6. Production of a Single Good

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Chapter Outline

1. Profit and Cost with a Single Output
2. Cost with Many Inputs
3. Cost Can be Minimized: The Cost Theorem
4. Maximizing Profit

Production of any product involves both inputs and outputs. Some intermediate goods may be produced within the firm and later used in the production process. For example, a coal mine may burn some of the coal produced in order to run their machinery. For the most part we will not explicitly model those intermediate products that are not marketed, although we will later consider the linear activity model where that is possible. We usually treat the firm as a black box, looking only at its net inputs and outputs and ignoring its internal structure.

Figure 6.0.1: Inputs flow into the firm, resulting in outputs coming out. Although we can see how the inputs and outputs relate (e.g., production function), we don't know the details of what happens inside firm. It remains a black box.

6.0.1 Introduction

Section one begins by studying production as in a micro principles class. We start with a production function that tells how much of a single output can be produced from a single input. Profit is revenue minus cost. Revenue is earned by selling the output produced at the market price. Cost is the cost of the single input at its market price.

The firm chooses an input level that maximizes profit. Alternatively, we can rewrite the firm's problem in terms of output by introducing the cost function—giving cost as a function of output. Profit is then maximized by choosing an appropriate level of output.

The next step is to allow multiple inputs as in intermediate microeconomics. In section two, there is again a single output and a production function that describes the production possibilities. We take the product of the vector of factor prices and the vector of factor inputs to determine the cost of the inputs. For each output, we find the cost minimizing input vector that produces that output. This determines the cost function. We follow with some examples.

Section three takes a look at some properties of the cost function, the section ends by considering homogeneous production, where the cost function factors into a homogeneous term and the unit cost function, the cost of producing a single unit of output.

The final section considers profit when there are many inputs. We form profit using the cost function. This divides profit maximization into two parts, finding the minimum cost way to produce any quantity, and choosing the profit-maximizing quantity. We pursue this method in section three. After looking at properties of the profit function, we focus on homogeneous production functions by using the unit cost function and homogeneity. When there are constant returns to scale, the unit cost function suffices to analyze production.

6.1 Price-taking Firms and Profit Maximization

We treat firms as profit-maximizers. This chapter focuses on the price-taking firm. We will start with a firm that is a price-taker in both the input and output markets. Price-taking firms operate under the assumption that they cannot affect the prices they pay or receive. Price-taking firms are typically small compared to the market, but may be very large in absolute terms. The price-taking firm may be a large oil company in a gigantic market that every day resets its wholesale price based on the current price in the spot market, or it may be a one or two person operation in a medium-sized market.[1]

We will start by considering a firm that uses a single input to produce a single output. There are two market prices to consider, the output price p and the input price w. A *production function* describes the firm's production possibilities. It tells how much output can be produced from a given input quantity $z \geq 0$. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be the production function. In this section we will assume that f is twice continuously differentiable on $\mathbb{R}_{++}$, that the *marginal product* $MP = f'$ is strictly positive, and that $f(0) = 0$.

Profit is

$$\pi(z) = pf(z) - wz.$$

Here $pf(z)$ is the revenue earned from selling output $f(z)$ at price p and wz is the cost of using z amount of the input at price w.

[1] Even a large firm such as Standard Oil (Amoco) acted as a price-taker in the wholesale gasoline market. Every day, they determined their price by using the morning spot price for gasoline.

6.1.1 A Simple Production Function

Let's take a look at a simple production function with a single input and single output.

Example 6.1.1: Suppose the production function is $f(z) = z^\gamma/\gamma$ where $\gamma > 0$. As required, the production function is twice continuously differentiable on $\mathbb{R}_{++}$, with strictly positive marginal product $MP = f' = z^{\gamma-1}$. Moreover, $f(0) = 0$.

There are three cases of interest, depending on whether $\gamma < 1$, $\gamma = 1$, or $\gamma > 1$.

Case one has $\gamma < 1$. Here $f'' = (\gamma - 1)z^{\gamma-2} < 0$ when $z > 0$. The marginal product is decreasing in z. Then $f'(0+) = +\infty$, ensuring that $z = 0$ is not a solution. Profit is

$$\pi(z) = p\left(\frac{z^\gamma}{\gamma}\right) - wz \quad \text{and} \quad \pi' = pz^{\gamma-1} - w.$$

Marginal profit will be negative for large z, so letting $z \to \infty$ does not maximize profit either. The profit-maximizing input z^* is determined by the first-order conditions $p(z^*)^{\gamma-1} = w$. The optimal input level is

$$z^* = \left(\frac{p}{w}\right)^{1/(1-\gamma)}$$

Then z^* yields maximum profit

$$\pi^* = \gamma^{-1}(1-\gamma)\left(\frac{p}{w^\gamma}\right)^{1/(1-\gamma)} > 0.$$

The second case is the linear case $\gamma = 1$. Here the marginal product is constant, $MP = 1$. Profit becomes $\pi(z) = pz - wz = (p - w)z$. If $p > w$, there is no maximum. Profit increases without bound as $z \to \infty$. If $p < w$, positive z always yields negative profit. Profit is maximized only at $z^* = 0$. Finally, if $p = w$, profit is zero regardless of the input level. Any $z \in \mathbb{R}_+$ maximizes profit.

In the third case $\gamma > 0$. Then $f'' = (\gamma - 1)z^{\gamma-2} > 0$, so the marginal product is increasing in z. Profit is now $\pi(z) = z(pz^{\gamma-1} - w)$. As both terms of the product increase without bound, there is no limit to how large profit may become. Therefor, there is no solution to the profit maximization problem when $\gamma > 1$. ◀

6.1.2 Can Profit Be Maximized?

What did we find out about profit maximization in Example 6.1.1?

1. When marginal product was decreasing, it was always possible to maximize profit.
2. When marginal product was constant, whether profit could be maximized depended on both input and output prices, or more specificially, on their ratio. At some price ratios the firm will not produce, at others profit is zero regardless of output, and sometimes profit cannot be maximized at all.
3. Finally, profit maximization was just impossible when the marginal product was increasing.

Similar results hold when there are multiple inputs and even multiple outputs. The profit maximization problem will usually not have solutions in the face of persistent increasing returns to scale, corresponding to increasing marginal product in Example 6.1.1. This is an environment where price-taking is not possible. In Chapter 18 we will see that price-setting firms can maximize profit in the face of persistent increasing marginal product.

What happens under constant returns to scale depends on prices. Some prices will lead to no solution, others yield only the zero solution, and still others allow an infinity of solutions.

Decreasing returns to scale will generally allow for a unique solution. When the technology is not sufficiently productive. Such firms will not produce anything. We will see that there are also cases with decreasing returns where profit maximization is not possible.

Of course, there are other possibilities. There is no requirement that a production function be either concave or convex. There might be regions of increasing and decreasing marginal product. Indeed, it is common when considering renewable resource problems to use a production function that has increasing marginal product at low levels of output, with diminishing marginal product at high level of output.

