21. The Edgeworth Conjecture

Walras imagined the economy as a collection of consumers and firms that reacted to the prices they faced. However, he was unclear about exactly how that worked. When arguing for existence of an equilibrium, he posited an auctioneer who organized the market. Walras was not so foolish as to think that such a mechanism for finding an equilibrium actually existed, but his use of the auctioneer highlights a deficiency in the Walrasian approach. It doesn't address price formation.

21.0.1 Edgeworth on Exchange

Instead of thinking of the economy in Walrasian terms, we go back to the ideas of Edgeworth (1881). Edgeworth's approach to equilibrium begins at the other end of the market from Walras. Rather than treating markets as impersonal seas of price-taking consumers and producers, Edgeworth made the market personal.

Edgeworth began his analysis by considering two people bargaining over the exchange of goods. Each person brings an endowment of goods to the bargaining table. They can either agree to trade with each other, or take their endowments home and consume them. Edgeworth then asked what trades such consumers would be willing to make.

So what happens if the consumers trade until all gains from trade have been realized? Edgeworth considered economies with more consumers and conjectured that in a large economy, the result of such trades would be a Walrasian equilibrium allocation. The first proof of such a result was the Debreu-Scarf Limit Theorem (1963), followed closely by a proof for continuum economies by Aumann (1964).

21.0.2 Chapter Outline

Section one starts by examining the incentives to trade. Section two builds on this to define the core and shows that any Walrasian equilibrium allocation is in the core. Section three shows how to enlarge the economy by replicating the consumers and demonstrates that consumers of the same type must be treated equally in the core. Section four proves the Debreu-Scarf Limit Theorem, showing that the allocations that are in the core of every replica are exactly the Walrasian equilibrium allocations.

Outline:

1. Incentives to Trade
2. The Core of an Economy
3. Replica Economies
4. Core Equivalence

21.1 Incentives to Trade

Edgeworth's bargaining-based bottom-up approach also contrasts with the top-down procedure of social welfare maximization examined in Chapter 20.4. When examining Pareto optimality, we look at the whole economy in relation to aggregate resources. The welfare optimum picks from the Pareto optima without regard for the individual endowments. Only the aggregate endowment matters.

In contrast, the endowment is vitally important for Edgeworth's method. It is the starting point for trade. This means the endowments play a key role by determining which trades are possible. The final results will depend on the initial distribution of resources.

Now consider consumer incentives to trade. Neither consumer will be willing to trade if the trade makes them worse off than their endowment. If they are not happy, they will simply take their endowment and go home. This is the requirement of *individual rationality*.

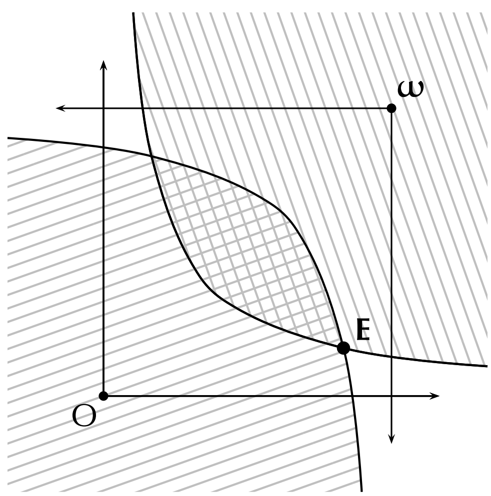

Figure 21.1.1: The endowment point is denoted E. The individually rational allocations are those in the cross-hatched region between both indifference curves. Any trade in that region benefits both consumers.

Any competitive equilibrium also obeys individual rationality. A consumer can always afford to keep their endowment. If they choose to buy some other basket of goods, it is because it makes them better off.

21.1.1 Individual Rationality

Suppose an economy has constant returns to scale production. We will presume that everyone has access to a common production technology Y.

Individual Rationality. An allocation $(\mathbf{x}^i, \mathbf{y})$ is called *individually rational* if there is no consumer j and net output $\mathbf{y} \in Y$ with $\boldsymbol{\omega}^j + \mathbf{y} \succ_j \mathbf{x}^j$.

In other words, an allocation is individually rational if it makes each consumer at least as well off as they could make themselves using their own endowment and the common technology. In exchange economies the common technology is $Y = \mathbb{R}^m_-$, and individual rationality means that each consumer gets something at least as good as their own endowment.

21.1.2 Pareto Optimality

Individual rationality is not the only requirement for Edgeworthian trading. If trades have not resulted in a Pareto optimum, more trading is still possible. They will not have exhausted the trading possibilities until they reach a Pareto optimum. We expect them to continue to trade until they end up at an allocation that is both Pareto optimal and individually rational.

This is the end of the story when there are two consumers. Edgeworth called the resulting allocations the *contract curve*. We call them *core allocations*. With only two consumers there are many possibilities on the contract curve, including the perfectly efficient monopoly case where one consumer or the other gains all of the surplus.

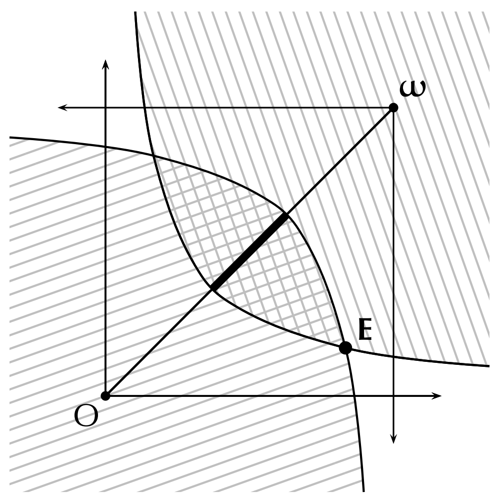

Figure 21.1.2: The endowment point is denoted E. The individually rational allocations are those in the cross-hatched region between both indifference curves. Any trade in that region benefits both consumers. The diagonal is the set of Pareto optima, where the trading possibilities have been exhausted. The allocations that are both Pareto optimal and individually rational form the contract curve C, shown by the heavy diagonal line.

Edgeworth's idea was that as more people join the economy, the increased competition would put more constraints on the terms of trade. He argued that the contract curve would shrink as the economy grows, with the competitive equilibrium emerging in the limit as the economy becomes large.

21.1.3 Trading between Two Cobb-Douglas Consumers

Let's consider an example with two Cobb-Douglas consumers.

Example 21.1.3: Cobb-Douglas: Trading with Two Consumers. There are two consumers with identical, equal-weighted Cobb-Douglas preferences, $u_i(\mathbf{x}^i) = (x_1^i)^{1/2}(x_2^i)^{1/2}$. Endowments are $\boldsymbol{\omega}^1 = (2.5, 0.5)$ and $\boldsymbol{\omega}^2 = (0.5, 2.5)$, yielding aggregate endowment $\omega = (3, 3)$. We know that the set of Pareto optimal allocations is the diagonal of the Edgeworth box. Thus $x_1^i = x_2^i \geq 0$ and $x_1^1 + x_1^2 = 3$ characterize the Pareto optima.

Individual rationality requires $u_i(\mathbf{x}^i) \geq u_i(\boldsymbol{\omega}^i) = \sqrt{1.25}$. The set of possible final trades is then

$$\{(x_1^1, x_2^1, x_1^2, x_2^2) : \sqrt{1.25} \leq x_1^i = x_2^i \leq 3 - \sqrt{1.25}, x_1^1 + x_1^2 = 3\}.$$

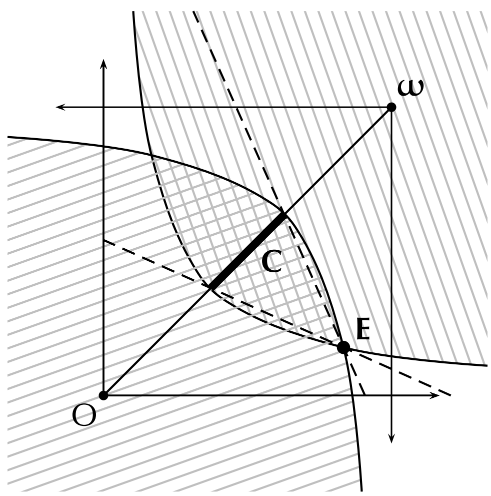

Figure 21.1.4: The endowment point is denoted E. The individually rational allocations are those in the cross-hatched region between both indifference curves. Any trade in that region benefits both consumers. With identical Cobb-Douglas preferences, the diagonal is the set of Pareto optima—the allocations where the trading possibilities have been exhausted. The allocations that are both Pareto optimal and individually rational form the contract curve C, shown by the heavy diagonal line.

