13. Production Sets and Profit

1. Production Sets
2. Asymptotic Cones and Returns to Scale
3. Transformation Functions
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5. Production Efficiency

In Chapter 6 we explored the production of a single good. In this chapter we allow the possibility that a firm or even production process may create multiple goods. We also allow for the production of these goods to be related. To accommodate this, we abandon the use of production functions, instead employing the more general concept of a production set. The production set is rather like a production possibilities set, except that it does not have a particular resource endowment built in. Rather, the firms will be able to purchase inputs from the market.

Just as consumer preferences have to obey certain axioms, so do production sets. This gives us a very flexible method for modeling firms. As with preferences, we may impose additional, economically motivated, restrictions. Production sets are explored in section one.

Although it is easy to describe constant returns to scale in terms of production sets, characterizing decreasing or increasing returns is more difficult. However, global increasing or decreasing returns can be connected to the asymptotic cone of the production set, which we do in section two.

Just as preferences can often be described by a utility function, production sets can often be described by a transformation function. The transformation function describes the frontier of the production set. A typical case where this is useful is joint production, when the same inputs are used to produce multiple outputs, as cattle are used to produce meat, leather, and a number of other products. Transformation functions are the subject of section three.

Profit and supply functions are the subject of section four. The main properties of profit and supply follow from the Support Function Theorem. The chapter closes with a brief look at production efficiency in section five.

13.1 Production Sets

There are limitations to the production function approach. As long as production of the products occurs independently, we can handle cases with production of multiple goods. But if production is independent, there's no particular reason for it to occur in the same firm. When production of two products is not independent, there may be advantages to producing both in the same firm. This could be due to joint production. Other types of interaction between production processes is also possible (e.g., workflow improvements or externalities). We need a framework that permits firms that produce production-related goods.

13.1.1 Net Output and Profit

Production sets provide such a framework. We describe a firm's production choice by a *net output vector* $\mathbf{y} \in \mathbb{R}^L$. The components of $\mathbf{y}$ can be either positive or negative. We interpret a positive component as denoting a good offered for sale and a negative component as a good that is demanded. The magnitude of the component denotes the quantity of that good supplied or demanded. Thus $\mathbf{y} = (-1, 2, -5)$ indicates that 2 units of good 2 are supplied while 1 unit of good 1 and 5 units of good 3 are demanded.

The convention that outputs are positive and inputs are negative allows us to write the profit from a net output vector in a simple form. Given a price vector $\mathbf{p}$, the profit obtained from a net output vector $\mathbf{y}$ is $\mathbf{p}\cdot\mathbf{y}$. We can see this with the net output vector $\mathbf{y} = (-1, 2, -5)$. The output is two units of good 2, yielding revenue $2p_2$. The inputs are one unit of good 1 and five units of good 3, with a total cost of $p_1 + 5p_3$. Thus profit is $2p_2 - (p_1 + 5p_3) = \mathbf{p}\cdot\mathbf{y}$.

More generally, we write

$$\begin{aligned}
\mathbf{p}\cdot\mathbf{y} &= \sum_{\ell} p_\ell y_\ell \\
&= \sum_{\{\ell : y_\ell \geq 0\}} p_\ell y_\ell - \sum_{\{\ell : y_\ell < 0\}} |p_\ell y_\ell| \\
&= \text{revenue} - \text{input cost} \\
&= \text{profit.}
\end{aligned}$$

13.1.2 Production Set: Definition

The production technology available to a firm is described by the *production set* or *technology set* which is a set consisting of all net output vectors that are technically feasible for that firm. We denote firm f's production set by $Y_f \in \mathbb{R}^L$.[1]

For theoretical purposes, the production set is the preferred way of modeling production. There are a number of properties we may require of the production set:

Basic Properties of Production Sets.

(T1) Non-emptiness: $Y \neq \emptyset$.
(T2) Closure: Y is a closed set.
(T3) No free lunch: $Y \cap \mathbb{R}^L_+ \subset \{0\}$. That is, if $y \in Y$ and $y \geq 0$ then $y = 0$.
(T4) Inaction: $0 \in Y$.
(T5) Free disposal: $Y = Y - \mathbb{R}^L_+$. That is, if $y \in Y$ and $y' \leq y$, then $y' \in Y$.[2]

At a minimum, a production set must be a non-empty, closed set that obeys the no free lunch condition (conditions T1–T3). This ensures that you can't produce something without some input. We will require a bit more than this for our standard production sets.[3] It would be unusual to violate the inaction axiom (which says you have the option of not producing anything). Free disposal, which says you can decrease output or increase inputs as much as you like, is also normally assumed. However, it may not be appropriate when modeling environmental issues. For our purposes, the following definition of a production set works best.

Production Set. A *production set* or *technology set* is a set $Y \subset \mathbb{R}^L$ that obeys properties (T1)–(T5). In other words, it is non-empty, closed, and satisfies the no free lunch, inaction, and free disposal conditions.

[1] The use of production sets grew out of the theory of linear programming, where production sets were described by a system of linear inequalities. Koopmans (1951b) linear activity model soon led to a more general model. General production sets appeared in Arrow and Debreu (1954), who explicitly recognize Koopmans' influence, and in Debreu (1954b).

[2] In other words, Y is a comprehensive set.

[3] There are variations in the literature, due to the requirements of the papers. So be careful when using results from other sources.

13.1.3 Production Functions and Sets

One way to describe a production set is to use a production function.

Example 13.1.1: When we have a continuous production function $f\colon \mathbb{R}^L_+ \to \mathbb{R}_+$, we can define an associated production set in $\mathbb{R}^{L+1}$ as follows. Label goods so that good $L+1$ is the output and $1, \dots, L$ are the inputs.[4] Here goods $1, \dots, L$ are always inputs, so their components must be non-positive. By allowing $q \le f(\boldsymbol{z})$, we allow free disposal of the output. The single input case is illustrated in Figure 13.1.2.