6.1.3 The Cost Function

We can write the profit maximization problem in terms of output rather than input. To do this, we introduce the cost function. It is easy to derive the cost function in the single input, single output case when f' is strictly increasing. Here output q and input z related by the production function f, $q = f(z)$. We can write input in terms of output by inverting the production function, $z = f^{-1}(q)$.

Profit can then be written as a function of output q. Abusing notation, we write $\pi(q) = pq - wf^{-1}(q)$.[2] Here pq is the revenue and $wf^{-1}(q)$ is the cost of producing q, the cost function $c(w, q)$. Thus $c(w, q) = wf^{-1}(q)$. We compute the marginal cost

$$MC = \frac{\partial c}{\partial q} = \frac{w}{f'(q)}$$

and its q-derivative

$$\frac{\partial^2 c}{\partial q^2} = -\frac{wf''(q)}{[f'(q)]^2}.$$

When the marginal product is increasing ($f'' > 0$), marginal cost is decreasing in q, while a decreasing marginal product yields increasing marginal cost. Of course, constant marginal product translates to constant marginal cost. The first-order condition for profit maximization is now that

$$p = MC = \frac{\partial c}{\partial q},$$

price equals marginal cost. The marginal cost obeys

$$MC = \frac{w}{f'} = \frac{w}{MP},$$

marginal cost is the factor price divided by the marginal product.

[2] The abuse of notation is that π already has a definition in terms of input z, so it now has two definitions, a possible source of confusion. Both notations should be regarded as temporary as we will later use π for the profit function.

6.2 Cost and Production

The formulation of profit in terms of cost is usually easier to use. It separates the choice of output level from decisions about how to make that output. This allows easy generalization to models with many inputs. The firm still sets price equal to marginal cost to choose output. While the input choice becomes more complex, the choice of output level remains the same, set price equal to marginal cost.

Using a cost function also allows us to more easily handle cases where the firm is a price-taker in the input markets, but is a price-setter in the output market. The cost function is unchanged. What does change is that the firm no longer faces a horizontal demand curve, but must take into account the negatively sloped demand curve. In order to sell more, the firm must lower its price. This is discussed further in Chapter 18.

As long as there is a single output, we can handle production with multiple inputs by using a production function that accommodates multiple inputs. Suppose there are m inputs. Let $\mathbf{z} \in \mathbb{R}^m_+$ be a vector of inputs and $f\colon \mathbb{R}^m_+ \to \mathbb{R}_+$ be a *production function*, a function that describes how much output can be produced from any input vector. Here we do not require that f be strictly increasing. This allows us to consider cases where there are capacity constraints or where the marginal product may eventually become negative at large input levels.

We solve the firm's profit maximization problem by first constructing the cost function. This requires finding the cheapest way to produce a given level of output. Once it knows the cost function, the firm can choose the output level that maximizes profit. Since the vector of factor prices is known, this choice implies a particular input demand vector.

6.2.1 The Cost Minimization Problem

The cost function tells us the minimum cost required to produce any feasible quantity of output. For this we need know nothing about the output market. In fact, a firm may still be interested in cost minimization even if it is not a price-taker in the output market. As long as the firm cannot affect factor prices, the cost minimization problem remains the same even if the firm is a monopolist or oligopolist. We can even take this a step further. Charities, universities, and other non-profit firms have an interest in cost minimization as it enables them to do more with their scarce resources.

We consider a firm described by a production function f. Given a vector of factor prices $\boldsymbol{w} \gg \mathbf{0}$ and a target output level $q > 0$, the firm's cost minimization problem is to find the cheapest way to produce output q or more. Cost minimization defines the *cost function* $c(\boldsymbol{w}, q)$ via

$$\begin{aligned} c(\boldsymbol{w}, q) = \inf_{\boldsymbol{z}} \ & \boldsymbol{w}\cdot\boldsymbol{z} \\ \text{s.t. } & f(\boldsymbol{z}) \geq q, \boldsymbol{z} \geq \mathbf{0}. \end{aligned}$$

The use of the infimum rather than the minimum is important in two ways. If there are no $\boldsymbol{z}$ with $f(\boldsymbol{z}) \geq q$ we are taking the infimum of an empty set, so $c(\boldsymbol{w}, q) = +\infty$. Secondly, even if there are $\boldsymbol{z}$ with $f(\boldsymbol{z}) \geq q$, there still may not be a minimum. Even so, the infimum ensures that the cost function is still defined.

When the cost minimization problem has a solution, the set of such solutions is called the *conditional factor demand*. We denote the set of conditional factor demand by $\boldsymbol{z}(\boldsymbol{w}, q)$. This correspondence will be treated as a function when it is always a singleton.[3]

[3] Mathematically, the cost minimization problem is exactly the same as the consumer's expenditure minimization problem. The only changes are cosmetic. The production function has replaced the utility function. Input prices have replaced the goods prices. The output level has replaced the utility level. The conditional factor demands are analogous to the Hicksian demands.

The only important difference is connected with the fact that quantity is cardinal while utility is ordinal. As a result, we will usually require that the production function be concave, whereas the utility function was only required to be quasiconcave. This implies that the cost function is convex in q whereas the expenditure function $e(\mathbf{p}, \bar{u})$ need not have any concavity property where $\bar{u}$ is concerned.

6.2.2 Aside: The Indirect Production Function

An alternative method of thinking about the firms' problem is to use the indirect production function. The indirect production function has the same properties as the indirect utility function, including full duality with the cost function and the ability to recover the production function. It can be useful for deriving certain properties of factor demand and the production function. The indirect production function is only rarely applied to the firm's profit maximization problem, as we usually assume the firm is not subject to a budget constraint, but can finance its purchases of inputs as needed.

Given a production function $f\colon \mathbb{R}^m_+ \to \mathbb{R}_+$, a vector of input prices $\boldsymbol{w} \gg \mathbf{0}$, and a budget m, we define the *indirect production function* g via the following maximization problem.

$$\begin{aligned} g(\boldsymbol{w}, m) = \max_{\boldsymbol{z}}\ & f(\boldsymbol{z}) \\ \text{s.t. } & \boldsymbol{w}\cdot\boldsymbol{z} \le m \\ & \boldsymbol{z} \ge \mathbf{0}. \end{aligned}$$

6.2.3 Cost Minimization: Leontief Production

Let's work through the cost minimization problem when production has the Leontief form.