The dashed lines indicate the limits of the implied prices. Most of these prices are not competitive equilibria, but indicate trades involving some degree of market power, with monopoly pricing by one side or another at either end.

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21.1.4 Equilibrium in Bilateral Monopoly

Before going on to consider markets with competition, we'll take a quick look at possibilities with only two individuals in the market. This situation is called *bilateral monopoly*. It occurs when we have two individuals that are the sole sellers of two different goods. That happens in a Edgeworth box when there are two individuals and a region of mutual gain. Edgeworth showed they will end up on the contact curve C. Where exactly they end up on the contract curve depends on the bargaining power and skill of the the two individuals.

By appealing to the Second Welfare Theorem, we can think of any point on the contract curve as an equilibrium with taxes and transfers. We can reinterpret that as one side selling to the other by means of a two-part tariff.

A *tariff* is a schedule of prices, so a *two-part tariff* is a pricing system with two prices. The first part is often called an entry fee or connection fee. By paying it, you get the right to buy the product a given price, the second part of the tariff. Here we have two goods, so we need to know the relative price, which will follow if we know the prices of the two goods.

So if equilibrium with taxes and transfers has individual one paying a tax, we treat that as the entry fee. By the budget balance condition it is the transfer to individual two. The then relative price of the two goods determines the second part of the equivalent two-part tariff.

21.1.5 Equilibrium with a Two-part Tariff

I've illustrated below how a point where individual two collects all of the surplus can be attained using a two-part tariff. As noted above, any point on the contract curve can result from an appropriate two-part tariff.

Here the contact curve always corresponds to a price vector of $(1, 1)$, and by putting a budget line through the desired point, we find the entry fee to collect by comparing with the value of the endowment. The relative price of 1 determined by the price vector provides the second part of the tariff.

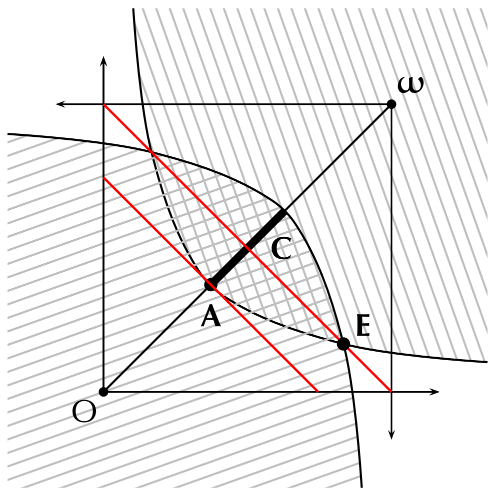

Figure 21.1.5: This diagram has been modified from Figure 21.1.4. As before, the endowment point is denoted E. Individual two can get the monopoly solution via a two-part tariff. This is at the end of the contract curve C, where individual two has maximimzed utility over the contract curve.

The first part of the tariff is the the difference in value of the two red budget lines. The second part is the price vector is $(1, 1)$. That tariff leads to individual one finding that the optimal choice under such a tariff, point A, is equally good as the endowment.

21.1.6 Making the Economy Larger with more Consumers

We turn this into a model of large economy by adding more and more people. Again we ask what trades they would be willing to make. The presence of more people increases competition and full monopoly pricing is no longer possible.

If Ada wants to make a trade with Bill and Chris, she not only has to offer them something more valuable than what they have. She has to offer them something more valuable than they could get by trading with each other. The larger the population is, the more potential competitors there are.

We are not limited to two-way trades either. Trades can be arranged among larger groups with many individuals. This possibility enforces a kind of Pareto optimality within each possible subgroup. There is not only competition between individuals for your business, but also between groups for your business.

With increased competition, people are no longer willing to settle for bad deals when they can get a better deal elsewhere. Edgeworth's idea was that in very large markets, all trades must take place on the same terms, and these terms of trade define the equilibrium prices.

21.2 The Core of an Economy

For the rest of the chapter we focus on economies with constant returns to scale production (including exchange economies).[1] The economy is defined by $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)_{i=1}^I$ where Y describes the aggregate technology. We require that Y is a non-empty closed set obeying inaction, constant returns to scale, and the no free lunch condition.[2]

[1] As shown in sections 14.3 and 16.7, we can translate diminishing returns technologies into constant returns by adding an "entrepreneurial factor". The use of a constant returns to scale production set considerably simplifies the proof of the Debreu-Scarf Limit Theorem.

[2] For the moment we do not require free disposal. This allows us to consider exchange economies with $Y = \{0\}$. It that case, allocations obey $\sum_i \mathbf{x}^i = \sum_i \boldsymbol{\omega}^i$.

21.2.1 Coalitions

We need to be able to deal with groups of consumers, coalitions. A *coalition* is a group of consumers.

Coalition. Let $\mathcal{I} = \{1, \ldots, I\}$ denote the set of consumers. A *coalition* is a non-empty subset of $\mathcal{I}$.

The total number of possible coalitions is $2^I - 1$, ranging from single person coalitions to the coalition $\mathcal{I}$, the *coalition of the whole*.

The endowment of a coalition S is denoted $\boldsymbol{\omega}^S$. It is the aggregate endowment of the individuals in S, so

$$\boldsymbol{\omega}^S = \sum_{i \in S} \boldsymbol{\omega}^i.$$

Given economy $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)$, the endowment of the whole economy $\boldsymbol{\omega}$ can be written $\boldsymbol{\omega}^{\mathcal{I}}$.

Each coalition is allowed free access to the common technology, limited only by their resources. Because the technology is additive, what one coalition does with their resources does not affect the production possibilities of other coalitions.[3]

In this economy, $(\mathbf{x}^1, \ldots, \mathbf{x}^I, \mathbf{y}) \in \mathbb{R}^{Im}_+ \times \mathbb{R}^m$ is a *feasible allocation*, if $\sum_i \mathbf{x}^i = \mathbf{y} + \sum_i \boldsymbol{\omega}^i$. The feasibility condition can be written more compactly in our notation: $\mathbf{x}^{\mathcal{I}} \in \boldsymbol{\omega}^{\mathcal{I}} + Y$.

[3] Diminishing returns to scale would create complications here. See Aliprantis, Brown, and Burkinshaw (1987).

21.2.2 Blocking Coalitions

The concept of the core formalizes Edgeworth's idea that bargaining proceeds until all trading possibilities have been exploited. We look for allocations of that kind—allocations that are the endpoint of Edgeworthian bargaining. These allocations are the *core*. At a core allocation, no group of individuals is able to make any trade amongst themselves, even if augmented by production, that makes each of them better off. If such an improving trade is possible for a coalition S, we say that S *blocks* or *improves on* the original allocation.

Block or Improve On. A coalition $S \subset \mathcal{I}$ can *block* or *improve on* the allocation $(\bar{\mathbf{x}}^i, \mathbf{y})$ if there are $\mathbf{x}^i \geq 0$ for $i \in S$ such that:

1. $\mathbf{x}^i \succ_i \bar{\mathbf{x}}^i$ for all $i \in S$.
2. $\mathbf{x}^S \in \boldsymbol{\omega}^S + Y$.[4]

[4] This condition needs modification when diminishing returns are allowed. Again see Aliprantis, Brown, and Burkinshaw (1987).

21.2.3 Core Allocations

The core consists of the allocations that cannot be improved on (blocked) by any coalition using only their own resources and the common technology.[5]

Core. The *core* of an economy $\mathcal{E}$ is the set of feasible allocations that cannot be improved on (blocked) by any coalition. Such allocations are called *core allocations*. We denote the set of core allocations in $\mathcal{E}$ by $\mathbf{C}(\mathcal{E})$.

[5] Edgeworth used the term *contract curve*. Gillies (1953, 1959) introduced the game theoretic concept of the core, which in this context, coincides with Edgeworth's contract curve.

21.2.4 Core and Individual Rationality

Core allocations are always individually rational.

Proposition 21.2.1. *Any core allocation is individually rational for every consumer.*

Proof. Suppose $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ is a core allocation. Let j be a consumer. That consumer constitutes the coalition $S = \{j\}$. Since $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ is in the core, the coalition $\{j\}$ cannot improve on $(\hat{\mathbf{x}}^i)$. In other words, $\hat{\mathbf{x}}^j \succsim_j \mathbf{x}^j$ for every $\mathbf{x}^j \in \boldsymbol{\omega}^j + Y$. Since j was arbitrary, the allocation $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ is individually rational. □

Individual rationality thus means that no singleton can block. If this is a one person economy, it is the only requirement to be in the core. There are no other coalitions to consider. In this Robinson Crusoe case, being in the core just means that the consumer maximizes utility given their endowment and the production technology.