As already noted, Y satisfies the free disposal condition. It is non-empty as $\big(\mathbf{0}, f(\mathbf{0})\big) \in Y$. The continuity of f implies Y is closed. If $f(\mathbf{0}) \ge 0$, it obeys inaction. Finally, if $f(0) = 0$ and f is increasing, there is no free lunch. ◄

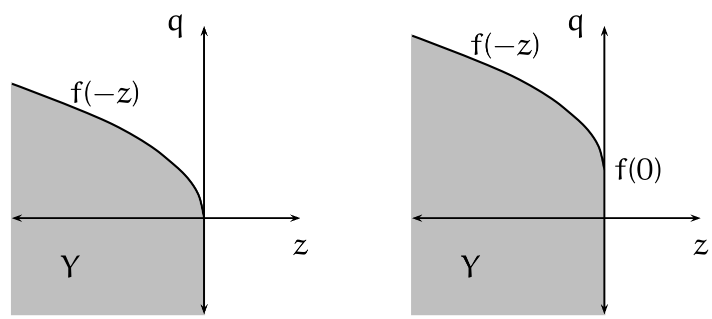

Figure 13.1.2: In both panels the production set Y is derived from a concave continuous production function. Good 1 is the input and good 2 the output. The production function f is flipped over and graphed in the NW quadrant because the input must be negative. The production set Y is the region below the production function.
The left panel shows a case where $f(0) = 0$. It obeys properties (T1)–(T9). The right panel has $f(0) > 0$. As a result, it violates no free lunch (T3) and irreversibility (T7), but obeys the rest of properties (T1)–(T9).

[4] This way of constructing the production set allows for more intuitive graphing of the production frontier when there is one input. Output is then on the vertical axis. An alternative convention, which we will sometimes use, labels the output as good 0. Then $Y = \{(q, -\boldsymbol{z}) : q \le f(\boldsymbol{z}), \boldsymbol{z} \in \mathbb{R}^L_+\}$. This makes it easier to add additional inputs (which we do later), but graphs the production function in the lower right quadrant by rotating it 90° clockwise. The production set is then $Y = \{(-\boldsymbol{z}, q) : q \le f(\boldsymbol{z}), \boldsymbol{z} \in \mathbb{R}^L_+\}$. This is rather unintuitive as output is on the horizontal axis, but it is sometime convenient to label the output good 0.

13.1.4 Supplemental Properties of Production Sets 2/3/22

There are a number of additional properties that are often required of production sets.

Supplemental Properties of Production Sets.

(T6) Productivity: Something can be produced. There is $\mathbf{y} \in Y$ with $\mathbf{y} \not\leq \mathbf{0}$.

(T7) Irreversible: $Y \cap (-Y) \subset \{\mathbf{0}\}$. That is, if both $\mathbf{y}, -\mathbf{y} \in Y$, then $\mathbf{y} = \mathbf{0}$.

(T8) Convex: The set Y is convex. In other words, if $\mathbf{y}, \mathbf{y}' \in Y$ and $0 \leq \alpha \leq 1$, then $\alpha\mathbf{y} + (1-\alpha)\mathbf{y}' \in Y$.

(T9) **Divisibility: If $\mathbf{y} \in Y$ and $0 \leq \alpha \leq 1$, then $\alpha\mathbf{y} \in Y$.

(T10) Expandability: If $\mathbf{y} \in Y$, $t\mathbf{y} \in Y$ for all $t \geq 1$.

(T11) Additive: $Y + Y \subset Y$, which means that if $\mathbf{y}, \mathbf{y}' \in Y$, then $\mathbf{y} + \mathbf{y}' \in Y$.

(T12) Constant returns to scale (CRS): Y is a cone, meaning that $tY \subset Y$ for every $t > 0$. Equivalently, if $\mathbf{y} \in Y$ and $t > 0$, then $t\mathbf{y} \in Y$.

We will not generally require that all of these conditions be satisfied, although they can be (see Figure 13.1.3).

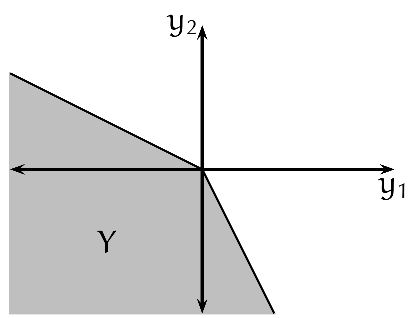

Figure 13.1.3: Let $Y = \{(y_1, y_2) : 2y_1 + y_2 \leq 0 \text{ and } y_1 + 2y_2 \leq 0\}$. This constant returns production set satisfies all 12 production conditions. Note that either good can be used as an input to produce the other good as an output.

13.1.5 More on Production Sets

Productivity (T6) is probably the most important of the supplemental properties. It says production is possible. Without it, we don't really have a production economy. By letting it fail we can include exchange economies as special cases of production economies. That allows us to accommodate both productive economies and exchange economies in the same model. For this reason we have not included productivity in the basic requirements of a production set.

The remaining supplemental properties may not always be employed, but will sometimes be useful. Irreversibility says that we can't take the outputs we just produced and use them to get back to our original inputs. It plays an important role in the construction of aggregate production sets.

Divisibility says that reducing inputs and outputs in the same proportion is possible. Divisibility rules out increasing returns to scale. Expandability is the opposite, it says increasing inputs and outputs in the same proportion is possible. It rules out decreasing returns to scale. It is sometimes referred to as *non-decreasing returns to scale*. If divisibility and expandability are both satisfied, we have constant returns to scale.

These axioms are not all independent. For example, divisibility implies inaction, while convexity and inaction imply divisibility. Additivity and divisibility imply both convexity and constant returns to scale. Constant returns to scale implies both divisibility and expandability. And of course, inaction implies non-emptiness.

13.1.6 Returns to Scale and Other Properties

Several of the supplemental properties of production sets can also be applied to net outputs, or indeed, to any vector. The most important of these are productivity and constant returns to scale. We say a net output $\mathbf{y} \in Y$ is *productive* if has an output, a component that is positive—that $\mathbf{y} \not\leq \mathbf{0}$. The other properties that can be applied to individual net outputs require examining other elements of the set. One such property is constant returns to scale.

Constant Returns to Scale (CRS) Element. Given a production set Y, a net output vector $\mathbf{y} \in Y$ is a *CRS element of* Y if $t\mathbf{y} \in Y$ for all $t > 0$. We denote the set of CRS elements of Y by *CRS*(Y).

Similarly, we can define *irreversible*, *divisible*, and *expandable* elements of Y based on (T7), (T9), and (T10). If an element of Y is both divisible and expandable, it is a CRS element of Y.

If Y is a production set, the combination of inaction and free disposal implies that every net output in the negative orthant will be a CRS element. This is the smallest the set of CRS elements can be. At the other extreme, when a production set obeys constant returns to scale, every element of the production set is a CRS element.