Example 6.2.1: Let $f(\boldsymbol{z}) = \min_i\{\alpha_i z_i\}$ with each $\alpha_i > 0$. Since $\boldsymbol{w} \gg \mathbf{0}$, any excess input will increase cost without increasing production. To produce q requires $q \leq \alpha_i z_i$ for every i. To minimize cost we must use only the minimum amount of each input needed to produce q, $z_i^* = q/\alpha_i$. Then

$$c(\boldsymbol{w}, q) = \sum_i w_i z_i^* = q\left(\sum_i \frac{w_i}{\alpha_i}\right)$$

and the condition factor demand is $\boldsymbol{z}(\boldsymbol{w}, q) = q(1/\alpha_1, \ldots, 1/\alpha_m)$. ◄

6.2.4 Cost Minimization with Smooth Production

When the problem has a little more structure (e.g., f is quasiconcave and $f \in \mathcal{C}^1$), we can use Kuhn-Tucker to find the conditional factor demands. Define the Lagrangian by

$$\mathcal{L} = \boldsymbol{w}\cdot\boldsymbol{z} - \lambda\big(f(\boldsymbol{z}) - q\big) - \boldsymbol{\mu}\cdot\boldsymbol{z}.$$

The first-order conditions are

$$w_i = \lambda\frac{\partial f}{\partial z_i} + \mu_i$$

for each i. Here each $\mu_i \geq 0$. This can be more compactly written $\boldsymbol{w} = \lambda\, Df + \boldsymbol{\mu}$ with $\boldsymbol{\mu} \geq \mathbf{0}$. The solution must also obey the complementary slackness conditions $\mu_i z_i = 0$ and $\lambda\big(f(\boldsymbol{z}) - q\big) = 0$.

It follows that $\boldsymbol{w} \geq \lambda\, Df$, or equivalently $w_i \geq \lambda\, \partial f/\partial z_i = \lambda\, MP_i$. When $z_i > 0$, the complementary slackness conditions imply

$$w_i = \lambda\,\frac{\partial f}{\partial z_i} = \lambda\, MP_i\,. \tag{6.2.1}$$

Then if $\partial f/\partial z_i > 0$, we can write $\lambda = w_i/\, MP_i > 0$, so $f(\boldsymbol{z}) = q$ by complementary slackness.

If both $z_i, z_j \neq 0$, we can eliminate λ from equation 6.2.1 by dividing the equation for i by the equation for j, resulting in

$$\frac{w_i}{w_j} = \frac{\partial f/\partial z_i}{\partial f/\partial z_j} = \frac{MP_i}{MP_j} = MRTS_{ij} \tag{6.2.2}$$

where MRTS is the *marginal rate of technical substitution*, the negative of the slope of the isoquant in $i{-}j$ space.

In other words, at the cost minimizing point, the relative input price w_i/w_j is equal to the marginal rate of technical substitution. We can get an alternative interpretation of the same fact by rearranging equation 6.2.2 a bit to obtain

$$\frac{MP_i}{w_i} = \frac{MP_j}{w_j},$$

which tells us the marginal product of a dollar's worth of input is the same for every i and j used in production. Moreover, if the marginal value per dollar of one input is lower than for the other inputs, it will not be used in production. Conversely, if an input is not used in production, its marginal value per dollar will be no higher than than the marginal value per dollar of those inputs that are in use.

6.2.5 Derivatives of the Cost Function

When the cost function is differentiable, the Envelope Theorem allows us to interpret the multiplier λ:

$$\frac{\partial c}{\partial q} = \frac{\partial \mathcal{L}}{\partial q} = \lambda.$$

In other words, the multiplier λ is the marginal cost. If $\partial f/\partial z_i > 0$, we can use equation 6.2.1 to obtain the marginal cost as

$$MC(q) = \frac{\partial c}{\partial q} = \frac{w_i}{MP_i}.$$

A second application of the Envelope Theorem yields

$$D_{\boldsymbol{w}} c = D_{\boldsymbol{w}} \mathcal{L} = \boldsymbol{z}(\boldsymbol{w}, q),$$

a result known as Shephard's Lemma.

6.2.6 Cost Minimization: Cobb-Douglas Production

We now solve the cost minimization problem for general Cobb-Douglas production.

Example 6.2.2: Let $f(\mathbf{z}) = \prod_i z_i^{\gamma_i}$ with $\gamma_i > 0$ for every i. Notice that zero input of any factor yields zero output. Then for $q > 0$, we don't have to worry about the constraints $\mathbf{z} \geq \mathbf{0}$ as they cannot bind. The first-order conditions can then be written

$$w_i = \lambda \gamma_i \frac{f(\mathbf{z})}{z_i}.$$

It follows that

$$\frac{w_i}{w_j} = \frac{\gamma_i}{\gamma_j}\frac{z_j}{z_i},$$

allowing us to write

$$z_i = \left(\frac{\gamma_i}{\gamma_j}\right)\left(\frac{w_j}{w_i}\right) z_j.$$

Setting $j = 1$ and using the constraint $f(\mathbf{z}) = q$, we obtain

$$q = \prod_i \left(\frac{\gamma_i}{\gamma_1}\right)^{\gamma_i}\left(\frac{w_1}{w_i}\right)^{\gamma_i} z_1^{\gamma_i} = \left(\frac{w_1}{\gamma_1}\right)^{\gamma} z_1^{\gamma} \prod_i \left(\frac{\gamma_i}{w_i}\right)^{\gamma_i}$$

where $\gamma = \sum_i \gamma_i$. The conditional factor demands are then

$$z_i(\mathbf{w}, q) = \frac{\gamma_i}{w_i} q^{1/\gamma} \prod_i \left(\frac{w_i}{\gamma_i}\right)^{\gamma_i/\gamma}.$$

and the cost function is

$$c(\mathbf{w}, q) = \gamma q^{1/\gamma} \prod_i \left(\frac{w_i}{\gamma_i}\right)^{\gamma_i/\gamma}.$$

Finally, the marginal cost becomes

$$\frac{\partial c}{\partial q}(\mathbf{w}, q) = q^{(1-\gamma)/\gamma} \prod_i \left(\frac{w_i}{\gamma_i}\right)^{\gamma_i/\gamma}.$$

It follows that if $\gamma < 1$, marginal cost is increasing, while if $\gamma > 1$, marginal cost is falling. When $\gamma < 1$ there are decreasing returns to scale, while $\gamma > 1$ yields increasing returns to scale.

It is easily verified that $\partial c/\partial w_i = z_i(\mathbf{w}, q)$. ◀

6.2.7 CES Production

We will do one more example of computing the cost function. This time we use a CES production function.

Example 6.2.3: When there are m inputs, *constant elasticity of substitution* (CES) production functions have the form

$$f(z) = \kappa \left(\sum_{i=1}^{m} \alpha_i z_i^{-\rho} \right)^{-\gamma/\rho}$$

where $-1 < \rho < \infty$ with $\rho \neq 0$, $\gamma, \kappa > 0$ and each $\alpha_i > 0$ with $\sum_i \alpha_i = 1$. When $\rho = -1$ and $\gamma = 1$ we have linear production.

The CES production function is homogeneous of degree γ. The other parameters are the substitution parameter ρ, the share parameters α_i, and the scale factor κ.

What about the limiting and excluded cases for ρ? When $\rho = -1$, the production function is $f(z) = \kappa\big(\sum_i \alpha_i z_i\big)^{\gamma}$. When $\gamma = 1$, this reduces to linear production.

