19. Pareto Optimality

March. 14, 2023

> "It is not from the benevolence of the butcher, the brewer, or the baker, that we can expect our dinner, but from their regard to their own interest. We address ourselves, not to their humanity but to their self-love, and never talk to them of our own necessities but of their advantage. – Adam Smith, *Wealth of Nations*, Bk.1, Chap.2"

The quote from Adam Smith reminds us that people engage in economic activity for their own benefit. It suggests a way to judge the performance of an economy. We ask two questions: (1) Do people gain from trade? (2) Have they realized all possible gains from trade?

Section one introduces the notion of Pareto optimality, focusing on Edgeworth boxes. In section two, we examine Pareto optimality in exchange economies. We consider economies with production in section three.

Outline:

1. Pareto Optimality in the Edgeworth Box
2. Pareto Optima in Exchange Economies
3. Pareto Optima in Production Economies

19.1 Pareto Optimality in the Edgeworth Box

We use the Edgeworth box to examine gains from trade. In the Edgeworth box, moves to the northeast or southwest transfer some amount of both goods to consumer one, or consumer two, respectively. These are not trades. They may be gifts, taxes, or payments of tribute, but they are not trades.

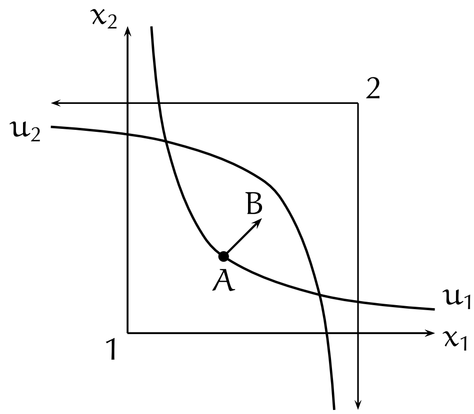

Figure 19.1.1: Moving from A to B makes consumer one worse off, and consumer two better off. Reversing that, moving back to A from B makes consumer one better off and consumer two worse off.

In contrast, moves to the northwest or southeast moves between allocations can be interpreted as trades. What consumer one gives up, consumer two receives, and vice-versa. Whether the consumer gains or loses from this trade depends on how the value of the goods received compares to the value of those given up.

19.1.1 Mutual Gains from Trade

Where does self-interest come in? We look for trades that each person prefers, trades that increase utility. We start from the current allocation and ask which trades are mutually advantageous. These **mutually advantageous** trades are referred to as *Pareto improvements*.

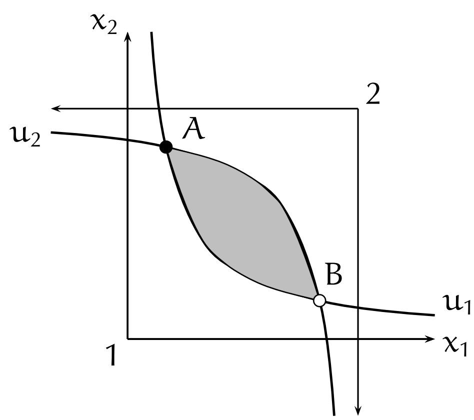

Figure 19.1.2: The economy starts at point A (the endowment). The shaded area shows the region of **mutual advantage**, of so-far unrealized gains from trade, potential Pareto improvements. The boundaries are included, but points A and B are not. Both consumer's have exactly the same utility at A and B. Neither consumer has gained.

19.1.2 Pareto Improvements

More formally, one allocation *Pareto improves* over another if two conditions hold.

1. Every person is at least as well off in the new allocation.
2. At least one person is better off.

Extending the definition to arbitrary economies gives us:

Pareto Improvement. Let $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ be an allocation in an economy $\mathcal{E}$. Another allocation $(\mathbf{x}^i, \mathbf{y}^f)$ is a *Pareto improvement* if

1. Every consumer i is at least as well off at $(\mathbf{x}^i, \mathbf{y}^f)$ as at $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$.
2. At least one consumer is better off.

I.e., $\mathbf{x}^i \succsim_i \hat{\mathbf{x}}^i$ for every i, and there is a consumer j with $\mathbf{x}^j \succ_j \hat{\mathbf{x}}^j$.

Returning to Figure 19.1.2, being at least as well off as at A means that each consumer is on or above their indifference curve. This is the shaded region. Point B is not a Pareto improvement. Although both consumers are at least as well off as at A, neither is better off than at A. They are both indifferent. Any movement into the unshaded area will make at least one of the consumers worse off.

19.1.3 Pareto Optima

Pareto improvements represent mutual gains from trade. If we can find an allocation where no Pareto improvement is possible, we will have exhausted all possible mutual gains from trade. Once you have reached this point, one party can only reap further gains by exploiting the other.

Pareto Optimal. A feasible allocation $(\hat{x}^i, \hat{y}^f)$ is *Pareto optimal* if no Pareto improvement is possible, i.e., if there is no other feasible allocation (x^i, y^f) with $x^i \succsim_i \hat{x}^i$ for all i and $x^j \succ_j \hat{x}^j$ for some j.

In general, a feasible allocation is Pareto optimal if there is no way to rearrange both consumption and production so that everyone is at least as well off and at least one person is strictly better off.

19.1.4 A Pareto Optimum in an Edgeworth Box

A Pareto optimal allocation in a two-person exchange economy is illustrated below.

