20. Equilibrium and Economic Welfare

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What do we mean by economic welfare? Economic welfare refers to the well-being of the individuals in the economy, the consumers. There are several ways of determining whether consumers are, in general, better or worse off as a result of an economic change. We've seen several methods of assessing welfare changes: Pareto improvements, changes in social welfare functions, and changes in compensating and equivalent variations all come to mind. We focus on the first two in this chapter.[1]

This naturally raises the question of how Walrasian equilibria rank both according to the Pareto criteria and according to some social welfare function. We will focus on several questions. Is any Walrasian equilibrium Pareto optimal? Can any Pareto optimum be realized as a Walrasian equilibrium? Is Walrasian equilibrium compatible with maximizing a social welfare criterion? How do these results change if public goods or other externalities are included?

Section one introduces the two welfare theorems before focusing on the First Welfare Theorem. The Second Welfare Theorem is covered in section two. The Second Welfare Theorem uses the notion of quasi-equilibrium. We study conditions that ensure a quasi-equilibrium is a true equilibrium in section three. Section four examines the relation between equilibria and social welfare maxima. Section five gives an application of social welfare methods to the problem of optimal production of a public good.

Outline:

1. The Welfare Theorems
2. The Second Welfare Theorem
3. When are Quasi-Equilibria True Equilibria?
4. Social Welfare and Equilibrium
5. Public Goods

[1] Diamond and McFadden (1974) show how the expenditure function may be used as a welfare criterion in three public finance problems including optimal income taxation.

20.1 The Welfare Theorems

The two welfare theorems (Arrow, 1951) spell out the relation between the set of equilibrium allocations and the set of Pareto optimal allocations. The two welfare theorems tell us that, under fairly general conditions, Walrasian equilibrium allocations are Pareto optima, and that most Pareto optimal allocations can occur as Walrasian equilibrium allocations, provided we can change consumer incomes.

20.1.1 The First Welfare Theorem in a Nutshell

We start with the First Welfare Theorem, which tells us that Walrasian equilibrium allocations are Pareto optimal allocations. The intuition behind the proof is simple. If you make a Pareto improvement, the consumers that gain must spend more than before, and the consumers who are at least as well off must spend at least as much as before. Aggregate spending must increase.

For this welfare increase to be feasible, we must increase spending. In this static model without international trade, GDP is both final spending by consumers and the income earned from aggregate output. This income comes from two sources, the value of the endowment and the profits of the firms. Since prices haven't changed, the value of the endowment remains the same. Firm profits are already at a maximum, and can not be increased.

Since neither endowment income nor profits can be increased, there is no way to increase GDP. That means our Pareto improvement is not feasible. We do not have sufficient resources to implement it. We will formalize this argument later in a bit more general setting.

20.1.2 The Second Welfare Theorem in Diagrams: I

The Second Welfare Theorem says that most Pareto optima can be realized as a type of competitive equilibrium. A Cobb-Douglas case is illustrated in the Edgeworth box of Figure 20.1.1. We start with an arbitrary interior Pareto optimum at the point A. We know that the two consumer's indifference curves have the same marginal rate of substitution at A. This defines a relative price and so a budget line. If the endowment is on the budget line, A is the equilibrium and we are done.

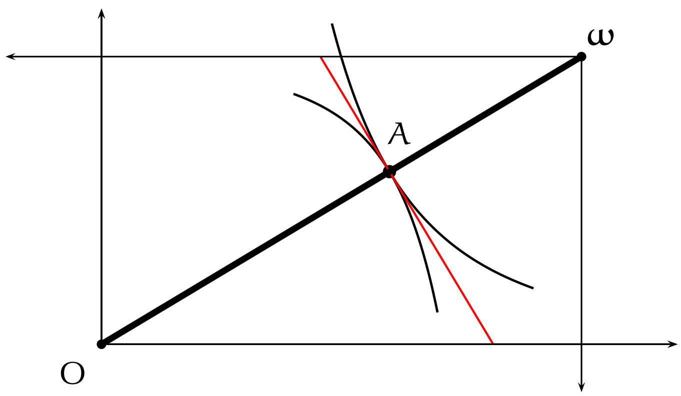

Figure 20.1.1: **The Second Welfare Theorem: I.** In this Cobb-Douglas example, the heavy diagonal line is the Pareto Set. Point A is a Pareto optimum, as demonstrated by the tangent indifference curves. The tangent line is shown in red. Its slope determines the relative price, which is equal to the MRS at A. Keep in mind that an endowment anywhere on that budget line will have equilibrium A.

20.1.3 The Second Welfare Theorem in Diagrams: II

If the endowment is somewhere else, call it point W, and consider the parallel budget line through the endowment W. With identical Cobb-Douglas utility, the point where that budget line intersects the Pareto set is an equilibrium point. The Pareto optimum A is *not* a Walrasian equilibrium allocation. Fixing that is where the taxes and transfers come in.

In order to end up at A we need to move the budget line. We do that with a tax and a transfer. The budget line through A requires additional income for consumer one (the transfer). This same amount of income is taken from consumer two (the tax). Since the amounts are the same, the government's budget balances. This shows us that by taxing consumer two and giving the proceeds to consumer one, we can make the Pareto optimal allocation A an equilibrium allocation.

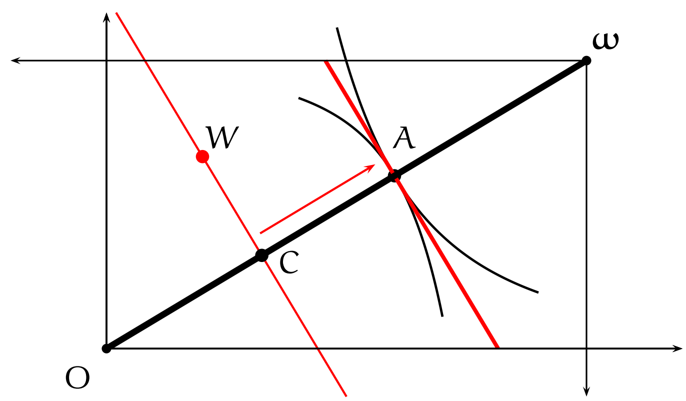

Figure 20.1.2: **The Second Welfare Theorem: II.** At the same prices, an endowment elsewhere, such as at point W, will yield a different equilibrium. Because we have identical Cobb-Douglas consumers, that point C is where the budget line through W crosses the Pareto set. But we want to get to the equilibrium at A. We do this by shifting the budget line. We give money to consumer one, financed by taking an equal amount from consumer two (a transfer to consumer one, a tax on consumer two).

20.1.4 Walrasian Equilibrium with Taxes and Transfers

Although we cannot expect a given Pareto optimum allocation to be an equilibrium allocation from an arbitrary endowment, we might be able to make it so by means of lump-sum taxes and transfers. With this in mind, we define a Walrasian equilibrium with lump-sum taxes and transfers. Unlike income or sales taxes, lump-sum taxes and transfers do not depend on economic behavior. One consequence is that lump-sum taxes do not create a tax wedge between buyers and sellers, a wedge which would distort relative prices.

Walrasian Equilibrium with Taxes and Transfers. In an economy $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y_f, \theta_f^i)$, an allocation $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ and price vector $\hat{\mathbf{p}} > 0$ form a *Walrasian equilibrium with taxes and transfers* if there exist wealth levels m^i obeying the *budget balance condition*

$$\sum_i m^i = \sum_i \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i + \sum_f \hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f$$

such that:

1. Firms maximize profit: $\hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f \geq \hat{\mathbf{p}}\cdot\mathbf{y}$ for all $\mathbf{y} \in Y_f$.
2. Consumers maximize utility: $\hat{\mathbf{x}}^i \succsim_i \mathbf{x}$ for all $\mathbf{x} \in \mathfrak{X}_i$ with $\hat{\mathbf{p}}\cdot\mathbf{x} \leq m^i$.
3. Markets clear: $\sum_i \hat{\mathbf{x}}^i \leq \sum_i \boldsymbol{\omega}^i + \sum_f \hat{\mathbf{y}}^f$.

20.1.5 Defining the Taxes and Transfers

The taxes and transfers are defined as the difference between spending at the Pareto optimum and income. Thus

$$t_i = m^i - \left(\hat{\mathbf{p}} \cdot \boldsymbol{\omega}^i + \sum_f \theta^i_f \hat{\mathbf{p}} \cdot \hat{\mathbf{y}}^f \right).$$

It follows that

$$m^i = t_i + \left(\hat{\mathbf{p}} \cdot \boldsymbol{\omega}^i + \sum_f \theta^i_f \hat{\mathbf{p}} \cdot \hat{\mathbf{y}}^f \right). \tag{20.1.1}$$

We can write the government budget balance condition using taxes and transfers as $\sum_{i=1}^I t_i = 0$.

The taxes and transfers provide a way to redistribute income among the consumers. A look at equation 20.1.1 shows that consumer i is subsidized by the transfer when $t_i > 0$ and consumer i is taxed by the transfer when $t_i < 0$. Walrasian equilibria are then the special case where $t_i = 0$ for all consumers i.

20.1.6 The First Welfare Theorem

We can now state the First Welfare Theorem. It says that the allocation of goods in any Walrasian equilibrium with taxes and transfers is a Pareto optimal allocation.