21.2.5 Core and Pareto Optimality

In any economy with two or more consumers there will be at least one other type of coalition to consider—the coalition of the whole. It is clear that any Pareto optimal allocation cannot be improved on by the coalition of the whole because such an improvement would be a feasible Pareto improvement. However, there can be allocations that are not quite Pareto optimal that can not be blocked by the coalition of the whole. The problem is that everyone in the economy must be made strictly better off for the coalition of the whole to improve on an allocation. This is not quite the same as making a Pareto improvement because it requires that **everyone** in the coalition be made strictly better off. A Pareto improvement for the coalition would only require that everyone in the coalition be at least as well off, and that at least one person be better off.

To examine this issue further we introduce a second type of Pareto optimality based on the concept of a strong Pareto improvement.

Strong Pareto Improvement. The type of improvement used in the definition of the core is not exactly the same as Pareto improvement. Instead of making one person better off without harming anyone else, it requires that all people be made better off. We say that $(\mathbf{x}^i, \mathbf{y})$ is a *strong Pareto improvement* over $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ if $\mathbf{x}^i \succ_i \hat{\mathbf{x}}^i$ for all i.

These two notions of Pareto improvement, ordinary and strong, lead to two notions of Pareto optimality. Pareto optimality requires that there are no feasible (ordinary) Pareto improvements. This type of Pareto optimality is also called *strong Pareto optimality*.

Weak Pareto Optimum. A feasible allocation is a *weak Pareto optimum* if there are no feasible strong Pareto improvements.[6]

[6] The terminology works this way because ruling out **any** Pareto improvement is a bit harder than merely ruling out strong Pareto improvements. Because of this it is easier to be a weak Pareto optimum than an ordinary (strong) Pareto optimum.

21.2.6 Weak Pareto Optimality

An example using the utility possibility set will help clarify the difference.

Example 21.2.2: Non-blockable, non Pareto optimal allocations. Define a two-person exchange economy with utility $u_1(\mathbf{x}) = x_1$ and $u_2(\mathbf{x}) = x_2 + \sqrt{x_1}$. Endowments are $\boldsymbol{\omega}^1 = \boldsymbol{\omega}^2 = (1,1)$. We turn our attention to the goods allocation $\mathbf{x}^1 = (2,1)$ and $\mathbf{x}^2 = (0,1)$ yielding utility allocation $\mathbf{u}^1 = (2,1)$. It is impossible to further increase consumer one's utility. It follows that the coalition of the whole cannot improve on $(\mathbf{x}^1, \mathbf{x}^2)$. However, the allocation $\mathbf{u}^1$ is not Pareto optimal as $\big((2,0),(0,2)\big)$, with utility allocation $\mathbf{u}^2 = (2,2)$ is a Pareto improvement. The utility allocation $\mathbf{u}^1$ is merely weakly Pareto optimal.

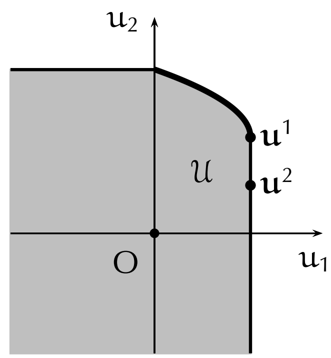

Figure 21.2.3: Here $\mathcal{U}$ is the utility possibility set for Example 21.2.2. The heavy line indicates set of standard (strong) Pareto optima. The solid lines on the top and side of $\mathcal{U}$ indicate additional utility allocations that are **weakly** Pareto optimal but **not strongly** Pareto optimal. The allocation $\mathbf{u}^2$ Pareto dominates the allocations directly below it, such as $\mathbf{u}^1$, but it does not strongly dominate them. As a result, those allocations are weakly Pareto optimal, but not Pareto optimal.

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21.2.7 Weak and Strong Pareto Improvements

March 23, 2023

When preferences are strongly monotonic, any Pareto improvement can be used to create a strong Pareto improvement.

Lemma 21.2.4. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)_{i=1}^I$ be an economy with a constant returns to scale aggregate production set Y, where each consumption set is $\mathfrak{X}_i = \mathbb{R}^m_+$, and each consumer has continuous, convex, and strongly monotonic preferences. If there is a feasible Pareto improvement over an allocation $(\mathbf{x}^i, \mathbf{y})$, then there is a feasible strong Pareto improvement over $(\mathbf{x}^i, \mathbf{y})$.*

Proof. Suppose $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ is feasible and a Pareto improvement over $(\mathbf{x}^i, \mathbf{y})$. Take a consumer h so that $\hat{\mathbf{x}}^h \succ_h \mathbf{x}^h$. By monotonicity and the fact that $\mathfrak{X}_h = \mathbb{R}^m_+$, $\mathbf{x}^h \succsim_h \mathbf{0}$. It follows that $\hat{\mathbf{x}}^h > \mathbf{0}$, that there is some good ℓ with $\hat{x}^h_\ell > 0$.

By continuity we may take $\varepsilon > 0$ small enough that $\bar{\mathbf{x}}^h = \hat{\mathbf{x}}^h - \varepsilon \mathbf{e}^\ell \succ_h \mathbf{x}^h$. For $i \neq h$, define

$$\bar{\mathbf{x}}^i = \hat{\mathbf{x}}^i + \left(\frac{\varepsilon}{I-1}\right)\mathbf{e}^\ell \succ_i \hat{\mathbf{x}}^i \succsim_i \mathbf{x}^i.$$

The strong preference is due to strong monotonicity. Then $\bar{\mathbf{x}}^i \succ_i \mathbf{x}^i$ for all consumers i. Because $\sum_i \bar{\mathbf{x}}^i = \sum_i \hat{\mathbf{x}}^i$, the allocation $(\bar{\mathbf{x}}^i, \hat{\mathbf{y}})$ is feasible. It follows that $(\bar{\mathbf{x}}^i, \hat{\mathbf{y}})$ is a feasible strong Pareto improvement over $(\mathbf{x}^i, \mathbf{y})$. □

As a corollary, weak and strong Pareto optima are the same for economies satisfying the hypotheses of Lemma 21.2.4.

Corollary 21.2.5. *If $\mathcal{E}$ satisfies the hypotheses of Lemma 21.2.4, the two types of Pareto optima are identical.*

21.2.8 The Core is Pareto Optimal

It immediately follows that any core allocation is Pareto optimal in such economies.

Proposition 21.2.6. *Suppose $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)_{i=1}^I$ be an economy with a constant returns to scale aggregate production set Y, where each consumption set is $\mathfrak{X}_i = \mathbb{R}^m_+$, and each consumer has continuous, convex, and strongly monotonic preferences. Then any core allocation is Pareto optimal.*

Proof. This satisfies the hypotheses of Lemma 21.2.4. Suppose $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ is a core allocation. If a Pareto improvement were feasible, Lemma 21.2.4 would yield a feasible strong Pareto improvement. But then the coalition of the whole would be able improve over $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$, contradicting the fact that $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ is in the core. This contradiction shows that no feasible Pareto improvements are possible—that $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ is Pareto optimal. □

21.2.9 The Core in a 2-person Economy

Suppose an economy has two consumers. In this case core allocations need only satisfy a few conditions. There are 3 possible coalitions—the two singletons $\{1\}$ and $\{2\}$, and the coalition of the whole $\{1, 2\}$. The requirement that neither $\{1\}$ nor $\{2\}$ alone can improve is individual rationality. The requirement that $\{1, 2\}$ cannot block is Pareto optimality. There are no other coalitions to consider. In a two-person economy, the core coincides with the set of allocations that are both Pareto optimal and individually rational.

Example 21.2.7: Leontief: Core with Two Consumers. We use the exchange economy as in Examples 15.3.6 and 19.2.8 but with endowments $\boldsymbol{\omega}^1 = (2, 0.5)$ and $\boldsymbol{\omega}^2 = (1, 1.5)$. The social endowment is $\boldsymbol{\omega} = (3, 2)$. Utility has the Leontief form $u_i(\mathbf{x}^i) = \min\{x_1^i, x_2^i\}$. Since this is a two person exchange economy, any individually rational Pareto optimum is in the core.