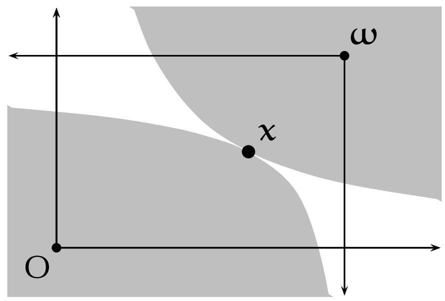

Figure 19.1.3: The allocation $(x, \omega - x)$ is Pareto optimal in this Edgeworth box. The region that represents an improvement for consumer 1 lies above and to the right of x while the region of improvement for consumer 2 lies below and to the left of x. Since these regions do not intersect, no Pareto improvements are possible, there are no unrealized gains from trade. Note that the indifference curves are tangent. If they cross, it is easy to see that the regions of improvement must intersect, making a Pareto improvement possible as in Figure 19.1.2

19.1.5 Individual Rationality

The individual endowments and firm shares play **no role** in determining whether an allocation is Pareto optimal. The only way they affect the problem is by determining feasibility.

Although Pareto optimality captures the idea of exhausting trading possibilities, it fails to incorporate an important component of the Smith quote at the head of the chapter. It does not speak to question (1). Do consumers actually gain by trade? That idea is covered by a related concept that pays attention to the individual endowments, individual rationality. In an exchange economy, an allocation is *individually rational* if each person is at least as well off as they would be with their endowment alone. We examine this idea further in Chapter 21.

19.1.6 Finding Pareto Optima

At a Pareto optimum, it is impossible to make one person better off without making someone else worse off. Thus we can find Pareto optima by considering the following maximization problems:

$$\begin{aligned} &\max u_j(\mathbf{x}^j) \\ &\text{s.t. } u_i(\mathbf{x}^i) \geq \bar{u}^i \quad \text{for } i \neq j \\ &\qquad (\mathbf{x}^1, \dots, \mathbf{x}^I) \text{ is a feasible allocation} \end{aligned}$$

The set of Pareto optima is referred to as the *Pareto set*. We denote the Pareto set of an economy $\mathcal{E}$ by $\mathcal{P}(\mathcal{E})$.

If the set of feasible allocations is compact, the continuity of utility ensures that we are maximizing over a compact set provided it is feasible to give utility at least $\bar{u}^i$ to each individual. In that case, this problem must have a solution by the Weierstrass Theorem. Thus, Pareto optima exist.

19.2 Pareto Optima in Exchange Economies

In the case of exchange economies with $\mathcal{C}^1$ utility, it is easy to characterize the **interior** Pareto optima, those inside the Edgeworth box, where each $x_k^i > 0$. For **interior** optima, we can ignore the boundary constraint and use the simplified Lagrangian[1]

$$\mathcal{L} = u_j(\mathbf{x}^j) + \sum_{i \neq j} \lambda_i \Big(u_i(\mathbf{x}^i) - \bar{u}^i \Big) - \sum_{\ell} \mu_\ell \left(\sum_{i=1}^{I} x_\ell^i - \omega_\ell \right).$$

The first-order conditions are then

$$0 = \frac{\partial \mathcal{L}}{\partial x_\ell^j} = \frac{\partial u_j}{\partial x_\ell^j} - \mu_\ell$$

and

$$0 = \frac{\partial \mathcal{L}}{\partial x_\ell^i} = \lambda_i \frac{\partial u_i}{\partial x_\ell^i} - \mu_\ell$$

for all $i \neq j$. By rearranging and dividing, we find that

$$\text{MRS}_{k\ell}^i = \text{MRS}_{k\ell}^j = \frac{\mu_k}{\mu_\ell},$$

for all k and ℓ. The marginal rates of substitution **must be the same for all consumers**. In the two-person case, this expresses the fact that the indifference curves must be tangent at interior Pareto optima. It embodies the idea of *consumption efficiency*, that there is no way to reallocate the existing consumption goods in a way that makes one person better off without making another worse off.

[1] We will include the boundary conditions later when we examine production economies.

19.2.1 Linear Utility

Armed with this fact, we can find the Pareto optima for linear utility functions.

Example 19.2.1: Pareto Set, Linear Utility. Suppose $u_1(\mathbf{x}^1) = 2x_1^1 + x_2^1$ and $u_2(\mathbf{x}^2) = x_1^2 + 2x_2^2$ and suppose the total endowment is $\boldsymbol{\omega} = (5, 3)$. Here $MRS_{12}^1 = 2$ and $MRS_{12}^2 = 1/2$. The indifference curves can never be tangent. It follows that the Pareto optima can only lie on the boundary of the Edgeworth box.

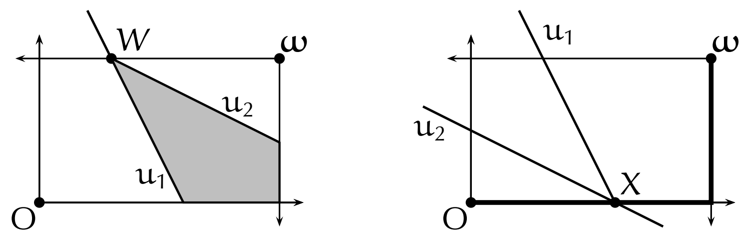

Figure 19.2.2: Here utility is linear. In the left panel, the point W **is not** Pareto optimal. All the shaded allocations to the southeast are Pareto improvements.

In the right panel, the point X **is** Pareto optima. Improvements for consumer 1 lie above the u_1 line while improvements for consumer 2 lie below u_2. If it were feasible, the region of mutual improvement would be below and to the right of X. As this is not feasible, X is Pareto optimal.

Of course, giving everything to either person 1 or person 2 will be Pareto optimal. All that is left is to determine what happens on the sides of the box.

On the upper boundary, consumer 1 has both goods while consumer 2 has only good 1. The subjective value of one unit of good 1 is $MRS_{12}^1 = 2$ units of good 2 to consumer 1 and $MRS_{12}^2 = 1/2$ units of good 2 to consumer 2. It follows that both consumers gain if consumer 1 gives up 1 unit of good 2 for one unit of good 1. The upper boundary cannot be Pareto optimal. Similar arguments apply on the left boundary. However, on the bottom, these trades are not possible, and both the bottom and right-hand side of the Edgeworth box are Pareto optimal.