First Welfare Theorem. *Suppose the economy* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y^f, \theta^i_f)$ *consists of* I *consumers and* F *firms. Suppose further that preferences are locally non-satiated and* $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f, \hat{\mathbf{p}})$ *is a Walrasian equilibrium with taxes and transfers. If* $\hat{\mathbf{p}} > \mathbf{0}$*, then the equilibrium allocation* $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ *is a Pareto optimal allocation. In particular, any Walrasian equilibrium allocation is Pareto optimal.*

Proof. **Suppose** a feasible allocation $(\mathbf{x}^i, \mathbf{y}^f)$ **is a Pareto improvement**. Then $\mathbf{x}^i \succsim_i \hat{\mathbf{x}}^i$. By Lemma 4.3.3, $\hat{\mathbf{p}}\cdot\mathbf{x}^i \geq m^i$. Moreover, for any j with $\mathbf{x}^j \succ_j \hat{\mathbf{x}}^j$, $\hat{\mathbf{p}}\cdot\mathbf{x}^j > m^j$ by utility maximization. Summing, we find

$$\hat{\mathbf{p}}\cdot\left(\sum_i \mathbf{x}^i\right) > \sum_i m^i.$$

Now $\mathbf{y}^f \in Y_f$, so $\hat{\mathbf{p}}\cdot\mathbf{y}^f \leq \hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f$ by profit maximization. Adding across all firms we find $\sum_f \hat{\mathbf{p}}\cdot\mathbf{y}^f \leq \sum_f \hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f$. It follows that

$$\sum_i \hat{\mathbf{p}}\cdot\mathbf{x}^i > \sum_i m^i \geq \hat{\mathbf{p}}\cdot\left(\sum_i \boldsymbol{\omega}^i + \sum_f \mathbf{y}^f\right)$$

and so

$$\sum_i \hat{\mathbf{p}}\cdot\mathbf{x}^i > \hat{\mathbf{p}}\cdot\left(\sum_i \boldsymbol{\omega}^i + \sum_f \mathbf{y}^f\right).$$

But that is impossible because $\sum_i \mathbf{x}^i \leq \sum_i \boldsymbol{\omega}^i + \sum_f \mathbf{y}^f$. This contradicts the supposition that $(\mathbf{x}^i, \mathbf{y}^f)$ was a Pareto improvement. In fact, no Pareto improvement is possible, meaning $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ is Pareto optimal. □

20.1.7 Welfare and GDP

The First Welfare Theorem shows how prices can be used to measure economic activity. The idea of Pareto improvement requires an increased dollar value of consumption. We use a measure of national income, the dollar value of the endowment together with profits (GDP). The proof then reduces to the argument that we do not have the resources to make a Pareto improvement since they would require increasing consumption spending beyond the GDP the economy can produce.

20.1.8 Envy-free Allocations

Pareto optimality is not the only way to evaluate economic performance. We can also ask about fairness of the outcomes. One simple notion of fairness (equity) is the idea that allocations should be envy-free, that no one prefers any one else's allocation to their own.

Envy-free Allocation. An allocation $(\mathbf{x}^i, \mathbf{y}^f)$ is *envy-free* if no consumer prefers someone else's consumption bundle to their own. Equivalently, if for every i and j, $\mathbf{x}^i \succsim_i \mathbf{x}^j$.

It is generally supposed that there is a trade-off between efficiency and equity. That is not always the case, at least when equity means envy-free. There can be allocations that are both envy-free and Pareto efficient, and we can use the First Welfare Theorem to show it.[2]

Theorem 20.1.3. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, -\mathbb{R}^m_+)$ be an exchange economy. Suppose for each $i = 1, \dots, I$, $\mathfrak{X}_i = \mathbb{R}^m_+$; preferences $\succsim_i$ are semi-strictly convex, monotonic, and continuous; $\boldsymbol{\omega}^i > 0$ and $\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}^i \gg 0$. Then $\mathcal{E}$ has an envy-free allocation that is Pareto optimal.*

Proof. Form another economy $\mathcal{E}'$ which is identical except that every consumer is given the average endowment $\overline{\boldsymbol{\omega}} = \boldsymbol{\omega}/I$. By Corollary 16.5.5, that economy has an equilibrium $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i)$.

Since both economies have the same consumption sets and aggregate endowment, they both have the same feasible allocations. By the First Welfare Theorem, $(\hat{\mathbf{x}}^i)$ is Pareto optimal in the economy $\mathcal{E}'$. Since the set of feasible allocations is the same for both economies, it is also Pareto optimal in $\mathcal{E}$.

Moreover, $(\hat{\mathbf{x}}^i)$ is envy-free. Consumer i has chosen a best point in the budget set $B_i = B(\hat{\mathbf{p}}, \hat{\mathbf{p}} \cdot \overline{\boldsymbol{\omega}})$. But this is the same budget set every other consumer has, and so every $\hat{\mathbf{x}}^j \in B_i$. Since $\hat{\mathbf{x}}^i \succsim_i \mathbf{x}$ for all $\mathbf{x} \in B_i$, $\hat{\mathbf{x}}^i \succsim_i \hat{\mathbf{x}}^j$ for all $j \neq i$. This shows that the allocation $(\hat{\mathbf{x}}^i)$ is envy-free. □

A similar trick, dividing all profits equally, can be used to generalize this result to production economies.

[2] Varian (1974) investigates this issue.

20.2 The Second Welfare Theorem

We now turn to the converse of the First Welfare Theorem—the problem of showing that any Pareto optimum is a Walrasian equilibrium with taxes and transfers. The basic idea is shown in the Edgeworth box of Figure 20.2.1. We start with a Pareto optimal allocation x. The indifference curves through x define a common marginal rate of substitution, a relative price. We use this to construct a budget line through the Pareto optimal allocation. Each consumer's income is given by the value of their allocation, and they maximize utility over the resulting budget set.

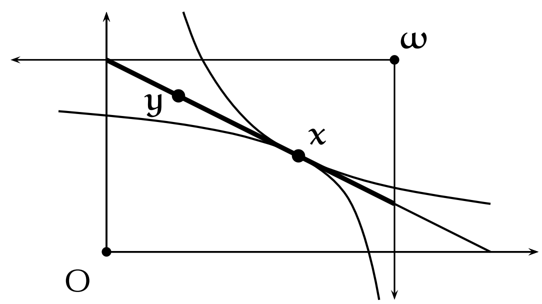

Figure 20.2.1: **Second Welfare Theorem as a New Endowment.** Start with the Pareto optimal allocation x. We form the line defined by the tangencies of the indifference curves. Via transfers, we give both consumers incomes that make this their budget line. The equilibrium allocation is then the Pareto optimum, x. We could accomplish the same thing by assigning an initial endowment such as y anywhere on the heavy line segment. This results in the same budget line and same equilibrium at x.

There is a second way to think about Figures 20.1.1 and 20.2.1. If the endowment is not on the generated budget line, one of the consumers pays the other for the right to trade at the given prices. At B in Figure 20.1.1, consumer two would pay consumer one. In other words, the equilibrium with taxes and transfers can be interpreted as a two-part tariff.

20.2.1 Arrow's "Exceptional Case"

Although Figure 20.2.1 shows how such a result may be obtained, it does not represent all possibilities. Indeed, the converse of the First Welfare Theorem is false. Arrow (1953) showed it is possible to find Pareto optima that are not equilibria, even when we allow taxes and transfers.

Example 20.2.2: Arrow's "Exceptional Case"—Optimal, but not Equilibrium. Let

$$u_1(\mathbf{x}^1) = \begin{cases} -(1-x_1^1)^2 + x_2^1 & \text{when } x_1^1 \leq 1 \\ x_2^1 & \text{when } x_1^1 > 1 \end{cases}$$

and $u_2(\mathbf{x}^2) = x_1^2 + \sqrt{x_2^2}$. Both utility functions are continuous, differentiable, and concave. Take as endowments $\boldsymbol{\omega}^1 = (1,2)$ and $\boldsymbol{\omega}^2 = (1,0)$. The situation is illustrated in Figure 20.2.3.[3]

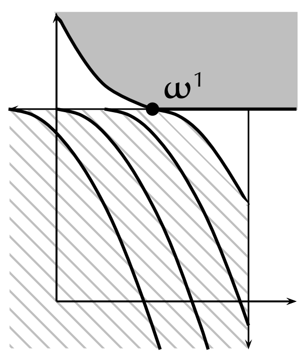

Figure 20.2.3: **Arrow's "Exceptional Case".** The allocation $(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2)$ is Pareto optimal in this Edgeworth box. It is clear that the only line separating the regions of improvement is horizontal. This means the price vector must have the form $(0, p)$. Although consumer one cannot improve on $\boldsymbol{\omega}^1$ within his budget set, consumer two can improve by choosing a point further out on his horizontal axis (to the left in the diagram, as the additional indifference curves illustrate). Thus consumer two cannot maximize utility with those prices, and no equilibrium with taxes and transfers exists.

[3] This example is based on Arrow's (1951, pg. 528) "exceptional case", but with explicit utility functions.

20.2.2 Arrow's "Exceptional Case"' is not an Equilibrium

It is easy to see that the endowment point is Pareto optimal. Consumer one has utility 2 while consumer two has utility 1. The only points in the Edgeworth box that give consumer one utility $u_1 \geq 2$ are the points on the upper right portion of the boundary, $\{(x, 2) : x \geq 1\}$. Then consumer two has $(2 - x, 0)$ with $x \geq 1$. As this yields utility $u_2 = 2 - x \leq 1$, these are not Pareto improvements.

It is also easy to see that this allocation cannot be an equilibrium. Both consumers have $MRS_{12} = 0$ at $\mathbf{x}^1 = (1, 2)$, $\mathbf{x}^2 = (1, 0)$, so the only possible equilibrium prices have the form $(0, p)$ with $p > 0$. We normalize so $p = 1$. Consumer one has income \$2 and will use it to buy 2 units of good 2. Since the price of good 1 is zero, consumer one can buy as much as he likes. Consumer one becomes satiated in good 1 when consuming 1 unit, so $(1, 2)$ maximizes utility. Consumer two is a different story. Income is \$0, but he can afford to buy any amount of good 1 that he wishes due to the zero price. Consumer two does not become satiated and his demand is undefined. Transfers cannot improve matters. Consumer two has no income to take, so the only possible transfer is from consumer one to consumer two. But consumer one must have income at least \$2 in order to choose $(1, 2)$, which leaves no income for consumer two. Besides, any income for consumer two will not help matters as two will still have infinite demand for good 1. ◄

There is no equilibrium in Example 20.2.2 because there is no solution to consumer two's utility maximization problem. However, if we consider the dual expenditure minimization problems, both consumers solve those at the Pareto optimum. This is true for consumer one who also solves the utility maximization problem. It is also true for consumer two. The points with $u_2 \geq 1$ either have x_2^2 positive (and so cost more than $(1, 0)$) or have $x_2^2 = 0$ and $x_1^2 \geq 1$. In the latter case they cost exactly what $(1, 0)$ does, nothing.