An intermediate case can occur when some elements of the production set enjoy constant returns to scale, while others do not. That is, some net output vectors are CRS, and some are not. We will encounter such a case later in Example 13.2.11.

13.1.7 Production Possibilities Set

If you took a principles of economics class, you may recall hearing about a production **possibilities** set. Such a set depends on both technology (embodied in the production set) and the resources available to the economy. An endowment vector $\boldsymbol{\omega}$ specifies now much of each primary resource, including labor services, is available to the economy. We combine that resource endowment with the various net output vectors in the production set. If the resulting vectors are non-negative, they represent feasible production choices—goods available for consumption that use no more resources in their production than the economy has available. In other words, these vectors form the production possibilities set.

Production Possibilities Set. Given a production set Y and endowment $\boldsymbol{\omega}$, the *production possibilities set* is $(\boldsymbol{\omega} + Y) \cap \mathbb{R}^L_+ = \{\mathbf{x} : \mathbf{x} = \mathbf{y} + \boldsymbol{\omega} \text{ for some } \mathbf{y} \in Y \text{ and } \mathbf{x} \geq 0\}$.

The condition $\mathbf{y} + \boldsymbol{\omega} \geq \mathbf{0}$ implies that for each good ℓ, $\omega_\ell \geq -y_\ell$, which shows that if ℓ is an input, we use no more of ℓ as an input than is available from the endowment ω_ℓ.

We illustrate the production possibilities set with a diagram.

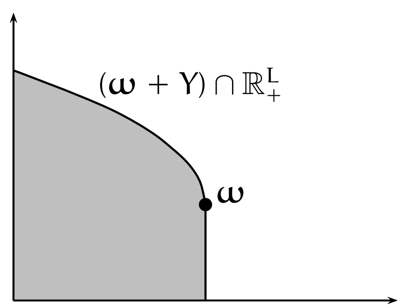

Figure 13.1.4: This production possibilities set $(\boldsymbol{\omega} + Y) \cap \mathbb{R}^L_+$ (shaded) is derived from the production set in the left panel of Figure 13.1.2 and endowment $\boldsymbol{\omega}$. Notice how $\mathbf{0}$ in Figure 13.1.2 has moved to $\boldsymbol{\omega}$ here.

13.1.8 Linear Activity Models I

So far, our main technique for describing a production set is to use a production function. Another way to obtain a production set is to use convexity, constant returns, and free disposal to generate the set from a finite number of basic activities—net output vectors. This is the method of linear activity analysis.[5]

Linear activity models provide a way of generating production sets. They are also a type of model that allows us to look inside the firm's black box and examine intermediate input and output flows within the firm.

We start with a fixed set of *activities* $\mathcal{A} = \{\mathbf{a}^1, \dots, \mathbf{a}^A\} \subset \mathbb{R}^L$ where each $\mathbf{a}^a$ has at least one negative component and at least one positive component. Each activity produces at least one output and requires at least one input.

Activities represent fixed methods (recipes) for combining inputs to obtain outputs. The activities may encompass different stages in the production of a product, or they may be wholly different production processes. The technology is constant returns to scale, so the activities can be run at any scale (*intensity*) $z_a \geq 0$. Scale 0 simply means the activity is not used.

[5] The activity model dates back at least to von Neumann (1937). It was soon adapted to a variety of problems, ranging from mathematical programming to efficient use of transportation. See the conference volume edited by Koopmans (1951a), and especially Koopmans (1951b) for more.

13.1.9 Linear Activity Models II

Activities may be freely combined (additivity) and free disposal is possible. The additivity means there are no externalities between the activities, either positive or negative. The resulting technology set is

$$Y = \left\{ \mathbf{y} : \mathbf{y} \le \sum_a z_a \mathbf{a}^a \text{ for some } z_a \ge 0 \right\}.$$

Define the $L \times A$ activity matrix by $\mathbf{A}$ with columns given by the activity vectors:

$$\mathbf{A} = \left[\ \mathbf{a}^1 \mid \mathbf{a}^2 \mid \cdots \mid \mathbf{a}^A\ \right].$$

We may then write the technology set more compactly as $Y = \{\mathbf{y} : \mathbf{y} \le \mathbf{A}\mathbf{z} \text{ for } \mathbf{z} \ge \mathbf{0}\}$.

The technology set Y is a closed, convex cone with $\mathbf{0} \in Y$. It also obeys free disposal. The no free lunch condition must be imposed separately. Otherwise, we might have activities $(-1, 2)$ and $(2, -1)$ that together allow for production without net inputs. However, the cone generated by $\mathbf{a}^1 = (1, 2, -1)$, $\mathbf{a}^2 = (2, 0, -1)$ and $\mathbf{a}^3 = (0, 5/2, -1)$ is perfectly okay since good 3 must always be an input. Also, the cone generated by $\mathbf{a}^1 = (-1, 2)$ and $\mathbf{a}^2 = (1, -3)$ is also fine.

We had already seen such a production set in Figure 13.1.3. That example was a linear activity model with activity matrix

$$\mathbf{A} = \begin{pmatrix} -2 & 1/2 \\ 1 & -1 \end{pmatrix}.$$

13.1.10 Production Possibilities in a Linear Activity Model

Example 13.1.5: Consider the production set generated by the activities $\mathbf{a}^1 = (1, 2, -1)^{\mathsf{T}}$, $\mathbf{a}^2 = (2, 0, -1)^{\mathsf{T}}$ and $\mathbf{a}^3 = (0, 5/2, -1)^{\mathsf{T}}$. It is $Y = \{\mathbf{y} : \mathbf{y} \le \sum_{a=1}^{3} z_a \mathbf{a}^a$ for some $z_a \ge 0\}$. Good 3 is always used as an input.