Individual rationality requires $u_i(\boldsymbol{\omega}^i) \le u_i(\mathbf{x}^i)$, which means $0.5 \le u_1(\mathbf{x}^1)$ and $1 \le u_2(\mathbf{x}^2)$. For person one, anything more than $(0.5, 0.5)$ will do, while person two requires $x_1^2, x_2^2 \ge 1$, which translates to $x_1^1 \le 2$ and $x_2^1 \le 1$.

Using Example 19.2.8, the core is $\{(\mathbf{x}^1, \mathbf{x}^2) \in \mathbb{R}_+^4 : 0.5 \le x_2^1 \le x_1^1 \le 1 + x_2^1,\ x_1^2 \ge 1,\ x_2^2 \ge 1 \text{ and } \mathbf{x}^1 + \mathbf{x}^2 \le (3, 2)\}$. Except for the points where good one is not fully allocated, this is illustrated in Figure 21.2.8.

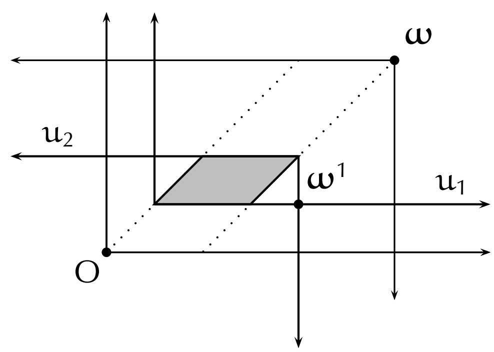

Figure 21.2.8: As in Example 19.2.8, the Pareto optimal allocations lie between the 45-degree lines through the origins for both consumer one and two. Individual rationality requires that the core be above the u_1 indifference curve and below the u_2 indifference curve. Combining these criteria, we find that the core is the shaded area.

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21.2.10 The Core in a 3-person Economy

In larger economies, Pareto optimality and individually rationality are both still required, but they are no longer sufficient to show an allocation is in the core. These two conditions account for the singleton coalitions and the coalition of the whole, but now, there are other coalitions.

These other coalitions will usually impose additional requirements on the core. In fact, in some economic settings those additional requirements will be so stringent that the core will be empty. That is not a concern if there is a Walrasian equilibrium.

The following example has three consumers, and the combination of Pareto optimality and individual rationality is not enough to determine the core.

21.2.11 The Core in a 3-person Cobb-Douglas Economy

Example 21.2.9: Cobb-Douglas: Core with Three Consumers. Suppose three consumers have identical Cobb-Douglas utility $u(\mathbf{x}) = (x_1)^{1/4}(x_2)^{3/4}$ and endowments are $\boldsymbol{\omega}^1 = (2,0)$, $\boldsymbol{\omega}^2 = (1,1)$, and $\boldsymbol{\omega}^3 = (1,3)$.

As we know, identical Cobb-Douglas utility implies that Pareto optimal points will have utility levels that sum to the utility of the aggregate endowment. Thus

$$u_1 + u_2 + u_3 = u(4,4) = 4. \tag{21.2.1}$$

The corresponding allocation gives each consumer a share of the aggregate endowment that is proportional to the utility level. Thus $\mathbf{x}^i = \big(u_i/u(\boldsymbol{\omega})\big)(4,4) = u_i(1,1)$. Moreover, $0 \le u_i \le 4$ for each consumer i.

Individual rationality imposes three constraints:

$$\begin{aligned} u_1 &= u_1(\mathbf{x}^1) \ge u(\boldsymbol{\omega}^1) = 0,\\ u_2 &= u_2(\mathbf{x}^2) \ge u(\boldsymbol{\omega}^2) = 1, \text{ and}\\ u_3 &= 4 - u_1 - u_2 \ge u(\boldsymbol{\omega}^3) = 3^{3/4}. \end{aligned} \tag{21.2.2}$$

Thus $u_1 \ge 0$ (which we already knew), $u_2 \ge 1$, and $u_1 + u_2 \le 4 - 3^{3/4} \approx 2.280$. The latter two constraints supersede the constraints $u_2 \ge 0$ and $u_1 + u_2 \le 4$ (i.e., $u_3 \ge 0$).

With three consumers, we also have to consider the constraints imposed by the requirement that no coalition of 2 people be able to improve. There are three of them:

$$\begin{aligned} u_1 + u_2 &\ge u(\boldsymbol{\omega}^{\{1,2\}}) = u(3,1) = 3^{1/4},\\ u_1 + (4 - u_1 - u_2) = 4 - u_2 &\ge u(\boldsymbol{\omega}^{\{1,3\}}) = u(3,3) = 3, \text{ and}\\ 4 - u_1 &\ge u(\boldsymbol{\omega}^{\{2,3\}}) = u(2,4) = 2^{7/4}. \end{aligned} \tag{21.2.3}$$

The second equation of (21.2.3) simplifies to $u_2 \le 1$.

Since $u_2 \ge 1$ and $u_2 \le 1$, we have pinned down the bundle that consumer two gets. It is $\mathbf{x}^2 = (1,1)$. Substituting in the other constraints in (21.2.3) , they reduce to $u_1 \le 4 - 2^{7/4} \approx 0.636$, $u_1 \le 3 - 3^{3/4} \approx 0.720$, and $u_1 \ge 3^{1/4} - 1 \approx 0.316$. The first constraint dominates the second, so this can be summed up by $3^{1/4} - 1 \le u_1 \le 4 - 2^{7/4}$. Thus the core is

$$\mathbf{C}(\mathcal{E}) = \Big\{ \big(u_1(1,1), (1,1), (4-u_1)(1,1)\big) : 3^{1/4} - 1 \le u_1 \le 4 - 2^{7/4} \Big\}.$$

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21.2.12 Walrasian Allocations are in the Core

There is an analog of the First Welfare Theorems for core. Any Walrasian equilibrium allocation is in the core. The proof of this is a slight modification of the proof of the First Welfare Theorem.

Theorem 21.2.10. *Suppose the economy* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)$ *consists of* I *consumers and one constant returns to scale firm. Let* $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}, \hat{\mathbf{p}})$ *be a Walrasian equilibrium. Then the allocation* $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ *is in the core.*

Proof. Suppose a coalition S can improve on the allocation $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$. Then, for each $i \in S$, there are $\mathbf{x}^i \in \mathfrak{X}_i$ with $\mathbf{x}^i \succ_i \hat{\mathbf{x}}^i$ and a $\mathbf{y} \in Y$ with $\mathbf{x}^S = \boldsymbol{\omega}^S + \mathbf{y}$. Due to constant returns to scale, $\hat{\mathbf{p}}\cdot\mathbf{y} \le \hat{\mathbf{p}}\cdot\hat{\mathbf{y}} = 0$. The equilibrium income of consumer i is then $\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i$. Now $\hat{\mathbf{p}}\cdot\mathbf{x}^i > \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i$ since $\hat{\mathbf{x}}^i$ maximizes utility over the budget set. Summing, we obtain

$$\hat{\mathbf{p}}\cdot\left(\sum_{i\in S}\mathbf{x}^i\right) > \hat{\mathbf{p}}\cdot\left(\sum_{i\in S}\boldsymbol{\omega}^i\right). \qquad ()$$

But $\mathbf{x}^S = \mathbf{y} + \boldsymbol{\omega}^S$. Applying the price vector $\hat{\mathbf{p}}$ yields

$$\hat{\mathbf{p}}\cdot\left(\sum_{i\in S}\mathbf{x}^i\right) = \hat{\mathbf{p}}\cdot\mathbf{y} + \hat{\mathbf{p}}\cdot\left(\sum_{i\in S}\boldsymbol{\omega}^i\right).$$

Together with equation , this implies $\hat{\mathbf{p}}\cdot\mathbf{y} > 0$, which contradicts the fact that maximum profit is zero due to constant returns to scale. We conclude that no coalition S can block. It follows that the allocation $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ is in the core. □

In the two-consumer exchange case, we can see that the converse fails. There will often be a unique Walrasian equilibrium allocation (as in all the Cobb-Douglas cases), but the core is larger. There may be extreme core allocations that give all of the gains from trade to one of the consumers. We saw this in Figure 21.1.4. This is the case of a perfectly price-discriminating monopolist, not Walrasian equilibrium.

21.3 Replica Economies

Edgeworth's idea is that market prices emerge as the market becomes more competitive, as there are more people trading. We use this idea to find a type of converse to Theorem 21.2.10, showing that core allocations in large competitive economies are Walrasian equilibrium allocations.

First, we have to decide how to make the market more competitive. We will make the economy larger by adding more people that are just like the people that already exist in our economy. Moreover, we will do it without changing the proportions of the various types of individuals, as defined by preferences and endowments.