20.2.3 Quasi-Equilibrium with Taxes and Transfers

Suppose that instead of having consumers maximize utility, we have them minimize expenditures. We call such a situation a quasi-equilibrium. A quasi-equilibrium is like a equilibrium except that consumers minimize expenditure at the equilibrium allocation rather than maximizing utility. More formally:

Quasi-equilibrium with Taxes and Transfers. In an economy $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y_f, \theta_f^i)$, an allocation $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ and price vector $\hat{\mathbf{p}} > 0$ are a *quasi-equilibrium with taxes and transfers* if there exist wealth levels m^i obeying the budget balance condition

$$\sum_i m^i = \sum_i \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i + \sum_f \hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f$$

such that:

1. Firms maximize profit: $\hat{\mathbf{p}}\hat{\mathbf{y}}^f \geq \hat{\mathbf{p}}\cdot\mathbf{y}$ for all $\mathbf{y} \in Y_f$.
2. Consumers minimize expenditure: For every consumer i, $\mathbf{x} \succsim_i \hat{\mathbf{x}}^i$ implies $\hat{\mathbf{p}}\cdot\mathbf{x} \geq m^i$.
3. Markets clear: $\sum_i \hat{\mathbf{x}}^i \leq \sum_i \boldsymbol{\omega}^i + \sum_f \hat{\mathbf{y}}^f$.

Of course, the feasibility of $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ implies that $\sum_i \hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i \leq \sum_i m^i$. When combined with the fact that $\hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i \geq m^i$, it implies $\hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i = m^i$. Expenditure is then minimized at $\hat{\mathbf{x}}^i$.

20.2.4 The Second Welfare Theorem

We are now ready to turn to the Second Welfare Theorem.

Second Welfare Theorem. *Suppose each Y_f is convex and preferences are convex and locally non-satiated. Let $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ be a Pareto optimal allocation. Suppose one of the following holds:*

1. *the Pareto optimal allocation is non-wasteful*
2. *at least one consumer's preferences are monotonic*
3. *some production set Y_f obeys free disposal*

Then there is a price vector $\hat{\mathbf{p}} \neq \mathbf{0}$ that makes $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f, \hat{\mathbf{p}})$ a quasi-equilibrium with taxes and transfers. Moreover, the price vector obeys $\hat{\mathbf{p}} > \mathbf{0}$ in cases (2) and (3).

20.2.5 Proof of Second Welfare Theorem I

Market clearing automatically holds because $(\mathbf{x}^i, \mathbf{y}^f)$ is an allocation.

We will use a separation argument to find $\hat{\mathbf{p}}$. Let

$$B = \left\{ \sum_i \mathbf{x}^i : \mathbf{x}^i \in \mathfrak{X}_i, \mathbf{x}^i \succ_i \hat{\mathbf{x}}^i \right\} = \sum_i B_i.$$

where $B_i = \{\mathbf{x} : \mathbf{x} \succ_i \hat{\mathbf{x}}^i\}$. This is the set of aggregate consumption vectors that can be distributed in a way that makes all consumers better off (a *strong Pareto improvement*). Each set B_i is convex because preferences are convex and B is convex as the sum of convex sets.

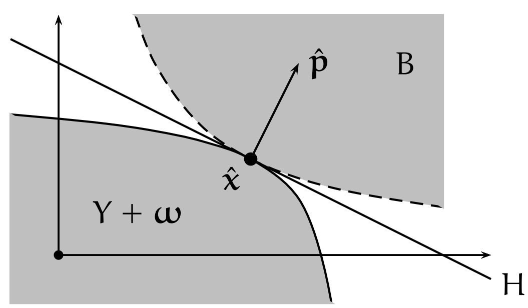

Figure 20.2.4: **Finding the Equilibrium Price.** The Pareto optimal consumption vector is $\hat{\mathbf{x}} = \sum_i \hat{\mathbf{x}}^i$. The production possibilities set is $Y + \{\boldsymbol{\omega}\}$. Here B is the set of consumption vectors that can be distributed in a Pareto improving fashion. The boundary of B is not part of B. We separate these two sets to find the equilibrium price vector $\hat{\mathbf{p}}$. The hyperplane H has equation $\hat{\mathbf{p}} \cdot \mathbf{x} = \alpha$.

Now consider B and the production possibilities set $Y + \{\boldsymbol{\omega}\}$ where $Y = \sum_f Y_f$. Both are non-empty convex sets. Pareto optimality of $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ implies they are disjoint. Separation Theorem B now gives us a price vector $\hat{\mathbf{p}} \neq \mathbf{0}$ and $\alpha \in \mathbb{R}$ with $\hat{\mathbf{p}} \cdot \mathbf{b} \geq \alpha$ for all $\mathbf{b} \in B$ and $\hat{\mathbf{p}} \cdot (\mathbf{y} + \omega) \leq \alpha$ for all $\mathbf{y} \in Y$.

20.2.6 Proof of Second Welfare Theorem II

Let $\{\mathbf{x}^i\}$ satisfy $\mathbf{x}^i \succsim_i \hat{\mathbf{x}}^i$ for each consumer i. By local non-satiation, we can find a sequence $\mathbf{x}^i_n \succ_i \mathbf{x}^i$ with $\mathbf{x}^i_n \to \mathbf{x}^i$. Now $\sum_i \mathbf{x}^i_n \in B$, and by separation, $\hat{\mathbf{p}}\cdot\big(\sum_i \mathbf{x}^i_n\big) \geq \alpha$. Letting $n \to \infty$ yields $\hat{\mathbf{p}}\cdot\big(\sum_i \mathbf{x}^i\big) \geq \alpha$ whenever $\mathbf{x}^i \succsim_i \hat{\mathbf{x}}^i$ for all consumers i. In particular, $\hat{\mathbf{p}}\cdot\big(\sum_i \hat{\mathbf{x}}^i\big) \geq \alpha$.

The argument temporarily breaks into two parts depending on the hypotheses used. Suppose (1) holds, the allocation is non-wasteful. Then $\sum_i \hat{\mathbf{x}}^i = \boldsymbol{\omega} + \sum_f \hat{\mathbf{y}}^f$. Applying $\hat{\mathbf{p}}$, we find $\alpha \leq \sum_i \hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i = \hat{\mathbf{p}}\cdot\boldsymbol{\omega} + \sum_f \hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f \leq \alpha$. It follows that

$$\alpha = \sum_i \hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i = \hat{\mathbf{p}}\cdot\boldsymbol{\omega} + \sum_f \hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f. \tag{20.2.2}$$

In this case we can say nothing about the positivity of $\hat{\mathbf{p}}$.

The other part covers both cases (2) and (3). Monotonicity of preference (2) implies that if $\mathbf{x}' \geq \mathbf{x}$ and $\mathbf{x} \in B$ then $\mathbf{x}' \in B$ (just give the extra to the guy with monotonic preferences). Thus B is anti-comprehensive. Alternatively, if there is a firm with free disposal (3) Y also enjoys free disposal, Y is comprehensive. Regardless of whether (2) and (3) holds, we may invoke Corollary 7.2.6 to show that $\hat{\mathbf{p}} > \mathbf{0}$.

In cases (2) and (3) some more work is required to obtain equation 20.2.2. Because the allocation is feasible, $\sum_i \hat{\mathbf{x}}^i \leq \boldsymbol{\omega} + \sum_f \hat{\mathbf{y}}^f \in Y + \{\boldsymbol{\omega}\}$. Since $\hat{\mathbf{p}} > \mathbf{0}$,

$$\hat{\mathbf{p}}\cdot\left(\sum_i \hat{\mathbf{x}}^i\right) \leq \hat{\mathbf{p}}\cdot\omega + \hat{\mathbf{p}}\cdot\left(\sum_f \hat{\mathbf{y}}^f\right) \leq \alpha.$$

Using the fact that $\alpha \leq \sum_i \hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i$, we find that each term of the inequality is equal to α, establishing that equation 20.2.2 also holds in cases (2) and (3).

20.2.7 Proof of Second Welfare Theorem III

The remainder of the proof is the same for both parts. From Equation 20.2.2 it follows that $\hat{\mathbf{p}} \cdot \left(\sum_f \mathbf{y}^f \right) \leq \hat{\mathbf{p}} \cdot \left(\sum_f \hat{\mathbf{y}}^f \right)$ whenever $\mathbf{y}^f \in Y_f$. By changing only the production vector for firm f, we see that any $\mathbf{y} \in Y_f$ must obey $\hat{\mathbf{p}} \cdot \mathbf{y} \leq \hat{\mathbf{p}} \cdot \hat{\mathbf{y}}^f$. Profits are maximized at $\hat{\mathbf{y}}^f$.

Define transfers by $m^i = \hat{\mathbf{p}} \cdot \hat{\mathbf{x}}^i$. Then $\sum_i m^i = \alpha$ by equation 20.2.2, establishing the budget balance condition.

Finally, suppose $\mathbf{x} \succsim_j \hat{\mathbf{x}}^j$. Define $\mathbf{x}^i = \hat{\mathbf{x}}^i$ for $i \neq j$ and $\mathbf{x}^j = \mathbf{x}$. Now $\sum_i \hat{\mathbf{p}} \cdot \mathbf{x}^i \geq \alpha = \sum_i \hat{\mathbf{p}} \cdot \hat{\mathbf{x}}^i$. Since only the j term is different, $\hat{\mathbf{p}} \cdot \mathbf{x} \geq \hat{\mathbf{p}} \cdot \hat{\mathbf{x}}^j = m^j$. Expenditure minimization holds and the proof is complete. $\square$

20.2.8 Can Prices be Negative?

An odd possibility arose in case (1). Prices could be negative. This could make sense if some commodities are bads, contributing negatively to utility. The following example shows what can happen.