The production possibilities frontier will contain points of the form $\mathbf{A}\mathbf{z}$ where $\mathbf{A}$ is the activity matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & \frac{5}{2} \\ -1 & -1 & -1 \end{bmatrix}$$

and $\mathbf{z} \ge \mathbf{0}$. Suppose the endowment is $\boldsymbol{\omega} = (1, 0, 4)$. The first and second components of $A\mathbf{z}$ will be non-negative. The critical thing is that the third component be non-negative. Since each activity uses one unit of good three as input, this requires that sum of the activity levels be no more than 4. If if is less than 4, some of good three will be available for consumption. We should also keep in mind that we also have an endowment of good 1. The production possibilities set is then $\{\mathbf{x} \in \mathbb{R}^3_+ : \mathbf{x} \le \mathbf{A}\mathbf{z} + (1, 0, 0)^{\mathsf{T}}$ for $z_1 + z_2 + z_3 \le 4\}$. ◀

13.2 Asymptotic Cones and Returns to Scale

Although we defined constant returns to scale technologies (T12), we didn't say a thing about decreasing or increasing returns to scale. Translating these concepts to production sets is a tricky matter. One problem is that a production set might include some production processes that have increasing returns to scale while other processes have decreasing returns to scale. Even if all production processes are of the same type, there is still the question of how to identify increasing or decreasing returns to scale for the entire production set.

Two thing seem clear enough. Decreasing returns should mean that we can always decrease the scale of operation, implying that production sets are divisible. Increasing returns should mean that we can increase the scale of operation, implying that production sets are expandable. These concepts are not enough as they also apply when there are constant returns to scale.

To help apply the concepts of increasing and decreasing returns to scale to the production set at a whole, we introduce the asymptotic cone.[6]

Asymptotic Cone. The *asymptotic cone* of a subset B of a normed vector space is $\mathbf{A}(B) = \{\mathbf{x} : \text{there exist } \mathbf{x}_n \in B \text{ and } t_n > 0,\ t_n \to 0 \text{ with } t_n\mathbf{x}_n \to \mathbf{x}$. The sequence $\{(t_n, \mathbf{x}_n)\}_{m=1}^{\infty}$ is called a *defining sequence of* $\mathbf{A}(B)$.[7]

[6] Asymptotic cones were introduced by Steinitz (1913). They have applications to convex analysis (e.g., Fenchel, 1953; Rockafellar, 1970). Debreu (1959) expanded the concept to non-convex sets when introducing them in economics.

[7] This definition is equivalent to that of Fenchel (1953). The definition used by Rockafellar (1970) for the *recession cone* is equivalent to this one for closed convex cones. Debreu used a different definition. Given a set B, let $B^k = \{\mathbf{x} \in B : \|\mathbf{x}\| \ge k\}$. Then let $\Gamma(B^k)$ be the smallest closed cone containing $B^k \cup \{0\}$. Then the asymptotic cone is $\cap_{k \ge 0}\Gamma(B^K)$. The current fashion seems to be to call this set (or its generalizations) the recession cone. There is often no difference. The recession cone and the asymptotic cone coincide on normed spaces. Zălinescu (1993) gives some general conditions for the two concepts to coincide.

13.2.1 Asymptotic Cone: Example 13.2.1

We give two examples of asymptotic cones. In the first example, the asymptotic cone is a proper subset of the original set.

Example 13.2.1: Consider the set defined by $B = \{(x, y) : x \geq 0, y \leq 1 + x\}$. We compute its asymptotic cone. Suppose $(x^*, y^*) \in \mathbf{A}(B)$ and let $\{(t_n, x_n, y_n)\}$ be a defining sequence. Then $y_n \leq 1 + x_n$. Multiply by t_n to obtain $t_n y_n \leq t_n + t_n x_n$. Then take the limit, showing that $y^* \leq x^*$. Further, since $x_n \geq 0$ and $t_n > 0$, $x^* \geq 0$. It follows that $\mathbf{A}(B) \subset \{(x, y) : x \geq 0, y \leq x\}$.

Now if $x \geq 0$ and $y \leq x$, let n be a positive integer and $x_n = nx$ and $y_n = ny$. Then $x_n \geq 0$ and

$$y_n = ny \leq nx = x_n < 1 + x_n.$$

It follows that $(x_n, y_n) \in B$. Setting $t_n = 1/n$, we find that $\{(1/n, nx, ny)\}$ is a defining sequence for (x, y). Then $\mathbf{A}(B) = \{(x, y) : x \geq 0, y \leq x\}$.

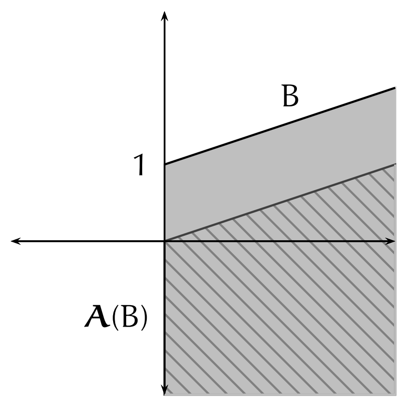

Figure 13.2.2: The set B is marked in solid gray and its asymptotic cone is hatched. Here B is larger than $\mathbf{A}(B)$.

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13.2.2 Asymptotic Cone: Example 13.2.3

The second asymptotic cone also includes vectors not in the original set.

Example 13.2.3: Define $B = \{(x,y) : x \ge 0, y \le -1 + \sqrt{x}\}$. Suppose $(x^*, y^*) \in \mathbf{A}(B)$ and let $\{(t_n, x_n, y_n)\}$ be a defining sequence. Then $t_n y_n \le -t_n + t_n^{1/2}\sqrt{t_n x_n}$. It follows that $y^* \le 0$. Of course, $x^* \ge 0$, so $\mathbf{A}(B) \subset \{(x,y) : x \ge 0, y \le 0\}$.

Now suppose $x > 0$ and $y \le 0$. Let $x_n = nx$, $y_n = ny$, and $t_n = 1/n$. I claim this is a defining sequence (x,y) for n large enough. Of course, $t_n x_n \to x$ and $t_n y_n \to y$ (in fact, $t_n x_n = x$ and $t_n y_n = y$). Now choose N so that $-1 + \sqrt{nx} \ge 0$ for $n \ge N$. Then $(nx, ny) \in B$ for $n \ge N$, showing that $(x,y) \in \mathbf{A}(B)$.

We also have to consider points of the form $(0,y)$ with $y \le 0$. Here Let $x_n = x$, so that $ny \le 0 = -1 + \sqrt{x_n}$. Then $(1/n, x, ny)$ is a defining sequence for $(0,y)$. Combining this with the previous paragraphs shows that $\{(x,y) : x \ge 0, y \le 0\} = \mathbf{A}(B)$.