19.2.2 Pareto Set: Two Identical Cobb-Douglas Consumers I

Example 19.2.3: Pareto Set: Cobb-Douglas Utility. Consider the setting of Example 15.3.4. Utility is identical for each consumer, with $u_1(\mathbf{x}^1) = (x_1^1)^\gamma (x_2^1)^{1-\gamma}$ and $u_2(\mathbf{x}^2) = (x_1^2)^\gamma (x_2^2)^{1-\gamma}$ with $0 < \gamma < 1$. The aggregate endowment is $\boldsymbol{\omega} = (5, 3)$.

Before solving this in general, note that if the allocation is on the boundary of the Edgeworth box, at least one consumer must get 0 of some good. That means their utility is zero. If their consumption of everything is not zero, we can make a Pareto improvement by giving the any excess goods to the other person, who will then have all of the goods and so have increase their utility.

There are only places where one consumer has nothing. The origin and the social endowment $\boldsymbol{\omega}$. Those are the only two Pareto optimal allocations on the boundary. All other Pareto optima must be interior allocations.

We can find all Pareto optima by solving

$$\begin{aligned} &\max u_1(\mathbf{x}^1) \\ &\text{s.t. } u_2(\mathbf{x}^2) \geq \bar{u}_2 \\ &\qquad \sum_i x_l^i = \omega_\ell, \ell = 1, 2 \end{aligned}$$

Using $\mathbf{x}^2 = \boldsymbol{\omega} - \mathbf{x}^1$, this reduces to

$$\begin{aligned} &\max u_1(\mathbf{x}^1) \\ &\text{s.t. } u_2(\omega - \mathbf{x}^1) \geq \bar{u}_2. \end{aligned}$$

19.2.3 Pareto Set: Two Identical Cobb-Douglas Consumers II

We have already considered boundary points, so we focus on interior Pareto optima. The Lagrangian for those is

$$\mathcal{L} = (x_1^1)^\gamma (x_2^1)^{1-\gamma} + \lambda\big((5 - x_1^1)^\gamma (3 - x_2^1)^{1-\gamma} - \bar{u}_2\big).$$

The first-order conditions are

$$\begin{aligned}(x_1^1)^{\gamma-1}(x_2^1)^{1-\gamma} &= \lambda(5 - x_1^1)^{\gamma-1}(3 - x_2^1)^{1-\gamma}\\ (x_1^1)^{\gamma}(x_2^1)^{-\gamma} &= \lambda(5 - x_1^1)^{\gamma}(3 - x_2^1)^{-\gamma}.\end{aligned}$$

Dividing, we find

$$\frac{x_2^1}{x_1^1} = \frac{3 - x_2^1}{5 - x_1^1} = \frac{x_2^2}{x_1^2}.$$

By cross-multiplying the left two terms and simplifying, we obtain $5x_2^1 = 3x_1^1$.

The Pareto optima are on the diagonal of the Edgeworth box. Interestingly, this doesn't depend on γ. No matter what value γ takes in $(0, 1)$, we get the same set of Pareto optima.

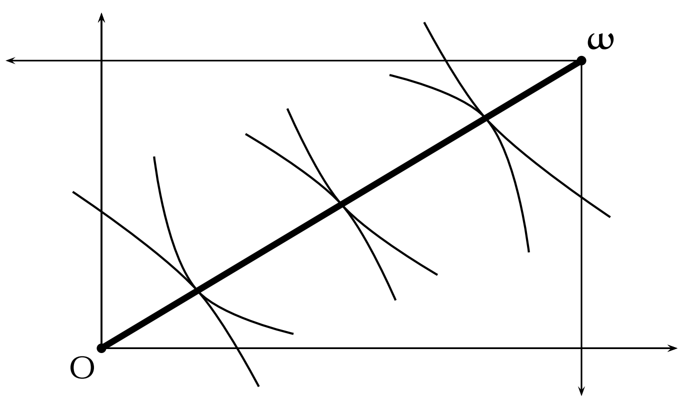

Figure 19.2.4: The heavy diagonal line is the Pareto Set of Example 19.2.3 for any γ. The indifference curve tangencies illustrated define three of its points for $\gamma = 2/3$. Although the Pareto optima are independent of $0 < \gamma < 1$, the shape of the indifference curves is affected by γ.

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19.2.4 Pareto Set: Two Identical Cobb-Douglas Consumers III

The parameter $\bar{u}_2$ (or equivalently, $\bar{u}_1$) determines which point we select in the Pareto set. We must have $(5 - x_1^1)^\gamma (3 - x_2^1)^{1-\gamma} = \bar{u}_2$. Since $5x_2^1 = 3x_1^1$ for all Pareto optima, we may substitute, obtaining $(3/5)^{1-\gamma}(5 - x_1^1) = \bar{u}_2$. Solving for x_1^1, we find

$$x_1^1 = 5 - (5/3)^{1-\gamma}\bar{u}_2.$$

The value of x_1^1 determines rest of the allocation:

$$\begin{aligned} x_1^2 &= (5/3)^{1-\gamma}\bar{u}_2, \\ x_2^1 &= 3 - (3/5)^\gamma \bar{u}_2, \\ x_2^2 &= (3/5)^\gamma \bar{u}_2. \end{aligned}$$

The utility levels are $u_1(\mathbf{x}^1) = 5^\gamma 3^{1-\gamma} - \bar{u}_2 = \bar{u}_1$ and $u_2(\mathbf{x}^2) = \bar{u}_2$ where $\bar{u}_2$ obeys $0 = u_2(\mathbf{0}) \le \bar{u}_2 \le u_2(\boldsymbol{\omega}) = 5^\gamma 3^{1-\gamma}$. Notice that $\bar{u}_1 + \bar{u}_2 = u(\boldsymbol{\omega})$. This happens whenever consumers have indentical Cobb-Douglas utility functions.