Example 20.2.5: Negative Prices. Consider an exchange economy with $\mathfrak{X}_i = \mathbb{R}^2_+$, $u_i(x^i_1, x^i_2) = -(x^i_1)^2 - (x^i_2)^2$ for $i = 1, 2$ and endowments $\boldsymbol{\omega}^i = (1, 1)$. Note that utility is concave. We don't allow disposal of goods and require $\sum_i \mathbf{x}^i = \sum_i \boldsymbol{\omega}^i$ to define feasible allocations. We could think of this as a problem of allocating two types of waste in a Pareto optimal fashion.

The allocation $\big((1, 1), (1, 1)\big)$ is then Pareto optimal. The Second Welfare Theorem applies, yielding a quasi-equilibrium with taxes and transfers—in fact it is an equilibrium without transfers. The prices are $\hat{\mathbf{p}} = (-1, -1)$! Note that the budget set is $\{(x^i_1, x^i_2) \in \mathbb{R}^2_+ : -x^i_1 - x^i_2 \le -2\}$. The budget constraint could be rewritten $x^i_1 + x^i_2 \ge 2$. Utility is least negative at $(1, 1)$. ◄

20.3 When are Quasi-Equilibria True Equilibria?

When does the Second Welfare Theorem give us a true equilibrium? The real problem in Arrow's "exceptional case" (Example 20.2.2) is that one of the consumers has no income in the quasi-equilibrium. More precisely, reducing the cost of his consumption is not an option. Where there is such an option, we say there is a cheaper point.

Cheaper Point Condition. The *cheaper point condition* holds for consumer i at $\mathbf{x}^i$ with price vector $\mathbf{p}$ if there is a consumption bundle $\bar{\mathbf{x}} \in \mathfrak{X}_i$ that is cheaper than $\mathbf{x}^i$. In other words, $\mathbf{p}\cdot\bar{\mathbf{x}} < \mathbf{p}\cdot\mathbf{x}^i$.

20.3.1 Cheaper Point Implies Equilibrium

If the cheaper point condition holds, cost minimization implies utility maximization. To show this, we need to show that anything better than the expenditure minimizing bundle costs more than it does.

Proposition 20.3.1. *Suppose a consumer has continuous preferences on a convex consumption set $\mathfrak{X}$. Suppose further that the consumption bundle $\hat{\mathbf{x}}$ minimizes expenditure at price vector $\hat{\mathbf{p}}$ over the weakly preferred set $\{\mathbf{x} : \mathbf{x} \succsim \hat{\mathbf{x}}\}$ and that the cheaper point condition holds at $\hat{\mathbf{x}}$. If $\mathbf{x} \succ \hat{\mathbf{x}}$ then $\mathbf{p}\cdot\mathbf{x} > \mathbf{p}\cdot\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}$ maximizes utility over all $\mathbf{x}$ with $\mathbf{p}\cdot\mathbf{x} \le \mathbf{p}\cdot\hat{\mathbf{x}}$.*

Proof. Let $\bar{\mathbf{x}}$ be a cheaper point at $\hat{\mathbf{x}}$. Then choose $0 < \alpha < 1$ such that $\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}} \succ \hat{\mathbf{x}}$. This is possible due to continuity. Dotting with $\mathbf{p}$, we obtain

$$\alpha\mathbf{p}\cdot\mathbf{x} + (1-\alpha)\mathbf{p}\cdot\bar{\mathbf{x}} \ge \mathbf{p}\cdot\hat{\mathbf{x}}.$$

Rearranging,

$$\begin{aligned}\alpha\mathbf{p}\cdot\mathbf{x} &\ge \mathbf{p}\cdot\hat{\mathbf{x}} - (1-\alpha)\mathbf{p}\cdot\bar{\mathbf{x}}\\ &> \mathbf{p}\cdot\hat{\mathbf{x}} - (1-\alpha)\mathbf{p}\cdot\hat{\mathbf{x}}\\ &= \alpha\mathbf{p}\cdot\hat{\mathbf{x}}.\end{aligned}$$

As $\alpha > 0$, we conclude $\mathbf{p}\cdot\mathbf{x} > \mathbf{p}\cdot\hat{\mathbf{x}}$.

In this case, there are no better points than $\hat{\mathbf{x}}$ in the budget set defined by $\mathbf{p}\cdot\mathbf{x} \le \mathbf{p}\cdot\hat{\mathbf{x}}$, so $\hat{\mathbf{x}}$ maximizes utility over that set. □

20.3.2 Applying the Cheaper Point Condition

We can use Proposition 20.3.1 to show that any interior Pareto optimum satisfies the cheaper point condition when $\hat{\mathbf{p}}$ is positive.

Corollary 20.3.2. *Let* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y_f, \theta^i_f)$ *be an economy where each* Y_f *is convex and preferences are convex and locally non-satiated. Let* $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ *be a Pareto optimal allocation and* $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ *form a quasi-equilibrium with taxes and transfers obeying* $\hat{\mathbf{p}} > \mathbf{0}$. *If each* $\hat{\mathbf{x}}^i \in \operatorname{int} \mathfrak{X}_i$, *then* $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ *is an equilibrium.*

Proof. Since each $\hat{\mathbf{x}}^i \in \operatorname{int} \mathfrak{X}_i$, there is a cheaper point $\bar{\mathbf{x}}^i \ll \hat{\mathbf{x}}^i$ with $\bar{\mathbf{x}}^i \in \mathfrak{X}_i$. □

The cheaper point condition is also satisfied when $m^i > 0$ and $\mathbf{0} \in \mathfrak{X}_i$.

Corollary 20.3.3. *Let* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y_f, \theta^i_f)$ *be an economy where all consumption sets* $\mathfrak{X}_i$ *are convex with* $\mathbf{0} \in \mathfrak{X}_i$ *and preferences are continuous. Then any quasi-equilibrium with taxes and transfers* $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ *where* $\hat{\mathbf{p}} > \mathbf{0}$ *and wealth levels obey* $m^i > 0$ *for all* i *is a Walrasian equilibrium with taxes and transfers.*

Proof. Here $\mathbf{0} \in \mathfrak{X}_i$ is always a cheaper point, so Proposition 20.3.1 applies. □

One consequence of this is that examples such as Arrow's "exceptional case" (Example 20.2.2) **must** involve boundary points. Even boundary points do not necessarily force us to a quasi-equilibrium. In Arrow's example consumer one has a cheaper point. It is only consumer two that fails to have a cheaper point, in this case because the value of the equilibrium allocation is zero. There are no cheaper points than that!

The cheaper point condition is a bit unsatisfactory in that it does not depend on economic primitives. Rather, we must find the quasi-equilibrium before we can tell whether there is a cheaper point. However, if we have conditions that guarantee a positive income, such as a strictly positive allocation ($\hat{\mathbf{x}}^i \gg \mathbf{0}$), then we know the cheaper point condition will hold regardless of which $\hat{\mathbf{p}} > \mathbf{0}$ yields the quasi-equilibrium.

20.3.3 What About Irreducibility? Skipped

Such a problem calls for irreducibility. The forms of irreducibility that we used in Chapter 16 are not quite what we need here. We need a form of irreducibility that focuses on trades that are actually feasible. For this, we turn to Boyd and McKenzie's (1993) strong irreducibility.

They used a strong form of irreducibility to show a core equivalence result (see Chapter 21). I have slightly modified the definition to align it better with the definition of modified irreducibility by requiring $\alpha \leq 1$. Although the results of Boyd and McKenzie (1993) still apply, use of this stronger form of irreducibility would narrow the scope of their results. However, this version is applicable to quasi-equilibria with taxes and transfers.

We will state the definition using consumption sets and endowments rather than the traditional net trading sets.[4] Before giving the definition we need two additional concepts.

A consumption bundle $\mathbf{x}^i \in \mathfrak{X}_i$ is *strongly individually rational* if the consumer is better off at $\mathbf{x}^i$ than at any consumption vector they could produce using their own resources ($\boldsymbol{\omega}^i$) and the common production technology Y. In other words, $\mathbf{x}^i \succ_i (\boldsymbol{\omega}^i + \mathbf{y}^i)$ for all $\mathbf{y}^i$ with $(\boldsymbol{\omega}^i + \mathbf{y}^i) \in (\boldsymbol{\omega}^i + Y) \cap \mathfrak{X}_i$.

[4] The traditional net trading method is used in section 16.4.

20.3.4 Extreme Points

Skipped

A point $\boldsymbol{x}$ is an *extreme point* of a convex set C if there do not exist distinct points $\boldsymbol{x}', \boldsymbol{x}'' \in C$ and a number α obeying $0 < \alpha < 1$ so that $\boldsymbol{x} = \alpha \boldsymbol{x}' + (1 - \alpha)\boldsymbol{x}''$. Of course, extreme points must be boundary points.

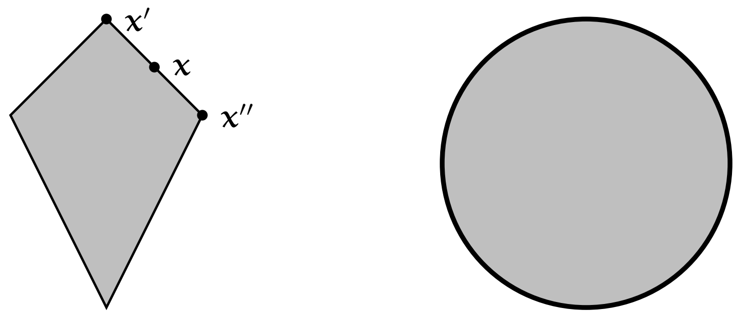

Figure 20.3.4: **Extreme Points.** In the left figure, corner points, including $\boldsymbol{x}'$ and $\boldsymbol{x}''$, are extreme points while points in-between such as $\boldsymbol{x} = \frac{1}{2}\boldsymbol{x}' + \frac{1}{2}\boldsymbol{x}''$ are not an extreme point. In the right figure, every boundary point is an extreme point.