This idea can be made precise by exactly duplicating the consumers that already exist. Since the production set is constant returns to scale, we may leave it unchanged. Nothing would gained by replicating it.

21.3.1 R-fold Replica Economies

The economy obtained by replicating each consumer R times is called the R-*fold replica economy*.

In the R-fold replica economies, we replace each consumer i by R clones of consumer i. Each clone's consumption set, preferences, and endowment are identical to that of consumer i. The r^{th} clone of consumer i is denoted $\#ri$ or (ri) with $r = 1, \ldots, R$. We also refer to (ri) as the r^{th} individual of type i.

R-fold replica economy. Suppose the economy $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)$ has I consumers with endowments $\boldsymbol{\omega}^i$ and preferences $\succsim_i$. The R-*fold replica* of $\mathcal{E}$ is an economy with $R \times I$ consumers. Consumer $\#ri$ has the same preferences, consumption set, and endowment as consumer i in the original economy:

1. For each i, $\succsim_{ri} = \succsim_i$ for $r = 1, \ldots, R$.
2. For each i, $\mathfrak{X}_{ri} = \mathfrak{X}_i$ for $r = 1, \ldots, R$.
3. For each i, $\boldsymbol{\omega}_{ri} = \boldsymbol{\omega}_i$ for $r = 1, \ldots, R$.

The production set is unchanged. We denote the R-replica of $\mathcal{E}$ by $\mathcal{E}^R$.

It is evident that this type of replication does not affect the proportions of the various types. Note that types with different i may have the same preferences and endowments.

21.3.2 Equal Treatment in Replica Economies

A key result is that core allocations in the replica economies must treat all consumers of type i equally. Since each consumer of type i gets the same allocation, we may indicate core allocations of $\mathcal{E}^R$ as if they were allocations in $\mathcal{E}$, with each type's consumption listed only once.

The basic idea behind equal treatment is that if consumers of the same type are not treated equally, we can form the coalition of the worst-off. By replacing their current basket of goods with the average basket that these types receive, they can make themselves better off, and so block the original allocation. There is a complication if all of the consumers of the same type receive the same allocation, so that the average is the same as the worst off consumer of that type. We solve that by using the same method we used to show that any Pareto improvement could be converted to a strong Pareto improvement by taking a bit from one of the better off consumers to make as strong Pareto improvement, as in Lemma 21.2.4.

We first state the theorem, then examine a relevant example which shows how the proof works, and finally provide the proof.

Equal Treatment Theorem. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)_{i=1}^I$ be an economy with a constant returns to scale aggregate production set Y, where each consumption set is $\mathfrak{X}_i = \mathbb{R}^m_+$, and each consumer has continuous, strictly convex, and monotonic preferences. If $(\hat{\mathbf{x}}^{ri}, \hat{\mathbf{y}}) \in \mathbf{C}(\mathcal{E}^R)$, then $\hat{\mathbf{x}}^{qi} = \hat{\mathbf{x}}^{ri}$ for all $i = 1, \dots, I$ and $q, r = 1, \dots, R$.*

21.3.3 Equal Treatment in a Cobb-Douglas Economy

The following example illustrates the essence of the proof even though it does not quite satisfy the conditions of the theorem. The preferences in the example are only strictly convex on the interior of each consumption set, but not on the entire consumption set.

Example 21.3.1: Equal Treatment. Start with an economy with three consumers with Cobb-Douglas utility $u_i(\mathbf{x}) = (x_1)^{\alpha_i}(x_2)^{\beta_i}(x_3)^{1-\alpha_i-\beta_i}$ for $i = 1, 2, 3$ where $0 < \alpha_i, \beta_i$ and $\alpha_i + \beta_i < 1$. Endowments are $\boldsymbol{\omega}^1 = \mathbf{0}$ and $\boldsymbol{\omega}^i = (1, 1, 1)$ for $i = 2, 3$.

We go to the three-fold replica, and examine the allocation where the type one consumers receive $\mathbf{x}^{r1} = \mathbf{0}$, the type two consumers receive

$$\mathbf{x}^{12} = \begin{pmatrix} 1 \\ 2/3 \\ 1 \end{pmatrix}, \mathbf{x}^{22} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}^{32} = \begin{pmatrix} 1 \\ 4/3 \\ 1 \end{pmatrix},$$

and the type three consumers receive

$$\mathbf{x}^{13} = \begin{pmatrix} 2/3 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}^{23} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}^{33} = \begin{pmatrix} 4/3 \\ 1 \\ 1 \end{pmatrix}.$$

We have labeled the consumers so that higher-numbered consumers of each type are better off: $\mathbf{x}^{ri} \precsim_i \mathbf{x}^{si}$ for $r < s$.

Now form a coalition S by taking the worst-off consumers of each type. The type one consumers are indifferent, so we just pick one of them, say #(11). For the other types it is clear that #(12) and #(13) are the worst-off consumers of types 2 and 3. This means that $S = \{(11), (12), (13)\}$ is a coalition of worst-off consumers of each type.

The average allocation received by each type is $\bar{\mathbf{x}}^1 = \mathbf{0}$ and $\bar{\mathbf{x}}^2 = \bar{\mathbf{x}}^3 = (1, 1, 1)^{\mathsf{T}}$. Since $\sum_i \bar{\mathbf{x}}^i = (2, 2, 2) = \boldsymbol{\omega}^S$, this is feasible for S, the coalition of the worst-off.

We now show that the coalition S can improve on their allocation. We first take the average received by each type, $\bar{\mathbf{x}}^i$. Notice that this is not only feasible for S, but that it strictly improves for #(12) and #(13). This is due to the strictly convexity of preferences and the averaging distinct commodity bundles. In the case of type one, everyone received the same consumption bundle, so the average consumption is no improvement.

We now use the same method as in Lemma 21.2.4. We will take a little bit from a better off consumer and distribute it to the consumer who has not gained. Choose $\varepsilon > 0$ so that $\bar{\mathbf{x}}^3 - \varepsilon\mathbf{e} \succsim_3 \mathbf{x}^{13}$. Set $\hat{\mathbf{x}}^{11} = \varepsilon\mathbf{e}$, $\hat{\mathbf{x}}^{12} = \bar{\mathbf{x}}^2$, and $\hat{\mathbf{x}}^{13} = \bar{\mathbf{x}}^3 - \varepsilon\mathbf{e}$. Then $\sum_i \hat{\mathbf{x}}^{i1} = \sum_i \bar{\mathbf{x}}^i = \boldsymbol{\omega}^S$, so this is feasible for the coalition S. Moreover, $\hat{\mathbf{x}}^{1i} \succsim_{1i} \mathbf{x}^{1i}$, so the coalition S can improve on $(\mathbf{x}^{1i})$. This means that $(\mathbf{x}^{ri})$ is not in the core. ◀

21.3.4 Proof of the Equal Treatment Theorem I

The method of forming the coalition of the worst-off works in general, and is the key step of the proof of the Equal Treatment Theorem. We restate the Equal Treatment Theorem before proving it.

Equal Treatment Theorem. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)_{i=1}^I$ be an economy with a constant returns to scale aggregate production set Y, where each consumption set is $\mathfrak{X}_i = \mathbb{R}^m_+$, and each consumer has continuous, strictly convex, and monotonic preferences. If $(\hat{\mathbf{x}}^{ri}, \hat{\mathbf{y}}) \in \mathbf{C}(\mathcal{E}^R)$, then $\hat{\mathbf{x}}^{qi} = \hat{\mathbf{x}}^{ri}$ for all $i = 1, \dots, I$ and $q, r = 1, \dots, R$.*

Proof of Equal Treatment Theorem. We prove this by contradiction, following the same procedure as in Example 21.3.1. First relabel the consumers so that higher numbered consumers of each type have higher utility. In other words, we number them so $\hat{\mathbf{x}}^{ri} \precsim_i \hat{\mathbf{x}}^{si}$ whenever $r < s$. This ensures that $(1i)$ is a worst-off consumer of type i (ties are allowed).

We will prove this by contradiction. Suppose equal treatment fails. Then there is a type h with individuals r', q' obeying $\hat{\mathbf{x}}^{r'h} \neq \hat{\mathbf{x}}^{q'h}$. Let S be the coalition of the worst-off, $S = \{(11), (12), \dots, (1I)\}$. We will show that the coalition of the worst-off can improve over $(\hat{\mathbf{x}}^{1i})_{(1i)\in S}$.