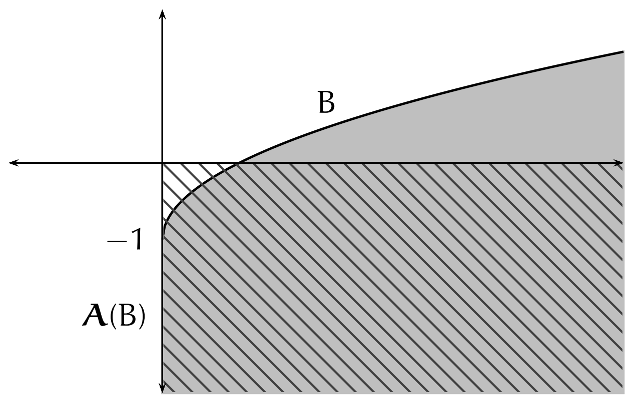

Figure 13.2.4: The set B is marked in solid gray and its asymptotic cone is hatched. In this example, B neither contains nor is contained in its asymptotic cone, $\mathbf{A}(B)$.

◀

13.2.3 Expandability and Asymptotic Cones

It is easy to show that the asymptotic cone of B contains every expandable element of B.

Proposition 13.2.5. *If* B *is a non-empty set and* $x \in B$ *is an expandable element, then* $x \in \mathbf{A}(B)$.

Proof. Since x is expandable, $nx \in B$ for all positive integers n. Then $(1/n, nx)$ is a defining sequence for x because $(1/n)(nx) = x \to x$. It follows that $x \in \mathbf{A}(B)$. □

We have an immediate corollary.

Corollary 13.2.6. *If all elements of a non-empty set* B *are expandable, then* $B \subset \mathbf{A}(B)$.

The corollary applies when there are constant returns. Once we define increasing returns, which will include expandability as part of the definition, the corollary will apply there too.

13.2.4 CRS Elements and Free Disposal

Although asymptotic cones have been used in economics for over 60 years, most of the focus has been on their mathematical properties rather than their economic properties.[8] As for the economic properties, asymptotic cones are useful when describing returns to scale of production sets.

Proposition 13.2.7. *Let* Y *be a set that obeys inaction and free disposal. Then every* $\mathbf{y} \le \mathbf{0}$ *is a CRS element of* Y*. In particular, if* Y *is a production set, every* $\mathbf{y} \le \mathbf{0}$ *is a CRS element of* Y*.*

Proof. Suppose $\mathbf{y} \le \mathbf{0}$ and take $t > 0$. Then $t\mathbf{y} \le \mathbf{0}$. Now $\mathbf{0} \in Y$ by inaction, so $t\mathbf{y} \in Y$ by free disposal. It follows that every element of the negative orthant is a CRS element of Y. Since production sets obey inaction and free disposal, this applies to every production set. $\square$

[8] Some of their key mathematical properties are investigated in subsections 13.2.2 and 13.2.3.

13.2.5 CRS Elements and the Asymptotic Cone

The following proposition shows the relation between the set of CRS elements of Y and its asymptotic cone. They are identical when production is divisible. We also show that the asymptotic cone of any production set must contain the negative orthant.

Proposition 13.2.8.

1. *If* Y *is a closed and divisible set, then* $\mathbf{A}(Y) = CRS(Y)$.
2. *If* Y *is a production set, then* $-\mathbb{R}^L_+ \subset \mathbf{A}(Y)$.

Proof. Part (1): First suppose $y \in \mathbf{A}(Y)$. Then there is a defining sequence with $y_n \in Y$ and $t_n \to 0$ with $t_n y_n \to y$. By divisibility, $t_n y_n \in Y$ for n large. Closure then implies $y \in Y$. Moreover, if $\lambda > 0$, $\lambda y = \lim(t_n\lambda)y_n$, showing that $\lambda y \in \mathbf{A}(Y)$. In other words, every member of the asymptotic cone of Y is a CRS element of Y.

Next suppose y is a CRS element of Y. As such, y is expandable. By Proposition 13.2.5 it is in the asymptotic cone of Y. It follows that the asymptotic cone is identical with the set of CRS elements of Y.

Part (2): Free disposal and inaction yield $-\mathbb{R}^L_+ \subset Y$. Since all elements of $-\mathbb{R}^L_+$ are CRS elements of Y, they are expandable. By Proposition 13.2.5, $-\mathbb{R}^L_+ \in \mathbf{A}(Y)$ □

13.2.6 Constant Returns and the Asymptotic Cone

Proposition 13.2.8 has given us an economic interpretation of the asymptotic cone, at least for closed divisible sets. We can put this to work by characterizing constant returns to scale production sets using the asymptotic cone.

Proposition 13.2.9. *Let Y be a production set. Then Y obeys constant returns to scale if and only if it is divisible and* $Y = \mathbf{A}(Y)$.

Proof. If Y is a constant returns to scale production set, it is closed and obeys divisibility. By Proposition 13.2.8 (1), $\mathbf{A}(Y) = CRS(Y) = Y$ because every element of Y is a CRS element.

Conversely, suppose a divisible production set obeys $\mathbf{A}(Y) = Y$. By Proposition 13.2.8, $\mathbf{A}(Y) = \{\text{CRS elements of } Y\}$. But since $\mathbf{A}(Y) = Y$, every element of Y is CRS, so Y is a cone. $\square$

13.2.7 Exploring Returns to Scale

What about other production sets? If the production set is convex, or even merely divisible, $\mathbf{A}(Y)$ is a subset of Y, consisting of the CRS elements of Y. Since we will focus on cases with convex production sets, we will generally be working with cases where the asymptotic cone has economic meaning.

To pursue this idea further, if $\mathbf{A}(Y)$ is smaller than Y, it is because some elements of Y do not enjoy constant returns to scale. Free disposal implies that the negative orthant is contained in both sets, so the elements of Y without constant returns must be productive. Divisibility then implies that $t\mathbf{y} \notin Y$ for t large. We cannot continue to scale up the outputs by proportional scaling the inputs. There must be decreasing returns to scale.

This suggests classifying returns to scale as follows. Decreasing returns to scale mean Y is divisible with $\mathbf{A}(Y) \subsetneq Y$, constant returns mean $\mathbf{A}(Y) = Y$, and increasing returns to scale would mean that Y is not divisible and $Y \subsetneq \mathbf{A}(Y)$. To see if that's reasonable, let's test it with Cobb-Douglas production.