20.3.5 Extra Strong Irreducibility **Skipped**

In Chapter 16 we introduced the following notation for sums over members of a set. Let J be a subset of the collection of consumers $\{1, \dots, I\}$. We use the notation $\mathbf{x}^J$ to indicate the sum of $\mathbf{x}^i$ over all members of J. That is, $\mathbf{x}^J = \sum_{i \in J} \mathbf{x}^i$. We are now ready to define extra strong irreducibility.

Extra Strong Irreducibility. The economy $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)$ is *extra strongly irreducible* if whenever I_1, I_2 form a non-trivial partition of the all consumers $\{1, \dots, I\}$, and $\hat{\mathbf{x}}^i \in \mathfrak{X}_i$ are given for every i and obey $\hat{\mathbf{x}}^{I_1} + \hat{\mathbf{x}}^{I_2} = \boldsymbol{\omega}^{I_1} + \boldsymbol{\omega}^{I_2} + \hat{\mathbf{y}}$ for some $\hat{\mathbf{y}} \in Y$, there are $\mathbf{x}^i$ obeying $\mathbf{x}^{I_1} + \mathbf{x}^{I_2} = \boldsymbol{\omega}^{I_1} + \boldsymbol{\omega}^{I_2} + \mathbf{y}$ with $\mathbf{y} \in Y$ so that:

1. $\mathbf{x}^i \succ_i \hat{\mathbf{x}}^i$ for $i \in I_1$.
2. $\mathbf{x}^i \in \mathfrak{X}_i$ when $i \in I_2$ and $\hat{\mathbf{x}}^i$ is both strongly individually rational and not an extreme point of $\mathfrak{X}_i$.
3. $\mathbf{x}^i \in (1-\alpha_i)\boldsymbol{\omega}^i + \alpha_i \mathfrak{X}_i$ with $0 < \alpha_i \le 1$ for $i \in I_2$, when $\hat{\mathbf{x}}^i$ either is an extreme point of $\mathfrak{X}_i$ or not strongly individually rational.

Item (2) says that that whenever $\hat{\mathbf{x}}^i$ is both strongly individually rational and not an extreme point of $\mathfrak{X}_i$, they we must able to actually supply goods that are useful to group I_1.

Item (3) covers the case where at least one of those conditions fails. Here, consumer i is not required to actually be able to supply the goods. It says we can write i's consumption vector as $\mathbf{x}^i = (1-\alpha_i)\boldsymbol{\omega}^i + \alpha_i \tilde{\mathbf{x}}^i$ for some $\tilde{\mathbf{x}}^i \in \mathfrak{X}_i$ and $0 < \alpha_i \le 1$. In this case, $\mathbf{x}^{I_1} + \sum_{i \in I_2} \alpha_i \tilde{\mathbf{x}}^i = \boldsymbol{\omega}^{I_1} + \sum_{i \in I_2} \alpha_i \boldsymbol{\omega}^i + \mathbf{y}$, which gives some extra flexibility, but not as much as allowing any $\alpha_i > 0$ as in Boyd and McKenzie (1993).

Like the standard definition of irreducibility, this requires that whenever we divide consumers into two groups and allocate consumption to each member in a way that can be produced using their collective endowments, then there is a way that people in one group can gain by using the resources of the other.

Unlike the standard definition, this definition requires in some cases that the I_2 consumers be able to improve the trades of the I_1 consumers by actual moves to other net trades in the I_2 possible trading sets. This is required in cases where $\hat{\mathbf{x}}^i$ is strongly individually rational, but not an extreme point. It is a possibility in other cases.

20.3.6 Irreducibility and the Second Welfare Theorem **Skipped**

This new irreducibility assumption does the trick. It converts quasi-equilibria with taxes and transfers into equilibria with taxes and transfers

Proposition 20.3.5. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y_f, \theta_f^i)$ be an economy obeying $\boldsymbol{\omega} \gg \mathbf{0}$, where all consumption sets $\mathfrak{X}_i$ are convex subsets of $\mathbb{R}_+^m$ containing $\mathbf{0}$, where each consumer i has continuous preferences and each production set Y_f satisfies free disposal. Suppose further that the economy is extra strongly irreducible. If $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f, \hat{\mathbf{p}})$ is a quasi-equilibrium with taxes and transfers m^i, then $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f, \hat{\mathbf{p}})$ is a Walrasian equilibrium with taxes and transfers m^i.*

20.3.7 Proof of Irreducibility Theorem I **Skipped**

Proof. In a quasi-equilibrium with taxes and transfers, $\sum_i m^i = \sum_i \hat{p}\cdot\hat{x}^i = \hat{p}\cdot\omega + \sum_f \hat{p}\cdot\hat{y}^f \geq \hat{p}\cdot\omega > 0$, so $m^i > 0$ for some i. This means that at least one consumer has positive income.

Because $0 \in \mathfrak{X}_i$ for all i, any consumer with positive income satisfies the cheaper point condition. Furthermore, since Y obeys free disposal, $\hat{p} > 0$. It follows that the cheaper point condition fails if and only if $m_i = 0$.

Let $I_1 = \{i : m_i > 0\}$, the set of consumers with a cheaper point. We know I_1 is not empty. Now suppose there is a least one consumer without a cheaper point and let $I_2 = \{i : m_i = 0\}$. Notice that $m_I = m_{I_1}$ since $m_{I_2} = 0$. By supposition, I_2 is not empty. Then for $i \in I_2$, we have both $m_i = 0$ and $\hat{p}\cdot\hat{x}^i = 0$.

Examining the definition of extra strong irreducibility, we can write $x^i = (1-\alpha_i)\omega^i + \alpha_i\tilde{x}^i$ with $\tilde{x}^i \in \mathfrak{X}_i$ in case (3). Notice that case (2) also fits this pattern with $\alpha_i = 1$. We now appeal to extra strong irreducibility which yields $y \in Y$ and x^i, $\tilde{x}^i$ obeying

$$x^{I_1} + \sum_{i\in I_2} \alpha_i\tilde{x}^i + \sum_{i\in I_2}(1-\alpha_i)\omega^i = y + \omega^{I_1} + \omega^{I_2}$$

with $x^i \succ_i \hat{x}^i$ for all $i \in I_1$. It follows that

$$x^{I_1} + \sum_{i\in I_2} \alpha_i\tilde{x}^i = y + \omega^{I_1} + \sum_{i\in I_2} \alpha_i\omega^i.$$

20.3.8 Proof of Irreducibility Theorem I **Skipped**

Proof continues. Note $\hat{p}\cdot x^i > m^i$ for each $i \in I_1$ by Proposition 20.3.1 since there is a cheaper point than $\hat{x}^i$ for every $i \in I_1$. Thus $\hat{p}\cdot x^{I_1} > m_{I_1}$. Now

$$\begin{aligned}
\hat{p}\cdot x^{I_1} + \sum_{i\in I_2} \alpha_i \hat{p}\cdot\tilde{x}^i &= \hat{p}\cdot y + \hat{p}\cdot\omega^{I_1} + \hat{p}\cdot\left(\sum_{I_2}\alpha_i\omega^i\right)\\
&\le \hat{p}\cdot\hat{y} + \hat{p}\cdot\omega^I + \hat{p}\cdot\left(\sum_{I_2}(\alpha_i-1)\omega^i\right)\\
&= m_{I_1} + \hat{p}\cdot\left(\sum_{I_2}(\alpha_i-1)\omega^i\right)\\
&< \hat{p}\cdot x^I + \hat{p}\cdot\left(\sum_{I_2}(\alpha_i-1)\omega^i\right)
\end{aligned}$$

The second line follows from profit maximization and rearrangement of the $\hat{p}\cdot\omega^i$ terms, the third by the definition of m_i and the fact that $m_{I_2} = 0$. Now since $\hat{p}\cdot x^I = \hat{p}\cdot x^{I_1}$, we have

$$0 = \sum_{i\in I_2}\alpha_i\hat{p}\cdot\tilde{x}^i < \hat{p}\cdot\left(\sum_{I_2}(\alpha_i-1)\omega^i\right). \tag{20.3.3}$$

Since $\alpha_i \le 1$, the right hand side of equation 20.3.3 is non-positive, which is impossible. This contradiction shows that I_2 is empty, that every consumer has a cheaper point.

By Proposition 20.3.1, $(\hat{x}^i, \hat{y}^f, \hat{p})$ is a Walrasian equilibrium with taxes and transfers m^i. □

Irreducibility cannot help in Arrow's "exceptional case" (Example 20.2.2) because the economy there is not irreducible.

20.4 Social Welfare and Equilibrium March 21, 2023

Pareto optimality is not the only way to evaluate allocations. Another method is to use a *Bergson-Samuelson social welfare function*, as used in Chapter 12. A Bergson-Samuleson social welfare function is an increasing mapping from utility vectors to the real numbers. They allow us to rank vectors of utilities, just as utility functions allow us to rank vectors of commodities.

Bergson-Samuelson Social Welfare Function. A *Bergson-Samuelson social welfare function* is a function $W\colon \mathbb{R}^I \to \mathbb{R}$ that is strictly increasing in each argument.[5]

A social welfare function can rank allocations that would be unrankable using the Pareto criterion. It can include other considerations such as inequality.

The older tradition in utility, dating back to Bentham (1776), considers utility as a cardinal concept. As such, it is perfectly proper to measure overall well-being by combining utility functions via a social welfare. Bentham (1776) based his philosophy of utilitarianism on the idea the "it is the greatest happiness of the greatest number that is the measure of right and wrong." This idea is often formalized by the *Benthamite social welfare function*, $W(\mathbf{u}) = u_1 + \cdots + u_I$.

This tradition has continued to the present day, with some authors using the theory of expected utility to justify cardinality. Nash's (1950, 1953) analysis of the bargaining game, both in cooperative and non-cooperative form, led to a solution that involves maximizing the product of utilities, using $W(\mathbf{u}) = \prod_{i=1}^{I} u_i$.