Keeping in mind that the utility here is homogeneous of degree one, we find

$$u\left(\frac{\bar{u}_i}{u(\boldsymbol{\omega})}\boldsymbol{\omega}\right) = \bar{u}_i.$$

We can now write the Pareto set as

$$\mathcal{P}(\mathcal{E}) = \left\{ \big(t_1\boldsymbol{\omega}, t_2\boldsymbol{\omega}\big) : t_i = \frac{\bar{u}_i}{u(\boldsymbol{\omega})}, 0 \le \bar{u}_i \le u_i(\boldsymbol{\omega}) \right\}.$$

19.2.5 Pareto Set: Identical Cobb-Douglas, Many Consumers I

We can consider the Pareto set in a general Cobb-Douglas setting where consumers have the same Cobb-Douglas utility function with the parameters summing to one. In that case, the Pareto set gives everyone a fraction of the endowment and the utilities again sum to the utility of the social endowment, as show in Example 19.2.5

Example 19.2.5: Pareto Set: Identical Cobb-Douglas Consumers. Suppose there are m goods and I consumers with identical Cobb-Douglas utility functions $u(x) = x_1^{\gamma_1} \cdots x_m^{\gamma_m}$ where each $\gamma_i > 0$ and $\sum_i \gamma_i = 1$. Let $\boldsymbol{\omega}^i$ be i's endowment and $\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}^i$ the social endowment. We assume $\boldsymbol{\omega} \gg \mathbf{0}$.

To find the Pareto set, we solve

$$\begin{aligned} &\max u(\mathbf{x}^1) \\ &\text{s.t. } u(\mathbf{x}^i) \geq \bar{u}_i, \quad i = 1, 2, \ldots, I \\ &\sum_i x_\ell^i = \omega_\ell, \quad \ell = 1, 2, \ldots, m \end{aligned}$$

As in Example 19.2.3, the boundary Pareto optima do not need special handling. We can use the Lagrangian

$$\mathcal{L} = (x_1^1)^{\gamma_1} \cdots (x_m^1)^{\gamma_m} + \lambda\big((x_1^2)^{\gamma_1} \cdots (x_m^2)^{\gamma_m} - \bar{u}_2\big) - \sum_\ell \mu_\ell \left(\sum_i x_\ell^i - \omega_\ell\right).$$

The resulting first-order conditions are:

$$\frac{\gamma_\ell u_1(\mathbf{x}^1)}{x_\ell^1} = \mu_\ell \quad \text{and} \quad \frac{\lambda \gamma_\ell u_i(\mathbf{x}^i)}{x_\ell^i} = \mu_\ell$$

for all $i = 2, \ldots, I$ and all $\ell = 1, \ldots, m$.

19.2.6 Pareto Set: Identical Cobb-Douglas, Many Consumers II

Dividing the first-order conditions for good ℓ by that for good 1, we obtain

$$\frac{x^1_\ell}{x^1_1} = \left(\frac{\mu_\ell}{\mu_1}\right)\left(\frac{\gamma_1}{\gamma_\ell}\right) \quad \text{and} \quad \frac{x^i_\ell}{x^i_1} = \left(\frac{\mu_\ell}{\mu_1}\right)\left(\frac{\gamma_1}{\gamma_\ell}\right).$$

We can rewrite these as $x^i_\ell = x^i_1 \xi_\ell$ by defining $\xi_\ell = (\mu_\ell/\mu_1)(\gamma_1/\gamma_\ell)$. Since the entire endowment is allocated,

$$\omega_\ell = \sum_i x^i_\ell = \xi_\ell \left(\sum_i x^i_1\right) = \xi_\ell \omega_1,$$

so $\xi_\ell = \omega_\ell/\omega_1$. We now set $t_i = x^i_1/\omega_1$ to obtain $x^i_\ell = t_i \omega_\ell$. Then each $t_i \geq 0$ and $\sum_i t_i = 1$. In other words, at every Pareto optimum, $\mathbf{x}^i = t_i \boldsymbol{\omega}$. Each consumer gets a share of the total endowment. The boundary cases merely have some shares being zero.

It follows that the Cobb-Douglas Pareto set is

$$\mathcal{P}(\mathcal{E}) = \left\{ (\mathbf{x}^1, \ldots, x^I) : \exists t_i \geq 0, \sum_i t_i = 1, \mathbf{x}^i = t_i \boldsymbol{\omega} \right\}.$$

19.2.7 Pareto Set: Identical Cobb-Douglas, Many Consumers III

We can also write the allocation in terms of the utility levels, $\bar{u}_i$. The $\bar{u}_i$ are known for $i = 2, \ldots, I$.

$$\bar{u}_i = \prod_\ell (x^i_\ell)^{\gamma_\ell} = t_i \left[\prod_\ell (\omega_\ell^{\gamma_\ell})\right] = t_i u(\boldsymbol{\omega}) \quad \text{for } i = 2, \ldots, I.$$

Moreover

$$u_1(x^1) = \prod_\ell (x^i_\ell)^{\gamma_\ell} = t_1 \left[\prod_\ell (\omega_\ell^{\gamma_\ell})\right] = t_1 u(\boldsymbol{\omega}),$$

so

$$u_1(x^1) + \sum_{i=2}^{I} \bar{u}_i = \sum_i t_i \left[\prod_\ell (\omega_\ell^{\gamma_\ell})\right] = \prod_\ell (\omega_\ell^{\gamma_\ell}).$$

The utility levels always sum to the utility of the total endowment, $u(\boldsymbol{\omega})$ since $\sum_i t_i = 1$,

$$\sum_i u(\mathbf{x}^i) = u(\boldsymbol{\omega}).$$

It follows that the utility allocations are in

$$\mathcal{U} = \left\{(u_1, \ldots, u_I) \in \mathbb{R}^I_+ : \sum_i u_i = u(\boldsymbol{\omega})\right\}.$$

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19.2.8 Pareto Set: Non-Identical Cobb-Douglas Utility

The last two examples yielded Pareto sets that are straight lines. This doesn't usually happen even if preferences are non-identical Cobb-Douglas.