Letting $\rho \to 0$ yields the Cobb-Douglas form $\kappa\big(\prod_i z_i^{\alpha_i}\big)^{\gamma}$. To see this, take the logarithm of f and use l'Hôpital's rule.

Finally, letting $\rho \to \infty$ yields $\kappa\big(\min\{z_i\}\big)^{\gamma}$, which is equivalent to the Leontief form. We obtain this by factoring $\min\{z_i\}$ out of the sum. What remains is bounded between some α_i and 1. Then take the limit to obtain the result.

The above shows how flexible the CES family of production functions is. It includes linear, Cobb-Douglas, and Leontief production functions as special cases.

6.2.8 CES Cost Function

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We now find the cost function and conditional factor demands. Since it is homogeneous of degree γ, $c(\boldsymbol{w}, q) = c(\boldsymbol{w}, 1)q^{1/\gamma}$ and $\boldsymbol{z}(\boldsymbol{w}, q) = q^{1/\gamma}\boldsymbol{z}(\boldsymbol{w}, 1)$. We focus on the case $q = 1$. The Lagrangian is $\mathcal{L} = \boldsymbol{w}\cdot\boldsymbol{z} - \lambda(f(\boldsymbol{z}) - 1)$ with first-order conditions

$$w_i = \alpha_i \lambda \kappa \gamma z_i^{-\rho-1} \left[\sum_i \alpha_i z_i^{-\rho}\right]^{-(\gamma+\rho)/\rho}$$

We divide to eliminate λ (as well as κ), yielding

$$\frac{w_i}{w_j} = \frac{\alpha_i z_i^{-\rho-1}}{\alpha_j z_j^{-\rho-1}} \quad \text{and so} \quad z_j = \left[\frac{\alpha_j w_1}{\alpha_1 w_j}\right]^{\frac{1}{1+\rho}} z_i.$$

Substituting back in the constraint $f(\boldsymbol{z}) = 1$, solving, and then scaling by $q^{1/\gamma}$, we obtain the conditional factor demands

$$z_i(\boldsymbol{w}, q) = \left(\frac{q}{\kappa}\right)^{1/\gamma} \left(\frac{\alpha_i}{w_i}\right)^{\frac{1}{1+\rho}} \left[\sum_i (\alpha_i w_i^{\rho})^{\frac{1}{1+\rho}}\right]^{1/\rho}.$$

The cost function is

$$c(\boldsymbol{w}, q) = \left(\frac{q}{\kappa}\right)^{1/\gamma} \left[\sum_i (\alpha_i w_i^{\rho})^{\frac{1}{1+\rho}}\right]^{\frac{1+\rho}{\rho}}.$$

6.2.9 Elasticity of Substitution is Constant

The *elasticity of substitution* measures how the marginal rate of technical substitution responds to changes in the ratio of factor inputs. It is defined by

$$\sigma_{ij} = -\frac{d \ln(\mathrm{MRTS}_{ij})}{d \ln(z_i/z_j)}.$$

In the CES case,

$$\mathrm{MRTS}_{ij} = \frac{\alpha_i}{\alpha_j}\frac{z_i^{-(1+\rho)}}{z_j^{-(1+\rho)}} = \frac{\alpha_i}{\alpha_j} e^{-(1+\rho)\ln(z_i/z_j)}.$$

so

$$\ln \mathrm{MRTS} = (\alpha_i/\alpha_j) - (1+\rho)\ln(z_i/z_j).$$

Then $\sigma_{ij} = 1+\rho$. Linear production has infinite elasticity of substitution, Cobb-Douglas has unit elasticity, and Leontief has zero elasticity of substitution. ◀

6.3 The Cost Theorem

The firm's cost minimization problem is mathematically identical to the consumer's expenditure minimization problem. In both cases, the isocost (isoexpenditure) lines support the upper contour sets of the production (utility) function. We may prove that the cost minimization problem has a solution in the same way we proved the consumer's expenditure minimization problem has a solution.

Theorem 6.2.4. *Suppose* f *is continuous,* $\boldsymbol{w} \gg 0$*, and there is a* $\bar{\boldsymbol{z}}$ *with* $f(\bar{\boldsymbol{z}}) \geq q$*. Then the cost minimization problem has a solution*

Proof. Since the minimum cannot cost more than using inputs $\bar{\boldsymbol{z}}$, the minimization problem is equivalent to minimizing $\boldsymbol{w}\cdot\boldsymbol{z}$ over the compact set $A = \{\boldsymbol{z} \in \mathbb{R}^m_+ : \boldsymbol{w}\cdot\boldsymbol{z} \leq \boldsymbol{w}\cdot\bar{\boldsymbol{z}} \text{ and } f(\boldsymbol{z}) \geq q\}$. Since $\boldsymbol{z} \mapsto \boldsymbol{w}\cdot\boldsymbol{z}$ is a continuous function, it has a minimum on A by the Weierstrass Theorem. □

The condition that $f(\bar{\boldsymbol{z}}) \geq q$ is quite mild. It merely says that production of q is feasible. Without it, the cost minimization problem makes no sense, as is indicated by the infimum of $+\infty$. The condition that $\boldsymbol{w} \gg 0$ has more bite. It can be plausibly violated. As with the consumer's problem, if some price is zero, there may not be an optimum (see Exercise 6.2.7).

6.3.1 The Basic Cost Theorem

You should not be surprised to hear that the Basic Cost Theorem is very similar to the Basic Expenditure Theorem of Chapter 5. After all, cost minimization and expenditure minimization involve the same mathematical problem. Only the notation differs.

Economically, the important differences are connected with the fact that production is cardinal while utility is ordinal. Because of that, concavity of the production function is economically meaningful, while only quasiconcavity is meaningful for utility. The last two parts of the Cost Theorem rely on the cardinality of the production function and are not repeated in the Basic Expenditure Theorem we saw earlier in Chapter 5.