[5] See Bergson (1938) and Samuelson (1947).

20.4.1 Utility Possibility Set

To use a welfare function requires replacing allocations by their associated utility vectors.

We can do this when preferences are continuous. Continuity allows us to represent consumer i's preferences by a utility function u_i. Each allocation $(\mathbf{x}^i, \mathbf{y}^f)$ then defines a utility vector $\left(u_i(\mathbf{x}^i)\right)_{i=1}^{I} \in \mathbb{R}^I$. We form the utility possibility set from these vectors via free disposal.

Utility Possibility Set. The *utility possibility set* is

$$\mathcal{U} = \left\{\mathbf{u} \in \mathbb{R}^I : u_i \le u_i(\mathbf{x}^i) \text{ for some feasible allocation } (\mathbf{x}^i, \mathbf{y}^f)\right\}.$$

When the utility functions are concave, free disposal of utility helps ensure that the utility possibility set is convex.

Proposition 20.4.1. *Suppose each utility function u_i is concave and continuous and that each consumption set $\mathfrak{X}_i$ and production set Y_f is convex. Then the utility possibility set $\mathcal{U}$ is convex.*

Proof. Let $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$ and let $0 < t < 1$. Let $(\mathbf{x}^i, \mathbf{y}^i)$ and $(\mathbf{x}'^i, \mathbf{y}'^f)$ be corresponding feasible allocations with $u_i \le u_i(\mathbf{x}^i)$ and $u_i' \le u(\mathbf{x}'^i)$. Define $\hat{\mathbf{x}}^i = (1-t)\mathbf{x}^i + t\mathbf{x}'^i$ and $\hat{\mathbf{y}}^f = (1-t)\mathbf{y}^f + t\mathbf{y}'^f$. The convexity of Y_f shows $\hat{\mathbf{y}}^f \in Y_f$ for all f and the convexity of $\mathfrak{X}_i$ shows $\hat{\mathbf{x}}^i \in \mathfrak{X}_i$ for all i.

Now $u_i(\hat{\mathbf{x}}^i) \ge (1-t)u_i(\mathbf{x}^i) + tu_i(\mathbf{x}'^i) \ge (1-t)u_i + tu_i'$. This shows that $(1-t)\mathbf{u} + t\mathbf{u}' \in \mathcal{U}$. $\square$

20.4.2 Pareto Frontier

The *Pareto frontier* of $\mathcal{U}$ is $\{\mathbf{u} \in \mathcal{U} : \text{there does not exist } \mathbf{u}' \in \mathcal{U} \text{ with } \mathbf{u}' > \mathbf{u}\}$. The Pareto frontier is the set of feasible utility allocations corresponding to Pareto optimal allocations of goods.[6]

Different utility representations of the same preferences yield different utility possibility sets, as illustrated in Figure 20.4.2.

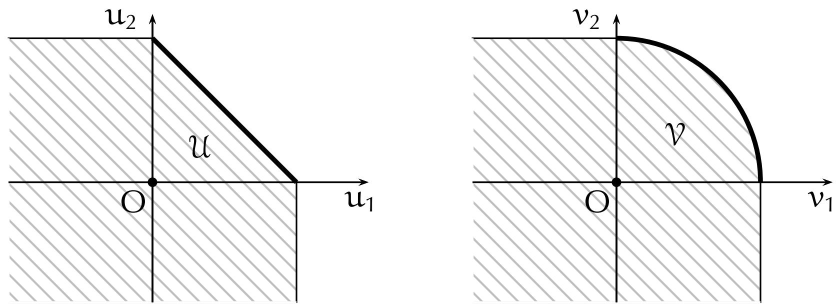

Figure 20.4.2: **Utility Possibility Sets.** In the left panel, $\mathcal{U}$ is the utility possibility set from Example 20.4.3. The heavy line indicates the Pareto frontier. In the right panel, $\mathcal{V}$ is the utility possibility set under the utility transformation $v_i = \sqrt{u_i}$. The preferences are the same in both cases. Only the utility representation has changed.

[6] Any utility allocation on the Pareto frontier of $\mathcal{U}$ must obey $u_i = u_i(\mathbf{x}^i)$ for some feasible allocation $(\mathbf{x}^i, \mathbf{y}^f)$.

20.4.3 Social Welfare and Cobb-Douglas Utility I

Here's a concrete example with the utility possibility sets of Figure 20.4.2.

Example 20.4.3: Social Welfare and Cobb-Douglas Utility. Let $\omega = (2, 8)$ and $u_i(\mathbf{x}) = (x_1^i)^\gamma (x_2^i)^{1-\gamma}$, $0 < \gamma < 1$. Applying Example 19.2.5 shows the Pareto frontier consists of the $(u_1, u_2) \geq (0, 0)$ with $u_1 + u_2 = 2^\gamma 8^{1-\gamma} = 2^{3-2\gamma}$. In this case, any Pareto optimum maximizes the Benthamite social welfare function $W_b(\mathbf{u}) = u_1 + u_2$.

Other social welfare functions may generate a different set of Pareto optima. They may even pick a unique optimum. For example, maximizing the social welfare function $W_1(\mathbf{u}) = u_1 u_2$ over the feasible set yields $u_1 = u_2 = 2^{2-2\gamma}$, corresponding to the allocation $\{(1, 4), (1, 4)\}$. More generally, the social welfare function $W_\alpha(\mathbf{u}) = u_1^\alpha u_2^{1-\alpha}$ with $0 < \alpha < 1$ yields optimum $(u_1, u_2) = (\alpha, 1 - \alpha)u(\boldsymbol{\omega}) = 2^{3-2\gamma}(\alpha, 1 - \alpha)$ with corresponding allocation $\mathbf{x} = \big(\alpha\boldsymbol{\omega}, (1 - \alpha)\boldsymbol{\omega}\big)$.

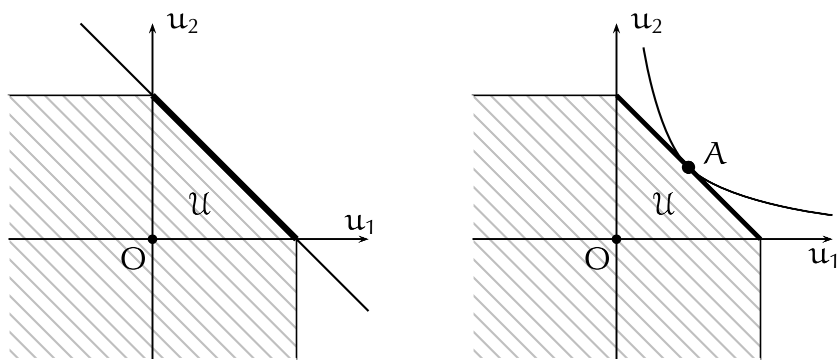

Figure 20.4.4: **Welfare Maxima.** In the left panel, all of the Pareto optima maximize the Benthamite social welfare function $W_b(\mathbf{u}) = u_1 + u_2$, as all of them lie on the same iso-welfare line.

In the right panel, we use the welfare function $W_1(u_1, u_2) = u_1 u_2$, which has a unique maximum at the Pareto optimal utility allocation $A = (3/2, 3/2)$.

20.4.4 Social Welfare and Cobb-Douglas Utility II

The situation shown here is somewhat atypical in that the Pareto frontier is usually not a straight line. If we replace the Cobb-Douglas utility function with an equivalent utility function that is strictly concave the Pareto frontier will no longer be a straight line. For example, using $v_i = \sqrt{u_i}$, turns the Pareto frontier into a quarter-circle, $\{(v_1, v_2) \geq 0 : v_1^2 + v_2^2 = 2^{3-2\gamma}\}$. Although there is no change in preferences, the same welfare function may now rank utility allocations differently and yield a different maximum. In this case, $W_b(\mathbf{u}) = u_1 + u_2$ has a unique maximum over $\mathcal{V}$ at $u_1 = u_2 = 2^{1-\gamma/2}$ which corresponds to the allocation $\{(1,4),(1,4)\}$. Keep in mind that the economic meaning of the welfare function $W_b(\mathbf{u}) = u_1 + u_2$ changes when we use a different utility representation. The function $W_2(\mathbf{u}) = u_1^{1/2} + u_2^{1/2}$ picks the same Pareto optima that $W_b(\mathbf{u}) = u_1 + u_2$ did prior to the transformation of utility.

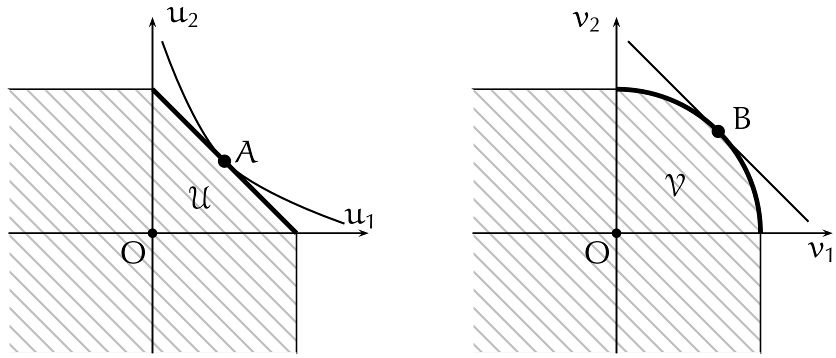

Figure 20.4.5: **More Welfare Maxima.** In the left panel, $\mathcal{U}$ is the same utility possibility as in Figure 20.4.5. But now we use the social welfare function $W_2(\mathbf{u}) = u_1^{1/2} + u_2^{1/2}$ to pick the same point $A = (3/2, 3/2)$.

In the right panel, $\mathcal{V}$ is the utility possibility set under the utility transformation $v_i = \sqrt{u_i}$ and we use the Benthamite social welfare function, which has a unique maximum at $B = (\sqrt{4.5}, \sqrt{4.5})$. The underlying preferences are the same in both cases, as is the resulting allocation of goods. However, we had to transform both utility and welfare to ensure that.