Let $\bar{\mathbf{x}}^i$ be the average consumption by consumers of type i, $\bar{\mathbf{x}}^i = \frac{1}{R}\sum_r \hat{\mathbf{x}}^{ri}$. By strict convexity of preferences, $\bar{\mathbf{x}}^i \succsim_i \hat{\mathbf{x}}^{1i}$ for all i and $\bar{\mathbf{x}}^h \succ_h \hat{\mathbf{x}}^{1h}$.

At this point we have a weak Pareto improvement for the coalition. To block, we need a strong Pareto improvement. We need to make every one of the worst-off better off. As in the proof of Lemma 21.2.4, we do that by taking ℓ with $\bar{\mathbf{x}}^h_\ell > 0$ and $\varepsilon > 0$ so $\bar{\mathbf{x}}^h - \varepsilon\mathbf{e}^\ell \succ_h \hat{\mathbf{x}}^{1h}$.[7] The excess is then spread among the other worst-off consumers. Define $\mathbf{x}^i = \bar{\mathbf{x}}^i + \frac{\varepsilon}{I-1}\mathbf{e}^\ell$ for $i \neq h$ and $\mathbf{x}^h = \bar{\mathbf{x}}^h - \varepsilon\mathbf{e}^\ell$. Then $\mathbf{x}^i \succ_i \hat{\mathbf{x}}^{1i}$ for every $i = 1, \dots, I$.

[7] Such a good ℓ must exist. First, $\bar{\mathbf{x}}^h \succ_h \mathbf{x}^{1h} \succsim_h \mathbf{0}$ by monotonicity. That shows that $\bar{\mathbf{x}}^h \neq \mathbf{0}$. Since $\mathfrak{X}_h = \mathbb{R}^m_+$, this implies $\bar{\mathbf{x}}^h > \mathbf{0}$.

21.3.5 Proof of the Equal Treatment Theorem II

Proof continues. All that is left is to show this is feasible for the coalition S of the worst-off. For that, we use the production vector $\frac{1}{R}\hat{\mathbf{y}} \in Y$. Then

$$\begin{aligned}
\sum_{i=1}^{I} \mathbf{x}^i &= \sum_{i=1}^{I} \bar{\mathbf{x}}^i + (I-1)\frac{\varepsilon}{I-1}\mathbf{e}^\ell - \varepsilon\mathbf{e}^\ell \\
&= \sum_{i=1}^{I} \bar{\mathbf{x}}^i = \sum_{i=1}^{I} \left(\frac{1}{R}\sum_{r=1}^{R} \hat{\mathbf{x}}^{ri}\right) \\
&= \frac{1}{R}\sum_{ri} \hat{\mathbf{x}}^{ri} = \frac{1}{R}\left(\hat{\mathbf{y}} + \sum_{ri} \boldsymbol{\omega}^{ri}\right) \\
&= \frac{1}{R}\left(\hat{\mathbf{y}} + \sum_{i=1}^{I} R\boldsymbol{\omega}^i\right) \\
&= \frac{1}{R}\hat{\mathbf{y}} + \sum_{i=1}^{I} \boldsymbol{\omega}^i = \frac{1}{R}\hat{\mathbf{y}} + \boldsymbol{\omega}^S.
\end{aligned}$$

Because this improving allocation is feasible, $(\hat{\mathbf{x}}^{ri})$ is not in the core. This contradiction shows that unequal treatment is impossible in the core. □

21.3.6 When Equal Treatment Fails

When preferences are convex, but not strictly convex, the equal-treatment property may fail when viewed in terms of consumption bundles. However, the same argument still shows there is equal treatment in terms of preference—all consumers of the same type receive indifferent consumption bundles. The following example illustrates this.

Example 21.3.2: Weak Equal Treatment with Convex Preferences. In this economy consumer one has utility $u_1(\mathbf{x}^1) = \min\{x_1^1, x_2^1\}$ and endowment $\boldsymbol{\omega}^1 = (1,1)$ and consumer two has utility $u_2(\mathbf{x}^2) = x_1^2 + x_2^2$ and endowment $\boldsymbol{\omega}^2 = (2,1)$. Here $\mathbf{p} = (1,1)$ is an equilibrium price vector and the initial allocation gives equilibrium consumption. The initial allocation is in the core.

In the 2R-fold replica economy take the allocation that gives $\mathbf{x}^{r1} = (1,1)$ to the type 1 individuals, while giving either $\mathbf{x}^{odd} = (3,0)$ or $\mathbf{x}^{even} = (1,2)$ to the type 2 individuals, depending on whether they are odd or even-numbered. This is an equilibrium allocation with price vector $\mathbf{p} = (1,1)$, and hence in the core. However, the type 2 consumers receive different consumption bundles depending on whether they are odd or even. Although their consumption bundles differ, all of the type 2 consumers receive the same utility. Notice that giving them the average amount does not increase their utility. ◀

21.3.7 The Core in Replica Economies

The Equal Treatment Theorem means that allocations in the core of $\mathcal{E}^R$ are replicas of allocations in $\mathcal{E}$ itself. We can find the corresponding allocation in the core of $\mathcal{E}$ by using the projection map where $\mathbf{x}^{ri} \mapsto \mathbf{x}^{1i}$ for every $r = 1, \dots, R$. By equal treatment, $\mathbf{x}^{ri} = \mathbf{x}^{1i}$ for every r.

This projection is invertible. We let $\hat{\mathbf{C}}(\mathcal{E}^R)$ denote the projection of $\mathbf{C}(\mathcal{E}^R)$, the core of $\mathcal{E}^R$. We can show $\hat{\mathbf{C}}(\mathcal{E}^R)$ is contained in the core of the original economy $\mathcal{E}$.

Proposition 21.3.3. *Suppose that $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)_{i=1}^I$ has strictly convex, monotonic, and continuous preferences defined on $\mathfrak{X}_i = \mathbb{R}^m_+$, and that Y is a convex, CRS production set. Then $\hat{\mathbf{C}}(\mathcal{E}^R) \subset \mathbf{C}(\mathcal{E})$.*

Proof. Let $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}) \in \hat{\mathbf{C}}(\mathcal{E}^R)$ and suppose that $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}) \notin \mathbf{C}(\mathcal{E})$, there is a coalition S that can improve over $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ with an allocation $(\mathbf{x}^i, \mathbf{y})$. Now form the coalition $S' = \{(1i) : i \in S\}$, for whom $\hat{\mathbf{x}}^{1i} = \mathbf{x}^i$ improves. Since S' is a coalition in $\mathcal{E}^R$, this shows that $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ cannot be in the core of $\mathcal{E}^R$. This contradiction means that $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ must be in the core of $\mathcal{E}$. $\square$

21.4 Core Equivalence

Let's take a closer look at the mapping between the core of a replica economy and the core of the the original economy. It is easy to see that any feasible allocation in the original economy that can be blocked in $\mathcal{E}^R$ can also be blocked in $\mathcal{E}^{R+1}$ (just use the same coalition). By Proposition 21.3.3, the projected cores $\hat{\mathbf{C}}(\mathcal{E}^R)$ are all subsets of the core of $\mathcal{E}$. Then we have $\hat{\mathbf{C}}(\mathcal{E}^R) \supset \hat{\mathbf{C}}(\mathcal{E}^{R+1})$. The core shrinks as we replicate. We sum this up in the following proposition.

Proposition 21.4.1. *Suppose that* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)_{i=1}^I$ *has strictly convex, monotonic, and continuous preferences defined on* $\mathfrak{X}_i = \mathbb{R}^m_+$*, and that* Y *is a convex, CRS production set. Then* $\hat{\mathbf{C}}(\mathcal{E}^{R+1}) \subset \hat{\mathbf{C}}(\mathcal{E}^R) \subset \mathbf{C}(\mathcal{E})$ *for every positive integer* R.

By Proposition 21.4.1, the core shrinks as we consider larger and larger replica economies. This confirms Edgeworth's intuition that competition in large economies will reduce the number of core allocations. The big question is: Does the core shrink to the set of Walrasian equilibrium allocations? Or are there other allocations that are always in the core?