Basic Cost Theorem. *Let f be a continuous production function. Let $\boldsymbol{w} \gg 0$ and suppose there is $\bar{z} \in \mathbb{R}^m_+$ with $f(\bar{z}) \geq q$. Then:*

(1) *There is a $\boldsymbol{z}(\boldsymbol{w}, q)$ solving the cost minimization problem for factor prices $\boldsymbol{w}$ and output q. If f is also strictly quasiconcave, then $\boldsymbol{z}$ is unique.*
(2) *The cost function $c(\boldsymbol{w}, q)$ is concave, upper semicontinuous and homogeneous of degree one in $\boldsymbol{w}$. In addition, $\boldsymbol{w} \mapsto c(\boldsymbol{w}, q)$ is continuous in $\boldsymbol{w}$ on the interior of its effective domain and weakly increasing in both $\boldsymbol{w}$ and q. If f is strictly quasiconcave, the conditional factor demand function $\boldsymbol{z}(\boldsymbol{w}, q)$ is continuous in $\boldsymbol{w}$.*
(3) **Shephard's Lemma**. *If $c(\boldsymbol{w}, q)$ is differentiable in $\boldsymbol{w}$, then the conditional factor demand obeys $\boldsymbol{z}(\boldsymbol{w}, q) = D_{\boldsymbol{w}}c(\boldsymbol{w}, q)$.*
(4) **Law of Factor Demand**. *If $\boldsymbol{z}_i \in \boldsymbol{z}(\boldsymbol{w}_i, q)$ for $i = 0, 1$, then $(\boldsymbol{z}_1 - \boldsymbol{z}_0)\cdot(\boldsymbol{w}_1 - \boldsymbol{w}_0) \leq 0$.*
(5) *The conditional factor demand $\boldsymbol{z}(\boldsymbol{w}, q)$ is homogeneous of degree zero in $\boldsymbol{w}$.*
(6) *If $c(\boldsymbol{w}, q)$ is $\mathcal{C}^2$ in $\boldsymbol{w}$, then $\boldsymbol{z}(\boldsymbol{w}, q)$ is differentiable in $\boldsymbol{w}$ and the matrix $D_{\boldsymbol{w}}\boldsymbol{z}(\boldsymbol{w}, q) = D^2_{\boldsymbol{w}}c(\boldsymbol{w}, q)$ is symmetric and negative semi-definite. Moreover, $\big[D_{\boldsymbol{w}}\boldsymbol{z}(\boldsymbol{w}, q)\big]\boldsymbol{w} = \big[D^2_{\boldsymbol{w}}c(\boldsymbol{w}, q)\big]\boldsymbol{w} = 0$.*
(7) *If f is concave, then $c(\boldsymbol{w}, q)$ is convex in q.*
(8) *If f is homogeneous of degree $\gamma > 0$, then $c(\boldsymbol{w}, q)$ and $\boldsymbol{z}(\boldsymbol{w}, q)$ are homogeneous of degree $1/\gamma$ in q.*

Property (3) was established by Shephard (1953) and is often referred to as *Shephard's Lemma*. The analogous property of the expenditure function is the *Shephard-McKenzie Lemma*. Notice that properties (3)–(5) do not require a concave production function.

6.3.2 Proof of the Basic Cost Theorem

Basic Cost Theorem. *Let* f *be a continuous production function. Let* $\boldsymbol{w} \gg 0$ *and suppose there is* $\bar{z} \in \mathbb{R}^m_+$ *with* $f(\bar{z}) \geq q$. *Then:*

(1) *There is a* $\boldsymbol{z}(\boldsymbol{w}, q)$ *solving the cost minimization problem for factor prices* $\boldsymbol{w}$ *and output* q. *If* f *is also strictly quasiconcave, then* $\boldsymbol{z}$ *is unique.*
(2) *The cost function* $c(\boldsymbol{w}, q)$ *is concave, upper semicontinuous and homogeneous of degree one in* $\boldsymbol{w}$. *In addition,* $\boldsymbol{w} \mapsto c(\boldsymbol{w}, q)$ *is continuous in* $\boldsymbol{w}$ *on the interior of its effective domain and weakly increasing in both* $\boldsymbol{w}$ *and* q. *If* f *is strictly quasiconcave, the conditional factor demand function* $\boldsymbol{z}(\boldsymbol{w}, q)$ *is continuous in* $\boldsymbol{w}$.
(3) Shephard's Lemma. *If* $c(\boldsymbol{w}, q)$ *is differentiable in* $\boldsymbol{w}$, *then the conditional factor demand obeys* $\boldsymbol{z}(\boldsymbol{w}, q) = D_{\boldsymbol{w}} c(\boldsymbol{w}, q)$.
(4) Law of Factor Demand. *If* $\boldsymbol{z}_i \in \boldsymbol{z}(\boldsymbol{w}_i, q)$ *for* $i = 0, 1$, *then* $(\boldsymbol{z}_1 - \boldsymbol{z}_0) \cdot (\boldsymbol{w}_1 - \boldsymbol{w}_0) \leq 0$.
(5) *The conditional factor demand* $\boldsymbol{z}(\boldsymbol{w}, q)$ *is homogeneous of degree zero in* $\boldsymbol{w}$.
(6) *If* $c(\boldsymbol{w}, q)$ *is* $\mathcal{C}^2$ *in* $\boldsymbol{w}$, *then* $\boldsymbol{z}(\boldsymbol{w}, q)$ *is differentiable in* $\boldsymbol{w}$ *and the matrix* $D\boldsymbol{z}(\boldsymbol{w}, q) = D^2 c(\boldsymbol{w}, q)$ *is symmetric and negative semi-definite. Moreover,* $\big[D\boldsymbol{z}(\boldsymbol{w}, q)\big]\boldsymbol{w} = \big[D^2 c(\boldsymbol{w}, q)\big]\boldsymbol{w} = 0$.
(7) *If* f *is concave, then* $c(\boldsymbol{w}, q)$ *is convex in* q.
(8) *If* f *is homogeneous of degree* $\gamma > 0$, *then* $c(\boldsymbol{w}, q)$ *and* $\boldsymbol{z}(\boldsymbol{w}, q)$ *are homogeneous of degree* $1/\gamma$ *in* q.

Proof. Parts (1)–(6) are essentially the same as the corresponding parts of the Basic Expenditure Theorem from Chapter 5. Since the cost and expenditure functions are mathematically the same, we can borrow the proofs from Chapter 5.

Part (1) combines Theorem 6.2.4 and Proposition 5.2.4. Part (2) is based on Theorems 5.1.5 and 33.6.4. Part (3) restates the Shephard-McKenzie Lemma. Part (4) is the Law of Compensated Demand. Part (5) is part (1) of Theorem 5.2.9, and part (6) is the remainder of Theorem 5.2.9.

For (7), let $q'' = \alpha q + (1-\alpha)q'$ with $0 < \alpha < 1$. If $f(\boldsymbol{z}) \geq q$ and $f(\boldsymbol{z}') \geq q'$, then $f(\alpha\boldsymbol{z} + (1-\alpha)\boldsymbol{z}') \geq \alpha f(\boldsymbol{z}) + (1-\alpha) f(\boldsymbol{z}') \geq q''$. Let $\boldsymbol{z}'' = \alpha\boldsymbol{z} + (1-\alpha)\boldsymbol{z}'$. Then $c_{q''}(\boldsymbol{w}) \leq \boldsymbol{w} \cdot \boldsymbol{z}'' = \alpha\boldsymbol{w} \cdot \boldsymbol{z} + (1-\alpha)\boldsymbol{w} \cdot \boldsymbol{z}'$. As this holds for all $\boldsymbol{z}$ with $f(\boldsymbol{z}) \geq q$ and $\boldsymbol{z}'$ with $f(\boldsymbol{z}') \geq q'$, we may minimize the right-hand side, obtaining $c(\boldsymbol{w}, q'') \leq \alpha c(\boldsymbol{w}, q) + (1-\alpha) c(\boldsymbol{w}, q')$, proving convexity.