Example 19.2.6: Curved Pareto Set. Suppose there are two consumers with endowments $\boldsymbol{\omega}^1 = \boldsymbol{\omega}^2 = (1,1)$ and utility functions $u_1(\mathbf{x}^1) = (x_1^1)^{1/2}(x_2^1)^{1/2}$ and $u_2(\mathbf{x}^2) = (x_1^2)^{1/3}(x_2^2)^{2/3}$. Using $x_1^1 + x_1^2 = 2$ and $x_2^1 + x_2^2 = 2$, we obtain the Lagrangian

$$\mathcal{L} = (x_1^1)^{1/2}(x_2^1)^{1/2} + \lambda\big((2 - x_1^1)^{1/3}(2 - x_2^1)^{2/3} - \bar{u}_2\big).$$

The first-order conditions are

$$\frac{1}{2}(x_1^1)^{-1/2}(x_2^1)^{1/2} = \frac{\lambda}{3}(2 - x_1^1)^{-2/3}(2 - x_2^1)^{2/3}$$

and

$$\frac{1}{2}(x_1^1)^{1/2}(x_2^1)^{-1/2} = \frac{2\lambda}{3}(2 - x_1^1)^{1/3}(2 - x_2^1)^{-1/3}.$$

Dividing to eliminate the multiplier λ yields

$$\frac{x_2^1}{x_1^1} = \frac{1}{2} \cdot \frac{2 - x_2^1}{2 - x_1^1} \quad \text{so} \quad x_2^1 = \frac{2x_1^1}{4 - x_1^1}$$

Here $0 \le x_1^1 \le 2$. The Pareto set is a convex curve connecting the corners of the Edgeworth box.

We complete this example by employing the utility constraint. Then $\bar{u}_2 = 2^{4/3}(2 - x_1^1)(4 - x_1^1)^{-2/3}$ pins down the value of x_1^1, and hence all the x_ℓ^i. Actually computing x_1^1 as a function of $\bar{u}_2$ involves solving a cubic equation.

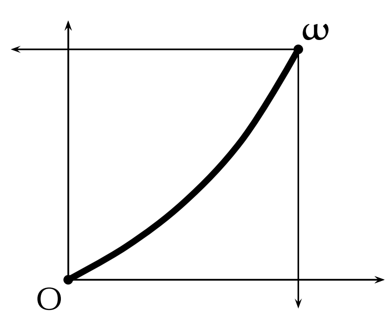

Figure 19.2.7: The heavy curve is the Pareto Set of Example 19.2.6.

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19.2.9 Pareto Set: Leontief Utility

Pareto sets need not be lines or curves. They can be thick sets.

Example 19.2.8: Pareto Optima with Leontief Consumers. We use the setting of Example 15.3.6 where $\boldsymbol{\omega} = (3, 2)$ and utility has the Leontief form $u_i(\mathbf{x}^i) = \min\{x_1^i, x_2^i\}$.

Suppose an allocation $(\mathbf{x}^1, \mathbf{x}^2)$ has $x_2^1 > x_1^1$ so $u_1(\mathbf{x}^1) = x_1^1$. Then $x_2^2 = 2 - x_2^1 < 3 - x_1^1 = x_1^2$ and $u_2(\mathbf{x}^2) = x_2^2$. We can increase the utility of person 2 by taking person 1's excess of good 2 and giving it to person 2. This Pareto improvement shows that no allocations with $x_2^1 > x_1^1$ can be Pareto optimal. Similarly, no allocation with $1 + x_2^1 < x_1^1$ can be Pareto optimal. Now consider allocations with $x_2^1 \leq x_1^1 \leq 1 + x_2^1$. Then $u_1(\mathbf{x}^1) = x_2^1$ and $u_2(\mathbf{x}^2) = x_2^2$. The only way that either consumer can increase utility is to obtain more of good 2. However, that will decrease the utility of the other consumer. It follows that the allocations where $x_2^1 \leq x_1^1 \leq 1 + x_2^1$ are the Pareto optimal allocations.

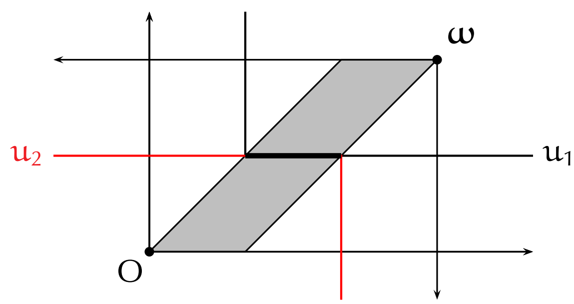

Figure 19.2.9: Indifference curves are shown for both consumers. The heavy line indicates the Pareto optimal points they generate. This happens as every level between the two diagonal line. As as result, the shaded area is the Pareto set for the Leontief Example 19.2.8.