13.2.8 Cobb-Douglas and Returns to Scale

Example 13.2.10: Consider the Cobb-Douglas production function $f(z_1, z_2) = z_1^\alpha z_2^\beta$ with $\alpha, \beta > 0$ and form the production set as in Example 13.1.1, so $Y = \{(-z_1, -z_2, q) : z_1, z_2 \geq 0, q \leq f(z_1, z_2)\}$. For convenience, we set $\gamma = \alpha + \beta$.

Case I: We first focus on the decreasing returns case ($\gamma < 1$). The production set is divisible since if $(-z_1, -z_2, q) \in Y$, $q \leq f(z_1, z_2)$. For $0 < t < 1$,

$$tq \leq t^\gamma q \leq t^\gamma f(z_1, z_2) = f(tz_1, tz_2),$$

showing that $(-tz_1, -tz_2, tq) \in Y$.

By part 1 of Proposition 13.2.8, the asymptotic cone of Y consists of the CRS elements of Y. Take $y = (-z_1, -z_2, q) \in Y$ with $z_1, z_2, q \geq 0$ and consider ty. Is $tq \leq f(tz_1, tz_2) = t^\gamma f(z_1, z_2)$? This requires $q \leq t^{\gamma-1} f(z_1, z_2)$. Letting $t \to \infty$, we find $q = 0$. The only CRS elements of Y are those that are not productive—are in the negative orthant. Since Y is convex, it is also divisible. By Proposition 13.2.8, $\mathbf{A}(Y)$ is the set of CRS elements, which is $-\mathbb{R}^L_+$.

Case II: When there are increasing returns to scale, the situation is quite different. When $\gamma > 1$, if $(-z_1, -z_2, q) \in Y$, so is $t(-z_1, -z_2, q)$ for $t > 1$. This implies that $(-z_1, -z_2, q)$ is in the asymptotic cone, which must contain many productive vectors. Moreover, $x_n = (-n, -n, n^\gamma) \in Y$ for every positive integer n. If we set $t_n = 1/n^\gamma$, we find $\lim t_n x_n = (0, 0, 1) \in \mathbf{A}(Y)$. The asymptotic cone contains vectors violating the no free lunch condition! ◀

13.2.9 Homogeneous Production and Returns to Scale

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In fact, these same results hold any time production is described by a homogeneous production function. Perhaps a stronger notion of returns to scale is needed. We could take decreasing returns to scale to mean that Y is divisible and $\mathbf{A}(Y) = -\mathbb{R}^L_+$. Increasing returns to scale could mean that Y not divisible and there are vectors in $\mathbf{A}(Y)$ that provide a free lunch.

For decreasing returns to scale, this definition is equivalent to requiring that that there are no productive CRS elements of Y. Since production is divisible, we can restate that to say that for every productive $\mathbf{y} \in Y$, there is a t_0 so that $t\mathbf{y} \notin Y$ for $t > t_0$.

For increasing returns to scale, the condition seems a bit stringent. It requires not only that for each productive $\mathbf{y} \in Y$ and $t > 0$, there is a $\mathbf{y}' \in Y$ with $\mathbf{y}' > t\mathbf{y}$ (more than linear increases in production are possible), but that the outputs must increase enough faster than the inputs that the input/output ratio goes to zero with t. This is satisfied in the Cobb-Douglas case when the ratio is $t^{1-\gamma}$ and $\gamma > 1$.

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13.2.10 Slowly Decreasing Returns to Scale

What if the returns to scale are not quite so relentless? We start with an example where there are decreasing returns, but the amount that the returns decrease itself decreases. In this case, there will be productive CRS elements, and an asymptotic cone that is larger than the negative orthant.

Example 13.2.11: Consider the production function

$$f(z) = 1 + z - \frac{1}{1+z}.$$

This function is $\mathcal{C}^\infty$ and for $z > 0$,

$$f' = 1 + \frac{1}{(1+z)^2} > 0, \quad \text{and} \quad f'' = \frac{-2}{(1+z)^3} < 0.$$

It also has the following properties: $f(0) = 0$ and for $z > 0$, $f(z) > z$. The latter follows because

$$f(z) = 1 + z - \frac{1}{1+z} = z + \left[1 - \frac{1}{1+z}\right] > z.$$

We have an increasing concave production function. The no free lunch condition holds and this defines a production set.

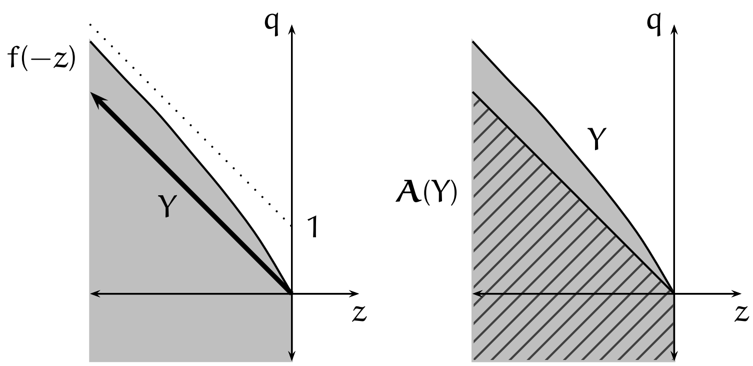

Figure 13.2.12: In the left panel, the production function is asymptotic to the dotted line through $(0, 1)$ with slope -1. Rays through the origin with slope -1 or less will remain in the production set Y. The ray through $(-1, 1)$ is illustrated.

The right side of the figure shows the asymptotic cone of Y.

The fact that $f'' < 0$ suggests this is a decreasing returns to scale case, and indeed, $f(tz) < tf(z)$ for large values of t. As a result, there are decreasing returns along the production frontier.

Because $f(z) > z$, the net output $(-z, z)$ is not on the production frontier. It is also not subject to the same decreasing returns. Indeed, because $(z, -z) \in Y$ for all $z \geq 0$, any $(z, -z)$ is a CRS element of Y.

We can not go any farther than this. The fact that $\lim_{z\to+\infty} f(z)/z = 1$, means that whenever $\alpha > 1$, $(\alpha z, -z)$ is **not** a CRS element.