◀

20.4.5 Arrow and the Problem of Social Welfare

As Example 20.4.3 shows, the meaning of a social welfare function is tied to a particular utility representation. Since there is no standard utility representation, we have no standard way of normalizing welfare. A natural question is whether there is a way of ranking allocations based solely on consumer preferences. Arrow (1950) examined this question and gave a list of basic criteria any such ranking should obey

Arrow was able to prove that the only such ranking method was to appoint one person social dictator. In other words, there is no reasonable way of creating such a ranking.

More precisely, Arrow assumed that each individual had preferences over the entire allocation, not just their own consumption. This allows for externalities and public goods as well as considerations such as envy.

A *social preference order* is a mapping from the set of all collections of individual preference orders over allocations to a single (social) ranking of allocations.

20.4.6 Arrow Impossibility Theorem

The original version of the Arrow Impossibility Theorem was proved by Arrow (1950). Later, Arrow (1963) strengthened the theorem to the following:

Arrow Impossibility Theorem. *There is no way to map arbitrary sets of individual preferences into a social preference order that obeys the following:*

1. *Unrestricted Domain: For any set of allocations and any set of individual preferences over those allocations, the social preference order will rank those allocations.*
2. *Unanimity. If everyone prefers one allocation to another, it is socially preferred.*
3. *Independence of irrelevant alternatives. If individual preferences are changed in a way that has no effect on the ranking of two allocations, the social preference between those allocations remains unchanged. This applies regardless of how the individual rankings of other allocations have changed.*
4. *Non-dictatorship. There is no individual whose ranking always determines the social ranking.*

Arrow's Impossibility Theorem stands as a warning to those who take any particular utility representation too seriously as a measure of intensity of preference.

20.4.7 Developments since Arrow

There is a literature on how one may work around this result to create social rankings, variously by restricting the range of situations the ranking applies to, by incorporating information other than consumer preferences (e.g., cardinal utility), or by using a different set of basic criteria.

Some of these ranking methods explicitly violate independence of irrelevant alternatives in order to provide a standard of comparison. Changes in preferences that affect the standard can change the ranking between two options even if every consumer's ranking of those options remains unchanged.

These ideas have also been applied to voting procedures by Gibbard (1973) and Satterthwaite (1975) who found that strategic voting is generally possible.

20.4.8 Social Welfare as a Tool

In spite of their limitations, social welfare functions are generally useful in economics. They may be used to prove the existence of competitive equilibrium (Negishi, 1960), to investigate optimal taxation, to characterize non-Walrasian equilibria (e.g., the Lindahl equilibrium of section 20.5), or as ways of summarizing the changes from a given policy. Finally, they can be use to characterize Pareto optima.

Any point on the Pareto frontier can also be thought of as maximizing some social welfare function. Under mild convexity assumptions, it is the case that points on the Pareto frontier not only maximize a social welfare function, they maximize a *linear* social welfare function.

Theorem 20.4.6. *Suppose the utility possibility set* $\mathcal{U}$ *is convex and* $\mathbf{u}^*$ *is a Pareto optimum. There are weights* $\boldsymbol{\lambda} \in \mathbb{R}^I$, $\boldsymbol{\lambda} > 0$ *so that* $\mathbf{u}^*$ *solves the problem of maximizing the weighted sum of utilities* $\boldsymbol{\lambda} \cdot \mathbf{u}$ *over* $\mathcal{U}$.

Moreover, if $\boldsymbol{\lambda} \gg 0$, *there is a strictly increasing function* W *so that* $\mathbf{u}^*$ *is the unique maximum of* W *over the set* $\mathcal{U}$.

20.4.9 Proof of Social Welfare Theorem

Proof. Let $B = \{\mathbf{u} \in \mathbb{R}^I : \mathbf{u} \gg \mathbf{u}^*\}$. The set B is convex. Because $\mathbf{u}^*$ is Pareto optimal, $B \cap \mathcal{U} = \emptyset$ and $\mathcal{U}$ is comprehensive by definition.

By the corollary to Separation Theorem B, we can find $\boldsymbol{\lambda} > \mathbf{0}$ and α with $\boldsymbol{\lambda}\cdot\mathbf{b} \geq \alpha \geq \boldsymbol{\lambda}\cdot\mathbf{u}$ for all $\mathbf{b} \in B$ and $\mathbf{u} \in \mathcal{U}$. Note that $\boldsymbol{\lambda}\cdot\mathbf{u}^* \leq \alpha$ since $\mathbf{u}^* \in \mathcal{U}$. Also, $\mathbf{u}^* + \frac{1}{n}\mathbf{e} \in B$, so

$$\boldsymbol{\lambda}\cdot\mathbf{u}^* + \frac{1}{n}\boldsymbol{\lambda}\cdot\mathbf{e} \geq \alpha.$$

Letting $n \to \infty$, we find $\boldsymbol{\lambda}\cdot\mathbf{u}^* \geq \alpha$. It follows that $\boldsymbol{\lambda}\cdot\mathbf{u}^* = \alpha \geq \boldsymbol{\lambda}\cdot\mathbf{u}$ for all $\mathbf{u} \in \mathcal{U}$. In other words, $\boldsymbol{\lambda}\cdot\mathbf{u}$ is maximized over $\mathcal{U}$ at $\mathbf{u}^*$.

The linear function $\boldsymbol{\lambda}\cdot\mathbf{u}$ may not have a unique maximum if the maximum occurs at a flat spot in the boundary of $\mathcal{U}$. However, this is easy to fix when $\boldsymbol{\lambda} \gg \mathbf{0}$, which we assume in the second part of the proof. Without loss of generality, we may assume $\mathbf{u}^* \gg \mathbf{0}$. We can convert the problem to that form by subtract a constant from each utility function. We do this by choosing $\mathbf{u}_0 \ll \mathbf{u}^*$ and replacing $\mathbf{u}$ by $\mathbf{v} = \mathbf{u} - \mathbf{u}_0$ below.

Define $\gamma_i = p_i u_i^*/\boldsymbol{\lambda}\cdot\mathbf{u}^* > 0$ and set

$$W(\mathbf{u}) = \begin{cases} \prod_{i=1}^I u_i^{\gamma_i} & \text{when } \mathbf{u} \geq \mathbf{0} \\ 0 & \text{otherwise.} \end{cases}$$

This Cobb-Douglas function is increasing, at least for $\mathbf{u} \gg \mathbf{0}$. We must show that W has a unique maximum over $\mathcal{U}$ at $\mathbf{u}^*$. To see that, we consider W's maximum over the set $C = \{\mathbf{u} \geq \mathbf{0} : \boldsymbol{\lambda}\cdot\mathbf{u} \leq \alpha\}$. Since W is strictly quasi-concave for $\mathbf{u} \gg \mathbf{0}$, it will have a unique maximum over C.

We need only check the first-order conditions to find the maximum over C. They are that the marginal rates of substitution MRS_{ij} are equal to the welfare weights λ_i/λ_j. Now the marginal rates of substitution at $\mathbf{u}^*$ are

$$MRS_{ij} = \frac{\gamma_i}{u_i^*}\frac{u_j^*}{\gamma_j} = \frac{\lambda_i}{\lambda_j},$$

so $\mathbf{u}^*$ is the unique maximizer of W over C.

Now $W(\mathbf{u}) = 0$ for any $\mathbf{u} \in \mathcal{U} \setminus C$, so $\mathbf{u}^*$ is also the unique maximizer of W over $\mathcal{U}$. □

20.4.10 Social Welfare Maxima

Theorem 20.4.6 gives us an alternate way to find Pareto optima. Look for solutions to a social welfare maximization problem of the form.

$$\begin{aligned} &\max W\big(u_i(\mathbf{x}^i)\big) \\ &\quad \text{s.t. } (\mathbf{x}^1, \dots, \mathbf{x}^I) \text{ is a feasible allocation} \end{aligned}$$

for some social welfare function W. When W is strictly increasing in each argument, it is easy to see that any feasible utility allocation that maximizes W must be Pareto optimal.

Theorem 20.4.7. *Suppose W is a Bergson-Samuelson social welfare function that is strictly increasing in each argument. Let* $\mathcal{E} = (\mathfrak{X}_i, u_i, \boldsymbol{\omega}^i, Y_f, \theta^i_f)$ *be a production economy and suppose the allocation* $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ *maximizes W among all feasible allocations for* $\mathcal{E}$. *Then* $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ *is a Pareto optimal allocation.*

Proof. If there is a feasible Pareto improvement $(\mathbf{x}^i, \mathbf{y}^f)$, $u_i(\mathbf{x}^i) \geq u_i(\hat{\mathbf{x}}^i)$ for all i and $u_j(\mathbf{x}^j) > u_j(\hat{\mathbf{x}}^j)$ for at least one j. This means that $W\big(u_1(\mathbf{x}^1), \dots, u_I(\mathbf{x}^I)\big) > W\big(u_1(\hat{\mathbf{x}}^i), \dots, u_I(\hat{\mathbf{x}}^I)\big)$ contradicting the fact that $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f)$ maximizes social welfare. □

The Second Welfare Theorem then tells us that any social welfare maximum is also a quasi-equilibrium with taxes and transfers. Social welfare maxima where both consumers have positive income will be Walrasian equilibria with transfers by Corollary 20.3.3. When there is a Walrasian equilibrium, it will be on the Pareto frontier.

This suggests an alternative method for finding a Walrasian equilibrium. Find a social welfare function where the transfers are zero. This can be done by adjust the weights in the welfare function (reduce the weight on consumers who get positive transfers, lower them for negative transfers, then use a fixed point theorem to find weights with no transfers). This type of argument was used by Negishi (1960) to show the existence of a Walrasian equilibrium.

In practice, Pareto optima are easier to find when the first-order conditions apply. Sometimes, equating the MRS^i and MRT^f can go a long way toward solving the problem.

20.4.11 Pareto Optima with CRS Production Revisited

We close this section by looking at Pareto optima as both social welfare maxima and equilibria with taxes and transfers in a model with production and Cobb-Douglas utility.