21.4.1 Edgeworth Equilibria

We do know that any Walrasian equilibrium is in the core, and that its replicas are in the core of the replica economies. It follows that

$$\bigcap_{R=1}^{\infty} \hat{\mathbf{C}}(\mathcal{E}^R)$$

is non-empty and contains all Walrasian equilibrium allocations. But does it contain anything else? Following Aliprantis, Brown, and Burkinshaw (1987), we define the set of *Edgeworth equilibrium allocations* as the intersection of the projections of the cores of the replicas into the original economy

$$\mathbf{EE}(\mathcal{E}) = \bigcap_{R=1}^{\infty} \hat{\mathbf{C}}(\mathcal{E}^R).$$

We can interpret Edgeworth's conjecture as saying that the set of Edgeworth equilibrium allocations is precisely the set of Walrasian equilibrium allocations.[8]

[8] There are other interpretations. For example, the Edgeworth conjecture could refer to the core of a continuum economy being equal to the set of equilibrium allocations. See Aumann (1964), Hildenbrand (1974) and Hildenbrand and Kirman (1988)

21.4.2 Core Shrinkage in a Cobb-Douglas Economy I

Although not commonly done, its fairly easy to make up examples that illustrate how the core shrinks under replication. Example 21.4.2 does that using Cobb-Douglas utility.

Example 21.4.2: Cobb-Douglas: The Core Shrinks to a Point. We return to the model of Example 21.1.3. The endowments in the original two-person economy are $\omega^1 = (2,1)$ and $\omega^2 = (1,2)$. Each person has equal-weighted Cobb-Douglas preferences. We know what the utility possibility set looks like for any coalition in any replica. If a coalition has m members of type 1 and n members of type 2, the aggregate endowment is $(2m+n, m+2n)$. Such coalitions exist if $R \geq \max\{m,n\}$. This yields maximum aggregate utility of $u(2m+n, m+2n) = (2n^2+5nm+2m^2)^{1/2}$, which can be divided among the coalition members in any fashion.

Now consider whether a utility allocation $u_1 = 3-x$, $u_2 = x$ is in the core for large replica economies $\mathcal{E}^R$. We consider a coalition S of m consumers of type 1 and $(m+1)$ consumers of type 2. This requires $R \geq m+1$. The utility of S's aggregate endowment is $(9m^2+9m+2)^{1/2}$. The coalition can improve on an allocation $(3-x,x)$ if the coalition's aggregate utility is larger than their current aggregate utility. That is, if

$$m(3-x) + (m+1)x < (9m^2+9m+2)^{1/2}.$$

This can be simplified to

$$x < (9m^2+9m+2)^{1/2} - 3m. \tag{21.4.4}$$

If equation (21.4.4) is satisfied, we can pick $u_1' > 3-x$ and $u_2' > x$ with

$$mu_1' + (m+1)u_2' < (9m^2+9m+2)^{1/2},$$

showing this allocation of utility is feasible for the coalition S. The coalition S can block anything where they get less.

Now set

$$f(m) = (9m^2+9m+2)^{1/2} - 3m.$$

Since anything with $x < f(m)$ can be blocked, the function $f(m)$ puts a lower bound on the utility x that consumers of type 2 can receive in the core. We need to investigate the properties of f. Of course, $f(m) > 0$ for $m = 0, 1, 2, \ldots$. The derivative obeys

$$f'(m) = \frac{1}{2}(18m+9)(9m^2+9m+2)^{-1/2} - 3 > 0,$$

as can be seen after some rearrangement and squaring. As a result, the lower bound on core utility rises as the size of the replica rises, shrinking the core.

21.4.3 Core Shrinkage in a Cobb-Douglas Economy II

The function $f(m)$ shows us how the core shrinks with every replication. In $\mathcal{E}$, only $m = 0$ gives a feasible allocation. This implies $u_i \geq \sqrt{2} \approx 1.4142$. In $\mathcal{E}^2$, $m = 1$ is feasible, which yields $u_i \geq \sqrt{20} - 3 \approx 1.4721$. From $m = 2$, we find $u_i \geq \sqrt{56} - 6 \approx 1.4833$. Then $m = 3$ yields $u_i \geq \sqrt{110} - 9 \approx 1.4881$ and $m = 4$ yields $u_i \geq \sqrt{182} - 12 \approx 1.4907$. Each time we replicate the economy, we slice another piece off the end of the core of the previous replica.

To prove that only $x = 3/2$ always remains in the core we compute the limit $\lim_{m\to\infty} f(m)$:

$$\begin{aligned}
\lim_{m\to\infty} f(m) &= \lim_{m\to\infty} 3m\left[\sqrt{1+\frac{1}{m}+\frac{2}{9m^2}}-1\right] \\
&= 3\lim_{m\to\infty}\left[\frac{(1+1/m+2/9m^2)^{1/2}-1}{1/m}\right] \\
&= 3\lim_{m\to\infty}\frac{-1/m^2-4/9m^3}{2\sqrt{1+1/m+2/9m^2}}\Bigg/\left(\frac{-1}{m^2}\right) \\
&= 3\lim_{m\to\infty}\frac{1+4/9m}{2\sqrt{1+1/m+2/9m^2}} \\
&= 3/2
\end{aligned}$$

with l'Hôpital's rule used to obtain (21.4.5). If the consumers of type 2 receive less that $3/2$ utils, there is a coalition of m consumers of type 1 and $(m+1)$ consumers of type 2 that can block the allocation in $\mathcal{E}^{m+1}$ and higher replicas. The only point that remains in core of all replicas is that one that gives utility $3/2$ to every clone of both types of consumers. This corresponds to the equilibrium allocation. ◄

21.4.4 The Debreu-Scarf Limit Theorem

The shrinking core is not an artifact of Cobb-Douglas preferences, but applies across a wide variety of economies. Preferences need not be Cobb-Douglas and they need not be identical. Debreu and Scarf (1963) were able to show that the core not only shrinks under replication, but that only the Walrasian equilibrium allocations remain in the limit.[9]

Debreu-Scarf Limit Theorem. *Let* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, Y)_{i=1}^I$ *be an economy with a constant returns to scale aggregate production set* Y*, where each consumption set is* $\mathfrak{X}_i = \mathbb{R}_+^m$*, and each consumer has continuous, strictly convex, and monotonic preferences. If an allocation* $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ *is an Edgeworth equilibrium, then there is a price vector* $\hat{\mathbf{p}} \gg \mathbf{0}$ *so that* $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}, \hat{\mathbf{p}})$ *is a Walrasian equilibrium. Conversely, if* $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}, \hat{\mathbf{p}})$ *is a Walrasian equilibrium, then* $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ *is an Edgeworth equilibrium.*

[9] The original Debreu-Scarf Limit Theorem applied only to exchange economies. The version here is based on that in Boyd and McKenzie (1993). For an example of the complexity of the proof with diminishing returns and distribution of profits, see Aliprantis, Brown, and Burkinshaw (1987).

21.4.5 Sketch of Limit Theorem Proof

The Debreu-Scarf Limit Theorem tells us that the Walrasian equilibrium allocations are the allocations in the intersection of the cores of the replica economies, the Edgeworth equilibrium allocations.

The proof here works somewhat differently than in the example. Of course, we use a separation argument to find the equilibrium price vector. The main issue is determining what to separate. For each individual we consider the set of net trades that make them better off. These are the sets $(B_i - \omega^i)$ of the proof. We form the convex hull B of these improving net trades. We then separate the convex hull B from the production set Y to obtain an equilibrium price vector.

For that, we need to show that B and Y do not intersect. We prove this by contradiction. If B and Y do intersect, we can find a way to block the core allocation. We take that convex combination of improving net trades that sits in Y. We approximate the coefficients in the convex combination by rational numbers. If the approximation is good enough, a coalition with those proportions will be able to block using those net trades in a sufficiently large replica economy. Since a coalition can block, it is not in the core of that replica. It is not an Edgeworth equilibrium. This contradiction shows that the sets cannot intersect.

Then we find a price vector that separates B and Y. This will be the equilibrium price vector. The remainder of the proof shows that we have a Walrasian equilibrium.

21.4.6 Proof of Debreu-Scarf Limit Theorem I

Proof of Debreu-Scarf Limit Theorem. The second part follows immediately from the fact that the R-fold replica of any Walrasian equilibrium of $\mathcal{E}$ is also a Walrasian equilibrium in $\mathcal{E}^R$. The first part requires a separation argument.

