf{z}}^f)$ where $(\hat{\mathbf{y}}^f, -\hat{\mathbf{z}}^f) \in \hat{Y}_f$ maximizes profit over $\hat{Y}_f$ for each firm $f = 1, \dots, F$. It follows that $(\hat{\mathbf{p}}, \hat{\mathbf{q}}) \in (\hat{Y}_f)^*$ for every firm f.

By construction, $\hat{z}^f \geq 0$. We now show that $\hat{q}_f = \pi_f(\hat{\mathbf{p}})$. There are two cases. If $\hat{z}^f > 0$, Proposition 14.3.11 tells us that $\hat{q}_f = \pi_f(\hat{\mathbf{p}})$. The other case is $\hat{z}^f = 0$. We can't have $\hat{q}_f > 0$ since the consumers will then collectively supply one unit of the entrepreneurial factor, creating excess supply. But in equilibrium, excess supply can only occur when the price is zero by Corollary 15.2.5, implying $\hat{q}_f = 0$. Thus $\hat{q}_f$ must be 0 when $\hat{z}^f = 0$. That forces profit to be zero since $0 \leq \pi_f(\hat{\mathbf{p}}) \leq \hat{q}_f = 0$. This case also has $\hat{q}_f = \pi_f(\hat{\mathbf{p}})$.

Proof continues...

16.7.6 Proof of Equivalence Theorem II

Rest of Proof. Now that we know $\hat{q}_f = \pi_f(\hat{\mathbf{p}})$ for each firm f, and that firm f is maximizing profit, we obtain $\hat{\mathbf{p}} \cdot \hat{\mathbf{y}}^f = \pi_f(\hat{\mathbf{p}})$. Each firm f in $\mathcal{E}$ maximizes profit at $\hat{\mathbf{y}}^f$ when the price vector is $\hat{\mathbf{p}}$.

Consumer i's endowment of entrepreneurial factor f is θ_f^i, so the income earned from the entrepreneurial factor is $\theta_f^i \hat{q}_f = \theta_f^i \hat{\mathbf{p}} \cdot \hat{\mathbf{y}}^f$. Consumer i's income in the CRS economy is $\hat{\mathbf{p}} \cdot \boldsymbol{\omega}^i + \sum_f \theta_f^i \hat{\mathbf{p}} \cdot \hat{\mathbf{y}}^f$. It is exactly the same as in the DRS economy. It follows that the consumer will choose the same bundle $\hat{\mathbf{x}}^f$ of non-entrepreneurial goods in either form of the economy.

Market clearing for the non-entrepreneurial goods in CRS($\mathcal{E}$) gives us market clearing in $\mathcal{E}$. Firms maximize profits, consumers maximize utility, and markets clear. Thus $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f, \hat{\mathbf{p}})$ is an equilibrium in $\mathcal{E}$.

For the converse, note profit maximization implies that $\hat{\mathbf{p}} \in Y_f^\circ$ for each f. Thus $(\hat{\mathbf{p}}, \boldsymbol{\pi}(\hat{\mathbf{p}})) \in \hat{Y}_f$ for all f. Since polars and duals are the same for cones, $(\hat{\mathbf{p}}, \boldsymbol{\pi}(\hat{\mathbf{p}})) \in \hat{Y}$ by Proposition 14.2.13. Now $(\hat{\mathbf{y}}^f, \pi_f(\hat{\mathbf{p}})\mathbf{e}_f)$ obeys the zero profit condition for $\hat{Y}_f$, so it maximizes profit for firm f. On the consumer side, consumers supply their entire endowments of entrepreneurial factors, so income is the same as in $\mathcal{E}$. It follows that $(\hat{\mathbf{x}}^i, \mathbf{0})$ maximizes consumer i's utility for each i. Market clearing for ordinary goods is the same as in $\mathcal{E}$. Since $\sum_i \theta_f^i = 1$, one unit of each entrepreneurial factor is supplied. One unit is also demanded, so the entrepreneurial factor markets clear. In sum, firms maximize profit, consumers maximize utility, and market clear. We have an equilibrium in CRS($\mathcal{E}$). $\square$

16.7.7 Understanding Equivalence

Since they yield the same equilibria in terms of ordinary goods, it does not matter whether we use the DRS or CRS versions of the economy. Without loss of generality, we can confine our attention to CRS economies which are technically a bit simpler.

We return to the production functions from Examples 14.2.1 and 14.3.12 one last time, now in an equilibrium context. We impose perfect substitution in an attempt to force the prices of input and output to be the same. That is, we try to force the equilibrium into the case where profit cannot be maximized in the diminishing returns version. The constant returns version has inaction as the only solution. As the example shows, this fails. But an unexpected equilibrium appears.