In this case, we have a weaker type of decreasing returns where $\mathbf{A}(Y)$ contains some productive net outputs. It is not just the negative orthant. In fact, $\mathbf{A}(Y) = \{\mathbf{y} : y_2 \leq -y_1\}$. ◀

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13.2.11 Slowly Increasing Returns to Scale

The weak decreasing returns in Example 13.2.11 meant that the asymptotic cone could not include the production function itself. This function fits our original idea of using the asymptotic cone to define decreasing returns to scale, that Y is divisible and $\mathbf{A}(Y) \subsetneq Y$.

Our next example is similar, but with increasing, not decreasing, returns.

Example 13.2.13: Consider the production function

$$f(z) = -1 + z + \frac{1}{1+z}.$$

This function is $\mathcal{C}^\infty$ and for $z > 0$

$$f' = 1 - \frac{1}{(1+z)^2} > 0, \quad \text{and} \quad f'' = \frac{2}{(1+z)^3} > 0.$$

Also, $f(0) = 0$, $f'(0) = 0$, and for $z > 0$, $f(z) < z$. The last follows because

$$f(z) = -1 + z + \frac{1}{1+z} = z - \left[1 - \frac{1}{1+z}\right] < z.$$

We have an increasing convex production function and the no free lunch condition holds.

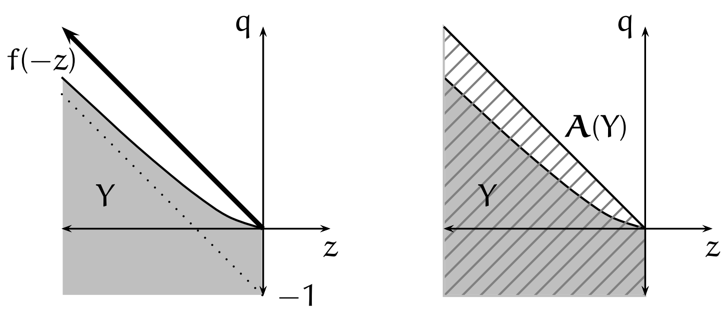

Figure 13.2.14: On the left, the production function is asymptotic to the dotted line through $(0, -1)$ with slope -1. Rays through the origin with slope -1 or less will remain in the asymptotic cone of Y, even though they are not in Y. The ray through $(-1, 1)$ is illustrated. If it is displaced by $(0, -1)$, it will always be in the production set. Since the asymptotic cone is unaffected by this displacement, the ray is in the asymptotic cone.

The right side of the figure shows the asymptotic cone of Y.

The fact that $f'' > 0$ suggests this is an increasing returns to scale case. Now $f'(0) = 0$, implying that no productive rays through the origin will be in the production set. Nonetheless, there are enough increasing returns that $(-1, 1)$ will be in the asymptotic cone. To see this, set $\mathbf{y}_n = \big(-n, f(n)\big) = \big(-n, n^2/(1+n)\big)$ and $t_n = 1/n$. Then

$$\lim_n t_n \mathbf{y}_n = \lim \left(-1, \frac{n}{n+1}\right) = (-1, 1).$$

Although $(-1, 1)$ is not in the production set, it is in the asymptotic cone. In fact, it defines the upper limit of the asymptotic cone, which is $\mathbf{A}(Y) = \{\mathbf{y} : y_2 \le -y_1 \text{ and } y_1 \le 0\}$. In this case the production set is a proper subset of the asymptotic cone. ◀

13.2.12 Defining Returns to Scale: Asymptotic Cones

The weak increasing returns meant that the asymptotic cone couldn't include the production function itself. As with decreasing returns, this fits our first notion of increasing returns, that Y is not divisible and $\mathbf{A}(Y) \supsetneq Y$.

Part of the difference in the definitions is the strength of the returns to scale. The weaker definition also allows differences in returns to scale in different directions. This leads us to the following definitions of both strong and weak returns to scale.

Returns to Scale.

(T12) Constant returns to scale: The set Y is a cone. If Y is a production set, this is equivalent to requiring that Y be divisible and $Y = \mathbf{A}(Y)$.

(T13) Weak decreasing returns to scale: The set Y is divisible and $\mathbf{A}(Y) \subsetneq Y$.

(T14) Weak increasing returns to scale: The set Y is not divisible and $Y \subsetneq \mathbf{A}(Y)$.

(T15) Strong decreasing returns to scale: The set Y is divisible and $\mathbf{A}(Y) = -\mathbb{R}^L_+$.

(T16) Strong increasing returns to scale: The set Y is not divisible and $\mathbf{A}(Y)$ violates the no free lunch condition. There is a $\mathbf{y} \in Y$ with $\mathbf{y} \not\leq \mathbf{0}$.

I've restated (T12) to write it in terms of the asymptotic cone. Although these definitions make sense, they are somewhat non standard, so don't expect other people to use them anytime soon.

The three possibilities for the relation between the production and its the asymptotic cone define weak increasing returns ($\mathbf{A}(Y) \subsetneq Y$), constant returns ($\mathbf{A}(Y) = Y$) and weak decreasing returns ($\mathbf{A}(Y) \supsetneq Y$). The two stronger cases involve making the returns hold more comprehensively, as happens with homogeneous production functions. With strong decreasing returns, any productive net output cannot be scaled beyond a certain point. With strong increasing returns, at least one output can have its production scaled up without limit.

13.3 Production Sets: Transformation Functions 2/8/22

NB: Problems 13.1.3, 13.1.5, 13.5.1, and 14.2.2 are due on Tuesday, February 15.

Of course, using an abstract production set is sometimes inconvenient. It does not easily lend itself to calculations and first-order conditions can not be used. It is often possible to gain the advantages of a production set without losing the advantages of a functional form by using a transformation function.

A *transformation function* $T\colon \mathbb{R}^L \to \mathbb{R} \cup \{+\infty\}$ describes the possible net output vectors.

Transformation Function. A function $T\colon \mathbb{R}^L \to \mathbb{R} \cup +\infty$ is a *transformation function* if it satisfies the following conditions.

1. T is lower semicontinuous on $\mathbb{R}^L$, and continuous on $\{\mathbf{y} : T(\mathbf{y}) < 0\}$.
2. T is globally non-decreasing, and increasing at every point in its effective domain, $\operatorname{dom} T = \{\mathbf{y} : T(\mathbf{y}) < +\infty\}$.
3. $T(\mathbf{0}) = 0$ and $T(\mathbf{y}) > 0$ for $\mathbf{y} > \mathbf{0}$.