Part (8) is established in Proposition 6.2.5, which follows this theorem. □

6.3.3 Cost Theorem: Homogeneous Production

Part (8) of the Cost Theorem tells us that homogeneity of the production function implies homogeneity of both cost and the conditional factor demands. However, it still lacks proof. This is remedied by Proposition 6.2.5. We only consider positively homogeneous production because negative homogeneity would decrease output as input increases.

Proposition 6.2.5. *Suppose the production function* f *is homogeneous of degree* $\gamma > 0$*. Then the cost function and conditional factor demands are homogeneous of degree* $1/\gamma$ *in quantity* q*. That is,* $c(\boldsymbol{w}, tq) = t^{1/\gamma} c(\boldsymbol{w}, q)$ *for* $t > 0$ *and* $\boldsymbol{z}(\boldsymbol{w}, tq) = t^{1/\gamma}\boldsymbol{z}(\boldsymbol{w}, q)$*. Moreover, we can write* $c(\boldsymbol{w}, q) = q^{1/\gamma}\, c(\boldsymbol{w}, 1)$ *and* $\boldsymbol{z}(\boldsymbol{w}, q) = q^{1/\gamma}\, \boldsymbol{z}(\boldsymbol{w}, 1)$*.*

Proof. Here

$$\begin{aligned} c(\boldsymbol{w}, tq) &= \min\{\boldsymbol{w}\cdot\boldsymbol{z} : f(\boldsymbol{z}) \geq tq\} \\ &= \min\{\boldsymbol{w}\cdot\boldsymbol{z} : f(t^{-1/\gamma}\boldsymbol{z}) \geq q\} \\ &= \min\{t^{1/\gamma}\boldsymbol{w}\cdot\boldsymbol{z}' : f(\boldsymbol{z}') \geq q\} \\ &= t^{1/\gamma} c(\boldsymbol{w}, q) \end{aligned}$$

where the substitution $\boldsymbol{z}' = t^{-1/\gamma}\boldsymbol{z}$ was used in the third line. This substitution tells us how the conditional factor demands relate in the two problems, thus $\boldsymbol{z}(\boldsymbol{w}, tq) = t^{1/\gamma}\boldsymbol{z}(\boldsymbol{w}, q)$. Now set $t = q^{-1/\gamma}$ to find $c(\boldsymbol{w}, q) = q^{1/\gamma}\, c(\boldsymbol{w}, 1)$ and $\boldsymbol{z}(\boldsymbol{w}, q) = q^{1/\gamma}\, \boldsymbol{z}(\boldsymbol{w}, 1)$. □

Now that we know that degree $\gamma > 0$ homogeneity of production leads to homogeneity of cost, we can factor the cost function into two parts, isolating the effects of scale (q) and factor prices ($\boldsymbol{w}$). We write $c(\boldsymbol{w}, q) = q^{1/\gamma} c(\boldsymbol{w}, 1)$.

6.3.4 The Unit Cost Function

We define the *unit cost function* $b(\boldsymbol{w})$ to be the cost of producing one unit of output when input prices are $\boldsymbol{w}$. In other words, $b(\boldsymbol{w}) = c(\boldsymbol{w}, 1)$. Of course, by part (2) of the Cost Theorem, the unit cost function is concave in $\boldsymbol{w}$.

The important fact about the unit cost function is that when production is homogeneous of degree γ, we can reconstruct the cost function from the unit cost function.

$$c(\boldsymbol{w}, q) = q^{1/\gamma} b(\boldsymbol{w}).$$

Two such cases are the Cobb-Douglas cost function computed in example 6.2.2 and the CES cost function from example 6.2.3. For Cobb-Douglas,

$$c(\boldsymbol{w}, q) = q^{1/\gamma} \left[\gamma \prod_i \left(\frac{w_i}{\gamma_i} \right)^{\gamma_i/\gamma} \right]$$

so

$$b(\boldsymbol{w}) = \gamma \prod_i \left(\frac{w_i}{\gamma_i} \right)^{\gamma_i/\gamma}$$

for Cobb-Douglas production functions.

The CES cost function has the same general form, $q^{1/\gamma} b(\boldsymbol{w})$, but now

$$c(\boldsymbol{w}, q) = q^{1/\gamma} \left\{ \kappa^{-1/\gamma} \left[\sum_i (\alpha_i w_i^{\rho})^{\frac{1}{1+\rho}} \right]^{\frac{1+\rho}{\rho}} \right\}.$$

This means

$$b(\boldsymbol{w}) = \kappa^{-1/\gamma} \left[\sum_i (\alpha_i w_i^{\rho})^{\frac{1}{1+\rho}} \right]^{\frac{1+\rho}{\rho}}$$

for CES production functions.

6.4 The Profit Maximization Problem

Now that we have the cost function under control, we return to the problem of profit maximization. Suppose a firm is a price-taker, facing a fixed output price p. Profit will be revenue minus cost, $\pi(q) = pq - c(\boldsymbol{w}, q)$. When the production function f is concave in inputs the cost function is convex in q according to the Basic Cost Theorem (8). This means profit is concave in output q. If positive profit is possible, $q = 0$ will not maximize profit unless the marginal profit is always negative. In most cases, if profit is also differentiable, we need only use the first-order condition to maximize profit. Thus $p = \partial c/\partial q$, price equals marginal cost.

Combining the profit maximization condition $p = \text{MC}$ with equation 6.2.1, we find that

$$\text{VMP}_i = p \times \text{MP}_i = w_i,$$

the value of the marginal product of k, (VMP_i), which price times marginal product, is equal to the factor price w_i for every $i = 1, \ldots, m$.

6.4.1 Profit Maximization: Separable Production

Let's see how this works with a somewhat different production function, one that is additive separable.

Example 6.3.1: Suppose the production function is additive separable. Then we can write $f(z) = f_1(z_1) + \cdots + f_m(z_m)$. Provided we have an interior solution, the profit maximization conditions become $pf_i'(z_i) = w_i$. The factor demand for input i is a function only of p and w_i. Letting $\phi_i = (f_i')^{-1}$, we can write $z_i = \phi_i(w_i/p)$ and optimal output is $q = f\big(\phi_1(w_1/p), \ldots, \phi_L(w_L/p)\big)$.

For example, let $f_i(z_i) = z_i^{\gamma_i}$, with $0 < \gamma_i < 1$. Then $\phi_i(y) = (\gamma_i/y)^{1/(1-\gamma_i)}$, so $z_i = (\gamma_i p/w_i)^{1/(1-\gamma_i)}$ and $q = \sum_i (\gamma_i p/w_i)^{\gamma_i/(1-\gamma_i)}$.