16.7.8 Example: An Unexpected Equilibrium I

Example 16.7.3: An Unexpected Equilibrium We consider the constant returns version of the production function from Examples 14.2.1 and 14.3.12. There are three goods. Good one is the output and goods two and three are inputs. We write the net output vector as $(q, -z_1, -z_2)$. The production set is described by the production function $F(z_2, z_3) = z_1 + z_2 + z_2^2/(z_1 + z_2)$.

There are two identical consumers, with utility $u_i(\mathbf{x}) = x_1 + x_2$ and endowments $\boldsymbol{\omega}^i = (1, 1, \frac{1}{2})^{\mathsf{T}}$. One might think that this would force goods one and two to have the same price, but if they did, the only profit maximizing net output would be zero. In that case there would be excess supply of good three, which would then have price zero in equilibrium. But the price of good three must be at least the common price of goods one and two, as we saw in Example 14.3.12. Since this fails, goods one and two must have different prices in equilibrium.

We can go further and conclude that z_1 must be more expensive. Consider the diminishing returns version of the production set as illustrated in Figure 16.7.4. Since z_1 can produce more that z_1 units of output q, q must be cheaper. As q and z_1 are perfect substitutes in consumption, only the cheaper q is consumed.

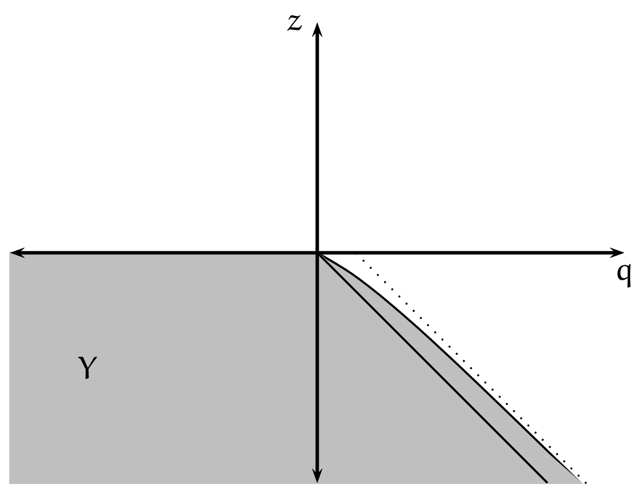

Figure 16.7.4: The diagram shows the production set $\{(q, -z) : q \leq f(z), z \geq 0\}$ where $f(z) = 1 + z - 1/(1 + z)$. The production set is asymptotic to the dotted line. The ray $(z, -z)$ is interior to Y, showing that the input z is can be used to produce more than one unit of output. Since the goods are perfect substitutes, z is more valuable to consumers in production than consumption. This is why the price of z must be more that the price of the output in equilibrium. Since q and z are perfect substitutes for consumers, this means that they only consume the cheaper output good, q.

16.7.9 Example: An Unexpected Equilibrium II

Now that we understand how the equilibrium works, take good one as numéraire and write the price vector as $\mathbf{p} = (1, w, p)$. The firm first-order conditions for profit maximization are

$$
\begin{aligned}
w &= 1 + \frac{z_2^2}{(z_1 + z_2)^2} \\
p &= 1 - \frac{2z_2}{z_1 + z_2} + \frac{z_2^2}{(z_1 + z_2)^2}.
\end{aligned}
$$

Clearly, the price of good two obeys $w > 1$. This means that the consumers will only consume good one. All of good two will be used for the production of good one. We also must have market clearing for good three. Thus $z_1 = 2$ and $z_2 = 1$. The resulting output is $8/3$, so the net output vector is $\mathbf{y} = (8/3, -2, -1)^{\mathsf{T}}$, leaving $\mathbf{x} = \mathbf{y} + \boldsymbol{\omega} = (14/3, 0, 0)^{\mathsf{T}}$. Since the consumers have the same incomes, each will consume half: $\mathbf{x}^1 = \mathbf{x}^2 = (7/3, 0, 0)^{\mathsf{T}}$.

We need to make sure they can afford this. We can now calculate the prices: $w = 10/9$ and $p = 4/9$. Consumer income is then $\mathbf{p}\cdot\boldsymbol{\omega}^i = 1 + w + p/2 = 7/3$, which is just enough to be able to afford $\mathbf{x}^1 = \mathbf{x}^2$.

Summing up, the equilibrium price vector is $\mathbf{p} = (1, 10/9, 4/9)$, net output is $\mathbf{y} = (8/3, -2, -1)$ and the consumption vectors are $\mathbf{x}^i = (7/3, 0, 0)$. Although we picked utility functions in an attempt to force goods one and two to have the same price, and prevent profit maximization, we failed. The equilibrium conditions prevent this. The prices of goods one and two are forced apart, with good one cheaper than good two. ◀

March 12, 2023