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19.3 Pareto Optima in Production Economies

We next consider Pareto optima in production economies. Suppose the production technology of firm f is described by a $\mathcal{C}^2$ transformation function T_f and preferences of consumer i are described by a $\mathcal{C}^2$ utility function, we can use the first-order conditions to analyze the Pareto set. The Pareto optimality problem is:

$$\begin{aligned} &\max u_j(\boldsymbol{x}^j) \\ &\text{s.t. } u_i(\boldsymbol{x}^i) \ge \bar{u}^i \text{ for } i \ne j \\ &\quad x_\ell^i \ge 0 \text{ for all } i, \ell \\ &\quad \sum_{i=1}^{I} x_\ell^i \le \omega_\ell + \sum_{f=1}^{F} y_\ell^f \quad \text{for all } \ell \\ &\quad T_f(\boldsymbol{y}^f) \le 0 \quad \text{for all } f. \end{aligned}$$

The Lagrangian is

$$\begin{aligned} \mathcal{L} = u_j(\boldsymbol{x}^j) &+ \sum_{i\ne j} \lambda_i \big(u_i(\boldsymbol{x}^i) - \bar{u}_i\big) \\ &+ \sum_\ell \mu_\ell \Big(\omega_\ell + \sum_f y_\ell^f - \sum_{i=1}^{I} x_\ell^i\Big) \\ &- \sum_f \nu_f T_f(\boldsymbol{y}^f) + \sum_{i=1}^{I} \sum_{\ell=1}^{m} \rho_\ell^i x_\ell^i. \end{aligned}$$

19.3.1 Pareto Optima in Production Economies II

The first-order conditions become:

$$\frac{\partial u_j}{\partial x_\ell} = \mu_\ell - \rho^j_\ell$$

$$\lambda_i \frac{\partial u_i}{\partial x_\ell} = \mu_\ell - \rho^i_\ell \quad (\text{for } i \neq j)$$

$$\nu_f \frac{\partial T_f}{\partial y^f_\ell} = \mu_\ell \quad (\text{for } i \neq j)$$

For **interior** solutions $\rho^i_\ell = 0$ and we can combine these to obtain

$$\text{MRS}^j_{k\ell} = \frac{\partial u_j/\partial x_k}{\partial u_j/\partial x_\ell} = \frac{\mu_k}{\mu_\ell}, \quad \text{MRS}^i_{k\ell} = \frac{\partial u_i/\partial x_k}{\partial u_i/\partial x_\ell} = \frac{\mu_k}{\mu_\ell},$$

$$\text{MRT}^f_{k\ell} = \frac{\partial T^f/\partial y^f_k}{\partial T^f/\partial y^f_\ell} = \frac{\mu_k}{\mu_\ell}.$$

19.3.2 Three Types of Efficiency

In other words, the Pareto optimum obeys:

(1) **Consumption Efficiency:** $\mathrm{MRS}^i_{k\ell} = \mathrm{MRS}^j_{k\ell}$ for all consumers i and j and all goods k and ℓ.

(2) **Production Efficiency:** $\mathrm{MRTS}^f_{k\ell} = \mathrm{MRTS}^g_{k\ell}$ for all firms f and g and all goods k and ℓ.

(3) **Product-Mix Efficiency:** $\mathrm{MRS}^i_{k\ell} = \mathrm{MRT}^f_{k\ell}$ for all firms f, consumers i, and goods k and ℓ.

19.3.3 Consumption Efficiency

Consumption efficiency means that there's no way to reallocate the existing consumption goods to make a Pareto improvement. This is the efficiency we see in the Edgeworth box where the total endowment consists of the consumption goods actually produced.

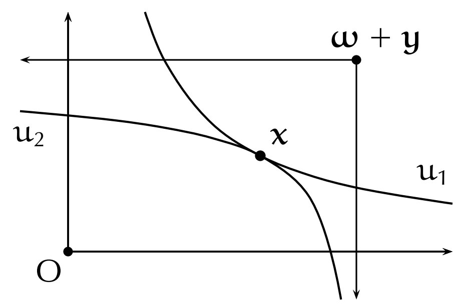

Figure 19.3.1: The allocation $(x, \omega + y - x)$ is Pareto optimal in this Edgeworth box. It is impossible to increase consumer one's utility without decreasing consumer two's utility. The consumers are efficiently consuming the goods available for consumption: $\omega + y$.

19.3.4 Production Efficiency

Production efficiency means there's no way to reallocate the inputs used to produce more of one good without producing less than another. When production is described by production functions, an Edgeworth box using isoquants can illustrate it as in Figure 19.3.2.

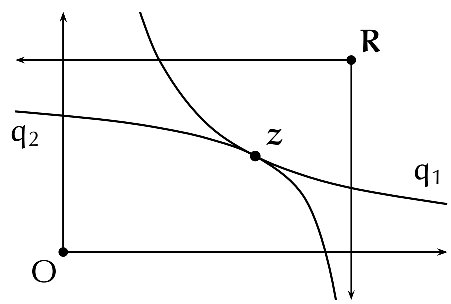

Figure 19.3.2: **Production Efficiency.** Let the vector $\mathbf{R}$ represent the available inputs to production and q_1, q_2 the production levels. The input allocation $(z, \mathbf{R}-z)$ is Pareto optimal in this Edgeworth box. Any increase in output of good 1 requires moving above the q_1 isoquant. Any increase in output of good 2 requires moving below the q_2 isoquant. Obviously, it is impossible to do both (production efficiency) due to the mutual tangency of the two isoquants. I.e., $MRTS^1_{12} = MRTS^2_{12}$.

19.3.5 Product-Mix Efficiency

Product-mix efficiency is more subtle and concerns the interaction between production and consumption. Even if the economy efficiently produces goods that are efficiently consumed, there's no guarantee that the goods produced are actually valuable to the consumers. Maybe it would be better to reallocate the productive resources to produce other goods, or even to direct consumption. Product-mix efficiency rules out such gains. It is best illustrated on a production possibilities diagram as in Figure 19.3.3.