If $\gamma_i = 1/2$ for all i, this becomes $z_i = (p/2w_i)^2$ and $q = \sum_i (p/2w_i) = (p/2)\sum_i 1/w_i$. The profit is

$$\begin{aligned} pq - \sum_i w_i z_i &= \frac{p^2}{2}\sum_i \frac{1}{w_i} - \frac{p^2}{4}\sum_i \frac{1}{w_i} \\ &= \frac{p^2}{4}\sum_i \frac{1}{w_i}. \end{aligned}$$

◀

6.4.2 Unit Cost and Profit Maximization

When the production function is homogeneous of degree $\gamma > 0$, we can factor the cost function into a homogeneous function and the unit cost function. Not all of these cost functions are compatible with profit maximization by a price-taking firm. There are three cases to consider, $\gamma > 1$, $\gamma < 1$, and $\gamma = 1$.

We start with the case $\gamma > 1$. Returns to scale are increasing and marginal cost is concave in q. The first-order conditions will then yield **minimum** profit, not maximum profit. Indeed, when profit is $\pi = pq - q^{1/\gamma}c(\boldsymbol{w}, 1)$ and $\gamma > 1$, increasing the scale of output will always increase profit. In fact, profit is $pq - q^{1/\gamma} = q^{1/\gamma}[pq^{(\gamma-1)/\gamma} - 1]$. Taking the limit as $q \to \infty$, and taking into account that p is positive, we find both terms converge to $+\infty$, implying that profit cannot be maximized.

Now consider the case $\gamma < 1$. Returns to scale are decreasing and profit maximization will be possible. There is a unique interior profit-maximizing solution defined by the first-order condition $p = \gamma q^{\gamma-1}b(\boldsymbol{w})$. Thus

$$q^* = \left[\frac{\gamma b(\boldsymbol{w})}{p}\right]^{\frac{1}{1-\gamma}}$$

maximizes profit.

The remaining case is $\gamma = 1$, constant returns to scale. Profit is $pq - c(\boldsymbol{w}, q) = [p - b(\boldsymbol{w})]q$ where $b(\boldsymbol{w})$ is the unit cost function. Profit is proportional to output. If $p > b(\boldsymbol{w})$, positive profit is possible. Profit cannot be maximized because higher output always increases profit further. Profit can be increased without bound.

If positive profit is not possible, maximum profit is zero, controlled by a type of complementary slackness condition $[p - b(\boldsymbol{w})]q = 0$. We refer to this as the *zero-profit condition*. There are two ways the zero-profit condition can be satisfied when $p \leq b(\boldsymbol{w})$. Whenever $b(\boldsymbol{w}) < p$, profit will be negative if anything is produced. Profit is maximized when $q^* = 0$. In the second case, the zero profit condition $b(\boldsymbol{w}) = p$ is satisfied. Then profit is always zero no matter what output level is chosen. Every output quantity $q^* \geq 0$ maximizes profit.

6.4.3 The Profit Function

For firms that are price-takers, with output price $p > 0$ and a vector of input prices $\mathbf{w} \gg \mathbf{0}$, we define the *profit function* $\pi(p, \mathbf{w})$

$$\pi(p, \mathbf{w}) = \sup\ pq - c(\mathbf{w}, q) \quad \text{s.t. } q \geq 0.$$

As we have just seen, there may not be a q^* that solves this problem. We may have a supremum, but not a maximum.[4]

► **Example 6.4.1: Profit with Homogeneous Production.** The previous discussion establishes that if the production function is homogeneous of degree $\gamma > 1$, $\pi(p, \mathbf{w}) = +\infty$, while if $\gamma = 1$, $\pi(p, \mathbf{w})$ is either 0 or $+\infty$. Finally, if $\gamma < 1$,

$$\begin{aligned}\pi(p, \mathbf{w}) &= pq^* - c(\mathbf{w}, q^*)\\ &= \left[\frac{\gamma b(\mathbf{w})}{p^\gamma}\right]^{\frac{1}{1-\gamma}} - \left[\frac{\gamma b(\mathbf{w})}{p^\gamma}\right]^{\frac{\gamma}{1-\gamma}} b(\mathbf{w})\\ &= (1-\gamma)\left[\frac{\gamma b(\mathbf{w})}{p^\gamma}\right]^{\frac{1}{1-\gamma}}.\end{aligned}$$

◄

[4] We will define a more general profit function in Chapter 13.

6.4.4 The Basic Profit Theorem

Just as we have theorems giving the basic properities of the cost and expenditure functions, we also have a theorem describing some properties of the profit functions. See also the Profit Theorem of Chapter 13. You'll notice that the Basic Profit Theorem does not assert the existence of a profit-maximizing solution. This should not surprise you at this point as we have seen that it is easy to find cases where there is no such solution.

Basic Profit Theorem. *Let* $f\colon \mathbb{R}^m_+ \to \mathbb{R}_+$ *be a continuous production function. Let* $p > 0$ *and* $\boldsymbol{w} \gg \mathbf{0}$. *Then:*

(1) *The profit function* $\pi(p, \boldsymbol{w})$ *is convex, lower semicontinuous and homogeneous of degree one in* $(p, \boldsymbol{w})$. *In addition,* π *is continuous on the interior of its effective domain, weakly increasing in* p *and weakly decreasing in* $\boldsymbol{w}$.

(2) **Hotelling's Lemma**. *If* $\pi(p, \boldsymbol{w})$ *is differentiable, the optimal output is* $q^*(p, \boldsymbol{w}) = D_p\pi(p, \boldsymbol{w})$ *and the optimal input is* $\boldsymbol{z}^*(p, \boldsymbol{w}) = -D_{\boldsymbol{w}}\pi(p, \boldsymbol{w})$.

(3) **Law of Supply**. *If* $(q_i, \boldsymbol{z}_i)$ *solve the profit maximization problem at* $(p_i, \boldsymbol{w}_i)$ *for* $i = 0, 1$, *then* $[(p_1, \boldsymbol{w}_1) - (p_2, \boldsymbol{w}_2)]\cdot[(q_1, -\boldsymbol{z}_1) - (q_, - \boldsymbol{z}_2)] \geq 0$.

(4) *The optimal output* $q^*(p, \boldsymbol{w})$ *and optimal input* $\boldsymbol{z}^*(p, \boldsymbol{w})$ *are both homogeneous of degree zero in* $(p, \boldsymbol{w})$.

(5) *If* $\pi(p, \boldsymbol{w})$ *is* $\mathcal{C}^2$ *in* $(p, \boldsymbol{w})$, *then* $\boldsymbol{y}(p, \boldsymbol{w}) = \big(q^*(p, \boldsymbol{w}), -\boldsymbol{z}^*(p, \boldsymbol{w})\big)$ *is differentiable in* $(p, \boldsymbol{w})$ *and the matrix* $D_{(p,\boldsymbol{w})}\boldsymbol{y}(p, \boldsymbol{w}) = D^2_{(p,\boldsymbol{w})}\pi(p, \boldsymbol{w})$ *is symmetric and positive semi-definite. Moreover,* $\big[D_{(p,\boldsymbol{w})}\boldsymbol{y}(p, \boldsymbol{w})\big](p, \boldsymbol{w}) = \big[D^2_{(p,\boldsymbol{w})}\pi(p, \boldsymbol{w})\big](p, \boldsymbol{w}) = 0$.