Example 20.4.8: Pareto Optima with CRS Production Revisited. Let's return to the setting of Examples 19.3.4 and 15.4.1. The two consumers have utility $(x_1^i)^{1/2}(x_2^i)^{1/2}$. Endowments are $\boldsymbol{\omega}^1 = (2,0)$ and $\boldsymbol{\omega}^2 = (1,0)$, yielding total endowment $\boldsymbol{\omega} = (3,0)$. The production set is $Y = \{(y_1, y_2) : y_2 \leq -y_1, y_1 \leq 0\}$.

We know the firm produces $(-3/2, 3/2)$ at any Pareto optimum and that total consumption is $(3/2, 3/2)$. Each consumer gets a fraction of this. We can write $\mathbf{x}^1 = t(3/2, 3/2)$ and $\mathbf{x}^2 = (1-t)(3/2, 3/2)$ for some t, $0 \leq t \leq 1$. The resulting utility allocation is $u_1 = 3t/2$ and $u_2 = 3(1-t)/2$. The utility possibility set is then

$$\mathcal{U} = \{(u_1, u_2) : u_1 + u_2 \leq 3/2, u_1, u_2 \leq 3/2\}.$$

Since the Pareto frontier is a line segment, all of these Pareto optima maximize the Benthamite social welfare function $W_b(\mathbf{u}) = u_1 + u_2$ over $\mathcal{U}$. The various Pareto optima also uniquely solve other social welfare maximization problems. The case where both consumers receive the same utility arises when maximizing $W(u_1, u_2) = \sqrt{u_1 u_2}$.

The other Pareto optima maximize $W_\alpha(\mathbf{u}) = u_1^\alpha u_2^{1-\alpha}$ for some α with $0 < \alpha < 1$. We interpret W_0 as u_2 and W_1 as u_1. These welfare functions have maxima at the utility allocations $\mathbf{u} = \big(3\alpha, 3(1-\alpha)\big)$. The corresponding allocations of consumption are $\mathbf{x}^1 = \alpha(3/2, 3/2)$ and $\mathbf{x}^2 = (1-\alpha)(3/2, 3/2)$ for $0 \leq \alpha \leq 1$.

We can also think of these allocations as Walrasian equilibria with taxes and transfers. Here the price vector is $\mathbf{p} = (1,1)$ (or any positive multiple thereof) and the wealth levels are $m_\alpha^1 = 3\alpha$ and $m_\alpha^2 = 3(1-\alpha)$. The values of the endowments are $\mathbf{p}\cdot\boldsymbol{\omega}^1 = 2$ and $\mathbf{p}\cdot\boldsymbol{\omega}^2 = 1$, so the actual transfers are $3\alpha - 2$ and $3(1-\alpha) - 1$. No transfers take place when $\alpha = 2/3$, which yields the original Walrasian equilibrium. ◀

20.5 Public Goods

We can use the social welfare function to characterize Pareto optima in cases where there are externalities or public goods. Let's take a simple case of an economy with one public good and one private good. The difference between the goods is that consumption of the private good is *rival*. Each unit of the private good consumed by one person is not available for anyone else in the economy. Once one person has eaten a slice of bread it cannot be eaten again by others. The public good is *non-rival*. Everyone can simultaneously consume all of the public good that is produced. One person's consumption does not reduce anyone else's consumption. Thus the same public radio broadcast can be listened to by many consumers.

Rivalry is not to be confused with excludability. A good is *excludable* if consumers can be prevented from using it at zero (or near zero) cost. Thus a radio signal that is encrypted is both non-rival (anyone can receive the signal) and excludable (you need the code to listen to it). Excludability does not play a role in finding the social optimum, but it could play a role in making the Lindahl equilibrium below actually work.

The term *congestion* is applied to goods that are neither fully rival nor fully non-rival. A freeway is an example. If there are few cars on the freeway, adding one more has little effect on the ability of the other drivers to make use of the freeway. However, as traffic grows, the costs to the other drivers start to increase. If there is too much traffic, a traffic jam ensues and the benefit to using the freeway falls considerably.

20.5.1 The Optimal Provision of a Public Good

We consider the simple case of one rival (private) good and one non-rival (public) good. Suppose there are I consumers. Let x_1^i denote the amount of the private good consumed by i and x_2 the (common) amount of the public good consumed by all. Each consumer will have an endowment ω^i of the private good. Let $\omega = \sum_i \omega^i$. A $\mathcal{C}^2$ production function $f\colon \mathbb{R} \to \mathbb{R}$ with $f' > 0$ and $f'' < 0$ describes production of the public good from the private good. Since the private good is rivalrous, $\sum_i x_1^i + z \leq \omega$ where z is the amount of the private good used as input for production of the public good. We will presume preferences are monotonic, so $\sum_i x_1^i + z = \omega$ at any Pareto optimum. An amount $q \leq f(z)$ of the public good is produced and available for consumption by all. Thus $x_2 = q$. This gives us the maximization problem:

$$
\begin{aligned}
\max\ & W\big(u_i(x_1^i, x_2)\big) \\
\text{s.t. } & \sum_i x_1^i + z = \omega \\
& f(z) = x_2
\end{aligned}
$$

20.5.2 Solution of the Public Good Problem

The Lagrangian is

$$\mathcal{L} = W\big(u_i(x_1^i, x_2)\big) + \lambda\left(\omega - z - \sum_i x_1^i\right) + \mu\big(f(z) - x_2\big).$$

The first-order conditions are:

$$\frac{\partial W}{\partial u_i}\frac{\partial u_i}{\partial x_1^i} = \lambda, \quad \sum_i \frac{\partial W}{\partial u_i}\frac{\partial u_i}{\partial x_2} = \mu, \quad \text{and} \quad \mu f'(z) = \lambda.$$

We divide the second equation by λ, noticing that

$$\frac{\partial W}{\partial u_i}\frac{\partial u_i}{\partial x_2}\Big/\lambda = \frac{\partial u_i}{\partial x_2}\Big/\frac{\partial u_i}{\partial x_1^i} = \text{MRS}_{21}^i \,.$$

This yields

$$\sum_i \text{MRS}_{21}^i = \mu/\lambda = \text{MRT}_{21}\,.$$

This is Samuelson's (1954) condition that the sum of the marginal rates of substitution must equal the marginal rate of transformation. It is possible to build an equilibrium notion around these first-order conditions.

20.5.3 Lindahl Equilibrium I

Suppose we use good one as numéraire. The marginal value of a unit of good 2 to consumer i is MRS^i_{21}. In terms of supply and demand, MRS^i_{21} is the demand price (marginal value) of good two. The traditional method for dealing with public goods to add the demand curves vertically rather than horizontally. The resulting aggregated demand curve has demand price $\sum_i \mathrm{MRS}^i_{21}$.

The supply curve is the marginal cost curve, so the supply price is $\mathrm{MC} = 1/f'(z)$. Thus $\mathrm{MC} = \sum_i \mathrm{MRS}^i_{21}$ determines the intersection of supply with the vertically summed demand curve. Once we have the intersection, the number MRS^i_{21} is the shadow price of good two to consumer i.

If we treat the marginal rates of substitution as individualized prices we are led to the Lindahl equilibrium. The definition of Lindahl equilibrium below is for the model considered here with one private and one public good.[7] It can be generalized to the case of many goods.

Lindahl Equilibrium. A *Lindahl equilibrium with taxes and transfers* consists of an allocation $\big(\mathbf{x}^i, (-q, z)\big)$ and prices $(p_1, p_2^i) \in \mathbb{R}^{I+1}_+$ together with wealth levels $(m^1, \dots, m^I) \in \mathbb{R}^I_+$ obeying $\sum_i m^i = \big(\sum_i p_2^i\big)q - p_1 z$ such that:

1. $q \leq f(z)$ and $\big(\sum_i p_2^i\big)q - p_1 z \geq \big(\sum_i p_2^i\big)q' - p_1 z'$ for all (q', z') with $z' \geq 0$ and $q' \leq f(z')$. (Profit maximization)
2. For all consumers i, (x_1^i, x_2^i) maximizes utility over $\{\mathbf{x}^i \in \mathfrak{X}_i : p_1 x_1 + p_2^i x_2 \leq m^i\}$. (Utility maximization)
3. $\sum_i x_1^i + z = \omega$ and $x_2^i = q$ for all i. (Market clearing)

[7] Lindahl (1919) was the first to solve this type of problem, proposing a solution that is similar to the modern Lindahl equilibrium.

20.5.4 Lindahl Equilibrium II

In the Lindahl equilibrium, each consumer faces an individualized price for the public good (good 2). The firm gets to collect the individualized price from each consumer, yielding a total revenue of $\sum_i p_1^i$ from each unit of good 2. The private good (good 1) has a uniform price paid by each consumer and the firm. Given this, firms maximize profit, consumers maximize utility, and markets clear.

We will not demonstrate that every Pareto optimum in a public goods economy is a Lindahl equilibrium (under suitable conditions). However, you will note that the first-order conditions for the firm and consumers ensure that $\text{MRT}_{21} = \sum_i \text{MRS}_{21}^i$, which is the Pareto optimality condition.

When the public good is excludable, one can imagine this equilibrium being used. When the public good is not excludable, there is no incentive to pay for it. However, the Lindahl equilibrium then describes an individualized tax system that pays for public good production.

In reality, there would still be incentive problems in terms of getting people to reveal their preferences, regardless of which case we're in. After all, if I say the good is worth \$1 to me when it is really worth \$2, it only affects production slightly and my gains from a 50% lower tax can easily outweigh the production losses. There is a literature on Clarke taxes and related mechanisms that attempt to solve this revelation problem. E.g., see Vickery (1961), Clarke (1971), and Groves (1973).

It is also possible to include externalities in the equilibrium concept. Boyd and Conley (1997) define a notion of Coasian equilibrium for an economy with both externalities and public goods, and prove First and Second Welfare Theorems for this Coasian equilibrium.

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