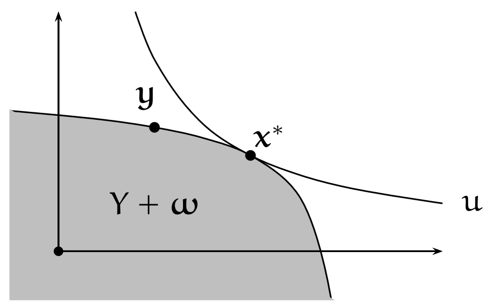

Figure 19.3.3: **Product-Mix Efficiency.** The production possibilities set is $Y + \omega$. It shows what consumption bundles may be produced. Utility of a consumer is maximized at x^*. At that point, the indifference curve and production set are tangent, MRS = MRT. Production at y would also be efficient, but would fail product-mix efficiency.

19.3.6 Pareto Optima: CRS Production

Example 19.3.4: Pareto Optima with CRS Production. Let's return to the setting of Example 15.4.1. Here the two consumers have utility $u_i(\mathbf{x}^i) = (x_1^i)^{1/2}(x_2^i)^{1/2}$. The social endowment is $\boldsymbol{\omega} = (3, 0)$. The production set is $Y = \{(y_1, y_2) : y_2 \le -y_1, y_1 \le 0\}$.

The firm must produce at all Pareto optima. Otherwise, both consumers would have none of good two. Both consumers would have zero utility. At least one of them would have some of good one, which could be given up in order to get good two, increasing utility.

Since the firm produces, $MRT_{12} = 1$. It follows that $MRS_{12}^i = 1$ also. Since $MRS_{12}^i = x_2^i/x_1^i$, we find $x_1^i = x_2^i$ for $i = 1, 2$. Both consumers consume equal amounts of each good. Any efficient production vector can be written $\mathbf{y} = (-z, z)$ where z is the input of good one. This means the goods available for consumption are $\mathbf{y} + \boldsymbol{\omega} = (3 - z, z)$.

Setting $3 - z = z$, we find that the only production vector yielding equal amounts of each good available for consumption is $(-3/2, 3/2)$, which yields $(3/2, 3/2)$. The consumers then consume $\mathbf{x}^1 = t(3/2, 3/2)$ and $\mathbf{x}^2 = (1 - t)(3/2, 3/2)$ for some t, $0 \le t \le 1$. The corresponding utility levels are $u_1 = \frac{3}{2}t$ and $u_2 = \frac{3}{2}(1 - t)$. ◄

19.3.7 Pareto Optima: DRS Production

Example 19.3.5: Pareto Optima with DRS Production. Now revisit Example 15.4.3. The two consumers have identical utility functions $u_i(\mathbf{x}) = (x_1^i x_2^i)^{1/2}$. The aggregate endowment is $\boldsymbol{\omega} = (2, 1)$. There is one firm with production function $f(z) = 2\sqrt{z}$ where z is the input of good 1. Net output is $\mathbf{y} = (-z, 2\sqrt{z})$, so $\boldsymbol{\omega} + \mathbf{y} = (2 - z, 1 + 2\sqrt{z})$ is available for consumption.

In this case, $MRT_{12} = MP = 1/\sqrt{z}$ and $MRS_{12}^i = x_2^i/x_1^i$. Once again, the consumers end up with the same proportions of the two goods. The only problem is to determine how much is produced. The proportions consumed are the same, so

$$\frac{x_2^i}{x_1^i} = \frac{1 + 2\sqrt{z}}{2 - z} = \frac{1}{\sqrt{z}}.$$

This implies $\sqrt{z} + 2z = 2 - z$ or $3z + \sqrt{z} - 2 = 0$. This is a quadratic in $\sqrt{z}$ with one positive root, $\sqrt{z} = 2/3$ which yields $z = 4/9$. The consumption bundles are a fraction of $(14/9, 7/3)$, so $\mathbf{x}^1 = t(14/9, 7/3)$ and $\mathbf{x}^2 = (1 - t)(14/9, 7/3)$ where $0 \le t \le 1$.

This example is not much different from the previous case. There is an optimal production amount. Consumers have identical Cobb-Douglas preferences, so they consume the goods in the proportions they are available. The various Pareto optima are distinguished from each other by the shares of the available goods each consumer receives.

One interesting feature concerns product-mix efficiency. Suppose we set $z = 1/4$, obtaining output 1. Production is efficient because we have obtained the maximum output possible from our input of $1/4$. This leaves $(7/4, 2)$ available for consumption purposes. With identical Cobb-Douglas utility, consumption goods will be allocated efficiently provided $\mathbf{x}^1 = t(7/4, 2)$ and $\mathbf{x}^2 = (1 - t)(7/4, 2)$ for some $t \in [0, 1]$. Even though both consumption and production are efficient, we are not at a Pareto optimum because the product-mix efficiency condition fails. The economy has too much of good 1 and not enough good 2.

To drive home the point, consider the allocation with $t = 1/2$. Then each consumer gets $(7/8, 1)$. Then $u_1 = u_2 = \sqrt{7/8}$, so utility is about 0.94. The consumers can both do better by consuming less of good 1 and more of good 2. Suppose we take the optimal input level $z = 4/9$, one option is for each consumer to receive $(7/9, 7/6)$, which has utility $7/3\sqrt{2}$, about 1.65. We have a Pareto improvement. Both are better off. ◄

19.3.8 Pareto Optima: Heterogeneous Consumers with Production

Remainder of Chapter Skipped

Example 19.3.6: Pareto Optima with Heterogeneous Consumers and Production. Now let $\boldsymbol{\omega} = (3,0)$, $u_1(\mathbf{x}^1) = (x_1^1)^{1/2}(x_2^1)^{1/2}$, $u_2(\mathbf{x}^2) = (x_1^2)^{1/3}(x_2^2)^{2/3}$ and suppose good 1 can be used to produce good 2 via a production function $f(z) = 2\sqrt{z}$.