Proof. Use the same methods we used to prove the Basic Expenditure Theorem and Basic Cost Theorem. □

We refer to $\boldsymbol{z}\big(\boldsymbol{w}, q^*(p, \boldsymbol{w})\big)$ as the *unconditional factor demand* and $\boldsymbol{y}(p, \boldsymbol{w}) = \big(q^*(p, \boldsymbol{w}), -\boldsymbol{z}(p, \boldsymbol{w})\big)$ as the *net supply*.

6.4.5 Profit Maximization: Cobb-Douglas Production

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One homogeneous case is Cobb-Douglas production.

Example 6.3.3: From example 6.2.2 the cost function is

$$c(\boldsymbol{w}, q) = q^{1/\gamma}\left[\gamma \prod_i \left(\frac{w_i}{\gamma_i}\right)^{\gamma_i/\gamma}\right] = q^{1/\gamma}\, b(\boldsymbol{w}),$$

resulting in a marginal cost of

$$\frac{\partial c}{\partial q} = \frac{1}{\gamma} q^{(1-\gamma)/\gamma}\, b(\boldsymbol{w}).$$

The profit maximization problem will not have solutions if $\gamma > 1$ (check the second-order conditions), while it will have an infinity of solutions if $\gamma = 1$ provided

$$p = \frac{b(\boldsymbol{w})}{\gamma} = \prod_i \left(\frac{w_i}{\gamma_i}\right)^{\gamma_i/\gamma}.$$

A higher output price results in no solution (infinite profits) while a lower price would yield $q = 0$ as the solution.

There is a unique solution when $\gamma < 1$. It is

$$q^*(p, \boldsymbol{w}) = \left(\frac{p\gamma}{b(\boldsymbol{w})}\right)^{\gamma/(1-\gamma)} = p^{\gamma/(1-\gamma)} \prod_i \left(\frac{\gamma_i}{w_i}\right)^{\gamma_i/(1-\gamma)}.$$

When $\gamma < 1$ the maximized profit is

$$\pi(q, \boldsymbol{w}) = (1-\gamma)p^{1/(1-\gamma)} \prod_i \left(\frac{\gamma_i}{w_i}\right)^{\gamma_i/(1-\gamma)}$$

and the corresponding unconditional factor demands are

$$\frac{\partial \pi}{\partial w_i} = p^{1/(1-\gamma)} \left(\frac{\gamma_i}{w_i}\right) \prod_i \left(\frac{\gamma_i}{w_i}\right)^{\gamma_i/(1-\gamma)} = z_i\big(q^*(p, \boldsymbol{w}), \boldsymbol{w}\big)$$

and firm supply is

$$\frac{\partial \pi}{\partial p} = p^{\gamma/(1-\gamma)} \prod_i \left(\frac{\gamma_i}{w_i}\right)^{\gamma_i/(1-\gamma)}.$$

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6.4.6 Profit Maximization: Constant Returns to Scale

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Constant returns to scale production functions can be handled in a slightly simpler fashion. Because profits scale linearly with outputs, the profit-maximizing output must satisfy the zero profit condition. If profit can be maximized, it will be zero. The following example illustrates this, as does the $\gamma = 1$ case of Example 6.3.3.

Example 6.3.4: The Leontief production function is $f(\boldsymbol{z}) = \min\{\alpha_i z_i\}$. We found that $c(\boldsymbol{w}, q) = q\big(\sum_i w_i/\alpha_i\big)$ with conditional factor demand $z(\boldsymbol{w}, q) = q(1/\alpha_1, \dots, 1/\alpha_L)$. The unit cost function is $b(\boldsymbol{w}) = \sum_i w_i/\alpha_i$.

As with all constant returns to scale models, the zero profit condition must be satisfied. When $p < \sum_i w_i/\alpha_i$, that means that optimal output is $q^* = 0$. Alternatively, when $p = \sum_i w_i/\alpha_i$, any non-negative level of output yields the maximum profit, which is zero. Finally, no q satisfies the zero profit condition when $p > \sum_i w_i/\alpha_i$. ◀

6.4.7 Unit Cost and Profit Maximization

We close the chapter with a graphical analysis of more general constant returns to scale cases. Here cost is proportional to output.

Example 6.3.5: Suppose a CRS production function describes the production of a good which sells for price p. Let $\boldsymbol{w}$ be the vector of input prices. We presume the output is not also an input to this firm's production. Let $b(\boldsymbol{w})$ be the unit cost function. Profit maximization will only be possible for factor prices $\boldsymbol{w}$ with $p \le b(\boldsymbol{w})$. Since b is concave, this is a convex region in factor price ($\boldsymbol{w}$) space. In that case the maximum profit is zero and the good will not be produced if $p < b(\boldsymbol{w})$.

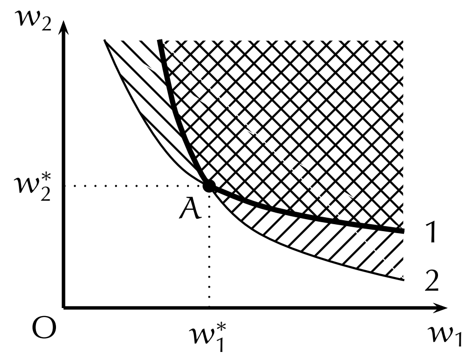

Figure 6.3.6: Each firm produces a unique product, but they use the same goods as inputs. Curve 1 plots the points where $p_1 = b_1(\boldsymbol{w})$ while curve 2 plots $p_2 = b_2(\boldsymbol{w})$. In equilibrium, only the factor prices in the cross-hatched area above both curves can occur. Infinite profit is possible if the factor prices are below either curve. That is not consistent with equilibrium in the output market. If factor prices are above a curve, that good is not produced. Thus only good 1 is produced along the lower right portion of the heavy curve, and only good 2 is produced on the upper left portion of the heavy curve. Both goods are produced only when factor prices are at the intersection of both curves, point $A = (w_1^*, w_2^*)$.

We can now consider the problem of equilibrium factor prices. Suppose there are multiple firms $f = 1, \ldots, F$ with unit cost b_f and output price p_f. Profit maximization will be possible when $p_f \le b_f(\boldsymbol{w})$. Prices with $p_f > b_f(\boldsymbol{w})$ are not possible in equilibrium due to the possibility of infinite profit. Moreover, when $p_f < b_f(\boldsymbol{w})$, firm f will not produce. By plotting the curves $p_f = b_f(\boldsymbol{w})$ in $\boldsymbol{w}$-space, we can see how factor prices determine which firms can produce in equilibrium. The two-firm, two-factor case is illustrated in Figure 6.3.6. ◄

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