Suppose $(\hat{\mathbf{x}}^i)$ is in the core of every replica of $\mathcal{E}$. Let $B_i = \{\mathbf{x} \in \mathfrak{X}_i : \mathbf{x} \succ_i \hat{\mathbf{x}}^i\}$ and take B as the convex hull of $\cup_i(B_i - \boldsymbol{\omega}^i)$. That is, B is the set of convex combinations of vectors in $\cup_i(B_i - \boldsymbol{\omega}^i)$. We will obtain the equilibrium prices by separating Y and B.[10]

We first show $Y \cap B$ is empty. **By way of contradiction**, we suppose $Y \cap B$ is not empty. Then there is some $\mathbf{y} \in B \cap Y$. This means there are $\alpha_i \geq 0$ and $\mathbf{z}^i \in (B_i - \boldsymbol{\omega}^i)$ with $\sum_i \alpha_i = 1$ and $\sum_i \alpha_i \mathbf{z}^i = \mathbf{y}$. Set $\mathbf{x}^i = \mathbf{z}^i + \boldsymbol{\omega}^i$ so that $\mathbf{x}^i \in B_i$. We will show that $\mathbf{y} \in B$ will imply $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ can be blocked in some replica economy, contradicting the hypothesis that $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ is in the core of every replica.

The necessary coalition will have approximately α_i of the consumers of type i. Let $J = \{i : \alpha_i > 0\}$. For $i \in J$ and any positive integer k, define β_i^k as the least integer greater than or equal to $k\alpha_i$. Then

$$\beta_i^k - 1 < k\alpha_i \leq \beta_i^k < k\alpha_i + 1.$$

It follows that $\beta_i^k \to \infty$ as $k \to \infty$ and $\lim_{k\to\infty} k\alpha_i/\beta_i^k = 1$. Moreover, $k\alpha_i/\beta_i^k \leq 1$. In other words, β_i^k is approximately $k\alpha_i$.

[10] We separated $\sum_i B_i$ and $\boldsymbol{\omega} + Y$, or equivalently, $\sum_i(B_i - \boldsymbol{\omega}^i)$ and Y, when we proved the Second Welfare Theorem. For the core, we must pay attention to who owns what, so we prefer the second form. Furthermore, the replica economies allow the possibilities of (rational) convex combainations, so we use the convex hull instead of the sum.

21.4.7 Proof of Debreu-Scarf Limit Theorem II

Proof continues. Define $\mathbf{x}_k^i$ by

$$\mathbf{x}_k^i = \frac{k\alpha_i}{\beta_i^k}\mathbf{x}^i + \left(1 - \frac{k\alpha_i}{\beta_i^k}\right)\boldsymbol{\omega}^i = \frac{k\alpha_i}{\beta_i^k}\mathbf{z}^i + \boldsymbol{\omega}^i.$$

Combine the facts that $\mathbf{x}^i, \boldsymbol{\omega}^i \in \mathfrak{X}_i$ and the convexity of $\mathfrak{X}_i$ to see $\mathbf{x}_k^i \in \mathfrak{X}_i$.

Using continuity, we can find a K with $\mathbf{x}_K^i \succ_i \hat{\mathbf{x}}^i$ for all $i \in J$. We will focus on the replica economy $\mathcal{E}^K$. Form a coalition S consisting of β_i^K agents of type i in the replica economy $\mathcal{E}^K$. This coalition prefers $(\mathbf{x}_K^i)$ to $(\hat{\mathbf{x}}^i)$. It is also feasible for the coalition S because

$$\begin{aligned}
\sum_{i\in J} \beta_i^K \mathbf{x}_K^i &= \sum_{i\in J} \beta_i^K \frac{K\alpha_i}{\beta_i^K}\mathbf{z}^i + \sum_{i\in J} \beta_i^K \boldsymbol{\omega}^i \\
&= \sum_{i\in J} K\alpha_i \mathbf{z}^i + \sum_{i\in J} \beta_i^K \boldsymbol{\omega}^i \\
&= K\sum_{i\in J} \alpha_i \mathbf{z}^i + \sum_{i\in J} \beta_i^K \boldsymbol{\omega}^i \\
&= K\mathbf{y} + \boldsymbol{\omega}^S.
\end{aligned}$$

By constant returns, $K\mathbf{y} \in Y$, establishing feasibility of $(\mathbf{x}_K^{ri})_{r=1,i\in J}^{\beta_K^i}$ for the coalition S.

We have shown that if $\mathbf{y} \in (B \cap Y)$, $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ can be blocked in $\mathcal{E}^K$. As $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}})$ is in the core, **this is impossible**, contradicting our assumption that $Y \cap B$ is non-empty. This completes the proof that $Y \cap B = \emptyset$.

21.4.8 Proof of Debreu-Scarf Limit Theorem III

Rest of Proof. The hard work is now done. The rest involves a typical separation argument. Each B_i is open by continuity of $\succsim_i$, which implies B is also open. We now employ Separation Theorem D on $B - Y$ (which is open) and $\mathbf{0}$ to find $\hat{\mathbf{p}} \neq \mathbf{0}$ obeying $\hat{\mathbf{p}}\cdot\mathbf{y} < \hat{\mathbf{p}}\cdot\mathbf{b}$, for all $\mathbf{b} \in B$ and $\mathbf{y} \in Y$.

Since $\mathbf{0} \in Y$, $\hat{\mathbf{p}}\cdot\mathbf{b} > 0$ for all $\mathbf{b} \in B$. Now take $\mathbf{b}' \in B$. We have $\hat{\mathbf{p}}\cdot\mathbf{b}' > \hat{\mathbf{p}}\cdot\mathbf{y}$ for all $\mathbf{y} \in Y$. Since profit is bounded above and Y is CRS, $\hat{\mathbf{p}}\cdot\mathbf{y} \leq 0$ for all $\mathbf{y} \in Y$. We conclude maximum profit is zero.

Now take $\mathbf{x}$ with $\mathbf{x} \succ_i \hat{\mathbf{x}}^i$. Then $\mathbf{x} - \boldsymbol{\omega}^i \in B$, which implies $\hat{\mathbf{p}}\cdot(\mathbf{x} - \boldsymbol{\omega}^i) > 0$. That is, $\hat{\mathbf{p}}\cdot\mathbf{x} > \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i$. By monotonicity, $\hat{\mathbf{x}}^i + \frac{1}{n}\mathbf{e} \succ_i \hat{\mathbf{x}}^i$, so

$$\hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i + \frac{1}{n}\hat{\mathbf{p}}\cdot\mathbf{e} > \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i.$$

Letting $n \to \infty$, we obtain $\hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i \geq \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i$. But $\sum_i \hat{\mathbf{x}}^i = \hat{\mathbf{y}} + \sum_i \boldsymbol{\omega}^i$, so

$$\sum_i \hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i = \hat{\mathbf{p}}\cdot\hat{\mathbf{y}} + \sum_i \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i \leq \sum_i \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i.$$

This implies $\hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i = \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i$. In other words, $\hat{\mathbf{x}}^i$ maximizes utility over the budget set. Substituting back in, we find $\hat{\mathbf{p}}\cdot\hat{\mathbf{y}} = 0$, so $\hat{\mathbf{y}}$ maximizes profit.

Since market clearing holds, all that we have left to show is that $\hat{\mathbf{p}} \gg \mathbf{0}$. This follows immediately since $\hat{\mathbf{x}}^i + \mathbf{e}^\ell \succ_i \hat{\mathbf{x}}^i$. The separation condition gives

$$\hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i + p_\ell > \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i = \hat{\mathbf{p}}\cdot\hat{\mathbf{x}}^i.$$

Thus $p_\ell > 0$ for every ℓ. $\square$

Under our strong monotonicity hypothesis, all equilibria must have strictly positive prices. There are other versions of the Debreu-Scarf Limit Theorem that hold under weaker conditions.

21.4.9 Edgeworth and the Existence of Walrasian Equilibria

The equivalence between Edgeworth equilibria and Walrasian equilibria suggests yet another way to prove the existence of Walrasian equilibria. We can show that the set of Edgeworth equilibria is non-empty.

Bondareva (1963) and Shapley (1967) developed a general condition (balancedness) for cooperative games that implies the existence of a core. It is easy to show balancedness in the Walrasian setting, implying that each $\hat{\mathbf{C}}(\mathcal{E}^R)$ is non-empty. Under appropriate conditions, these sets are also compact. Since they are nested, they must have a non-empty intersection, the set of Edgeworth equilibria.

A separation argument can be used to show that any Edgeworth equilibrium allocation is a competitive equilibrium allocation as in Aliprantis, Brown, and Burkinshaw (1987). Boyd and McKenzie (1993) applied this idea to a large class of infinite horizon models. They first showed that Edgeworth equilibria were a type of quasi-equilibrium, then used strong irreducibility to show that the quasi-equilibrium was actually an equilibrium.

These methods have one big advantage when dealing with infinite-dimensional commodity spaces. There may be several possible price spaces, and it may be unclear which is the right one. Generating a price system via Edgeworth equilibria provides a natural answer to which possible price space is the right one.

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