The marginal rates of substitution are $MRS_{12}^1 = x_2^1/x_1^1$, $MRS_{12}^2 = x_2^2/(2x_1^2)$ and the marginal rate of transformation is $MRT_{12} = 1/\sqrt{z}$. The resource constraints are $3 - z = x_1^1 + x_1^2$ and $2\sqrt{z} = x_2^1 + x_2^2$. Thus

$$\frac{x_2^1}{x_1^1} = \frac{x_2^2}{2x_1^2} = \frac{2\sqrt{z} - x_2^1}{2(3 - z - x_1^1)} = \frac{1}{\sqrt{z}}.$$

Solving the first and third terms for x_2^1 yields $x_2^1 = 2x_1^1\sqrt{z}/(6 - 2z - x_1^1)$. We then substitute for x_2^1 to obtain

$$\frac{x_2^1}{x_1^1} = \frac{2\sqrt{z}}{6 - 2z - x_1^1} = \frac{1}{\sqrt{z}}.$$

It follows that $x_1^1 = 6 - 4z$. Plugging back in for x_2^1, we find that $\mathbf{x}^1 = 2(3-2z)(1, 1/\sqrt{z})$. Also, $x_1^2 = 3 - z - x_1^1 = 3z - 3$, so $\mathbf{x}^2 = 3(z-1)(1, 2/\sqrt{z})$. The non-negativity constraints require $1 \le z \le 3/2$. The corresponding utility levels are $u_1(z) = 2(3z^{-1/4} - 2z^{3/4})$ and $u_2(z) = 2^{1/3}3[z^{2/3} - z^{-1/3}]$. Thus increases in z are bad for consumer 1 and good for consumer 2. We are on the Pareto frontier.

The Pareto set is then $\{\big(\mathbf{x}^1, \mathbf{x}^2, (-z, 2\sqrt{z})\big) : 1 \le z \le 3/2,\ \mathbf{x}^1 = 2(3 - 2z)(1, 1/\sqrt{z})$ and $\mathbf{x}^2 = 3(z - 1)(1, 2/\sqrt{z})\}$.

In this case the marginal rate of transformation is not fixed, nor is the level of factor input. At one extreme ($z = 1$), consumer 1 gets all of the goods and MRT $= 1$ while at the other extreme ($z = 3/2$), consumer 2 gets all of the goods and MRT $= \sqrt{2/3}$. For a given level of output, there is only one efficient allocation of goods that satisfies the product-mix efficiency condition. This contrasts with the previous example where there was a unique output level with many possible efficient allocations of consumption. ◀

19.3.9 Pareto Optima: Multiple Production Functions

Example 19.3.7: Pareto Optima and Multiple Production Functions. Now consider a three good example, where there is one non-produced good and two produced goods. There are two consumers with utility $u_i(\mathbf{x}) = (x_1^i)^{1/3}(x_2^i)^{1/3}(x_3^i)^{1/3}$. The aggregate endowment is $\boldsymbol{\omega} = (0, 0, 3)$. Goods 1 and 2 are produced from good 3. This is described by production functions $f_f(z^f) = \sqrt{z^f}$, where f_1 describes the production of good 1 and f_2 describes the production of good 2.

We can derive the aggregate transformation function as in Example 13.3.3. It is $T(\mathbf{y}) = y_1^2 + y_2^2 + y_3$. Since utility is Cobb-Douglas, all Pareto optima that give positive utility to both consumers will be interior Pareto optima. The first-order conditions reduce to the consumption efficiency condition $MRS^2_{k\ell} = MRS^2_{k\ell}$ and the product-mix efficiency condition $MRS^i_{k\ell} = MRT_{k\ell}$. Thus

$$\frac{x_1^1}{x_2^1} = \frac{x_1^2}{x_2^2}, \quad \frac{x_2^1}{x_3^1} = \frac{x_2^2}{x_3^2}, \quad \frac{x_3^1}{x_1^1} = 2y_1, \quad \frac{x_3^1}{x_2^1} = 2y_2$$

are the independent first-order conditions and $y_1^2 + y_2^2 + y_3 = 0$ is the production constraint, and $y_1 = x_1^1 + x_1^2$, $y_2 = x_2^1 + x_2^2$, and $3 + y_3 = x_3^1 + x_3^2$ are the adding-up constraints.

Letting $\alpha = x_1^2/x_1^1$, the first two first-order conditions yield $\mathbf{x}^2 = \alpha\mathbf{x}^1$. Substituting in the adding-up constraints for goods 1 and 2 we find $y_1 = (1+\alpha)x_1^1$ and $y_2 = (1+\alpha)x_2^1$. But then the last two first-order conditions yield

$$\frac{x_3^1}{2x_1^1} = (1+\alpha)x_1^1 \quad \text{and} \quad \frac{x_3^1}{2x_2^1} = (1+\alpha)x_2^1.$$

It follows immediately that $x_1^1 = x_2^1$ and so $y_1 = y_2$ by adding-up. The production constraint now yields $2y_1^2 = -y_3$, which we substitute in the adding-up constraint for good 3. This yields $3 - 2y_1^2 = (1+\alpha)x_3^1 = 2(1+\alpha)y_1x_1^1$. Since $2x_1^1 = y_1$, we obtain $y_1^2 = 3/4$.

It follows that $y_1 = y_2 = \sqrt{3/4}$ and $y_3 = -3/2$. Thus $\mathbf{x}^1 + \mathbf{x}^2 = \mathbf{y} + (0,0,3) = (\sqrt{3/4}, \sqrt{3/4}, 3/2)$. The total amount of each good available for consumption must be split between the two consumers. The fact that $\mathbf{x}^1$ and $\mathbf{x}^2$ are proportional (due to identical Cobb-Douglas utility) implies that the Pareto optima have the form $\mathbf{x}^1 = \beta(\sqrt{3/4}, \sqrt{3/4}, 3/2)$ and $\mathbf{x}^2 = (1-\beta)(\sqrt{3/4}, \sqrt{3/4}, 3/2)$ for $0 \le \beta \le 1$. ◄

March 15, 2023