Typically, we admit $+\infty$ as a possible value for T. It indicates that such net outputs are not feasible. It is sometimes helpful to use $+\infty$ as a value to ensure that T is quasiconvex.

The value $+\infty$ may seem to cause complications concerning the term "increasing". However, when T is increasing at every point in its effective domain, the fact that T is otherwise $+\infty$, means that if $T(\mathbf{y}) < 0$ and $\mathbf{y}' \gg \mathbf{y}$, then $T(\mathbf{y}') > T(\mathbf{y})$, regardless of whether $\mathbf{y}' \in \operatorname{dom} T$.

13.3.1 Production Sets from Transformation Functions

Any transformation function T defines a production set by $Y = \{y : T(y) \le 0\}$. Quasiconvexity of T is sufficient for convexity of Y. Further, the boundary of Y is the set of points where T is zero.

Proposition 13.3.1. *Let* T *be a transformation function on* $\mathbb{R}^L$. *Then*

1. $Y = \{y \in \mathbb{R}^L : T(y) \le 0\}$ *is a production set.*
2. $T(y) = 0$ *if and only if* $y \in \partial Y$.

Proof. Part 1: We first show that Y is a production set. Because $T(\mathbf{0}) = 0$ by assumption (3), $\mathbf{0} \in Y$. This shows that Y is both non-empty (T1) and obeys inaction (T4). By assumption (1), T is lower semicontinuous on $\mathbb{R}^L$. That means $Y = T^{-1}\big((-\infty, 0]\big)$ is closed (T2).

By assumption (3), if $y > \mathbf{0}$, then $T(y) > 0$, establishing the no free lunch condition (T3).

That leaves free disposal. Suppose $y \in Y$ and $y' \le y$. Then $T(y) \le 0$. By (2), T is globally non-decreasing. Then $T(y') \le T(y) \le 0$, so $y' \in Y$. In other words, Y also obeys free disposal (T5) and so is a production set.

Part 2: To show the boundary property first suppose $T(y) = 0$. As $y \in Y$ and Y is closed, we need only show y is in the closure of the complement of Y. Consider $y_n = y + \frac{1}{n}\boldsymbol{e} \gg y$. By assumption (2), T is increasing at every point of dom T. Now $y \in \operatorname{dom} T$, so $T(y_n) > 0$. Since $y = \lim_n y_n$, $y \in \overline{Y^c}$. As Y is closed, $y \in \partial Y$.

Conversely, let $y \in \partial Y$. Since $\partial Y \subset \overline{Y}$ and Y is closed, $\partial Y \subset Y$. It follows that $T(y) \le 0$. We must show that $T(y) = 0$. Because $y \in \partial Y$, there are $y_n \in Y^c$ with $y_n \to y$. If $T(y) < 0$, the continuity of T on $\{y : T(y) < 0\}$ shows there is an open neighborhood U of y with $T(y') < 0$ for all $y' \in U$. It follows that if $y_n \to y$, then $y_n \in U$ for n large, showing that y cannot be a limit of points in Y^c. Since y is such a limit, we cannot have $T(y) < 0$, so $T(y) = 0$. □

For example, suppose $L = 2$ and let $T(y) = y_1 + y_2$. This transformation function obeys the properties of a transformation function. The production set is $Y = \{(y_1, y_2) : y_1 + y_2 \le 0\}$. In this case either good can be an input (or output) as the vectors $(-1, 1) \in Y$ and $(2, -3) \in Y$ demonstrate.

Transformation functions are often required to satisfy additional properties. For example, T is often assumed to be convex, or at least quasiconvex. This implies that the derived production set Y is convex.

13.3.2 Marginal Rates of Transformation

When T is differentiable, we can define the *marginal rate of transformation* by

$$\mathrm{MRT}_{k\ell} = \frac{\partial T/\partial y_k}{\partial T/\partial y_\ell}.$$

When $dT \gg 0$, the production frontier is defined by $T(\mathbf{y}) = 0$. The marginal rate of transformation is the negative of the slope of the production frontier in $k - \ell$ space.

Example 13.3.2: We return to the case where there is a differentiable production function, with goods $1, \ldots, L$ used as inputs and good $L + 1$ as output. The corresponding transformation function (in term of $\mathbf{y}$) is

$$T(y) = \begin{cases} y_{L+1} - f(-y_1, \ldots, -y_{L-1}) & \text{for } y_1, \ldots, y_L \le 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Notice how the ability to use $+\infty$ as a value of T allows us to easily define T on all of $\mathbb{R}^L$. Moreover, T will be globally convex if f is concave.

Then $Y = \{\mathbf{y} : T(\mathbf{y}) \le 0\}$. Here, when $\ell = L+1$, the marginal rate of transformation is $\mathrm{MRT}_{k(L+1)} = \partial f/\partial y_k = \partial f/\partial z_k = \mathrm{MP}_k$ is the marginal product of k. And for $\ell < L+1$, $\mathrm{MRT}_{k\ell} = (\partial f/\partial y_k)/(\partial f/\partial y_\ell) = (\partial f/\partial z_k)/(\partial f/\partial z_\ell) = \mathrm{MRTS}_{k\ell}$ is the marginal rate of technical substitution. In other words,

$$\mathrm{MRT}_{k\ell} = \begin{cases} \mathrm{MRTS}_{k\ell} & \text{when } \ell < L + 1 \\ \mathrm{MP}_k & \text{when } \ell = L + 1. \end{cases}$$

◀

13.3.3 One Input and Two Outputs

The transformation function can handle cases where there are multiple outputs.

Example 13.3.3: Suppose good three can be used either to produce good one via a production function f_1 or to produce good two via the production function f_2. To find the transformation function, we first consider the input requirements to produce q_i of good i, $f_i^{-1}(q_i)$. The function f_i^{-1} is increasing and $f_i^{-1}(0) = 0$. We extend f_i to all of $\mathbb{R}$ by g_i:

$$g_i(x) = \begin{cases} f_i^{-1} & \text{when } x \geq 0 \\ 0 & \text{when } x < 0. \end{cases}$$

Since $f_i^{-1}(0) = 0$, g_i is continuous if f_i is.

The total input required to produce (y_1, y_2) is $g_1(y_1) + g_2(y_2)$. Feasibility requires $g_1(y_1) + g_2(y_2) \leq -y_3$ where $-y_3$ is the input of good three. Rewrite that as $g_1(y_1) + g_2(y_2) + y_3 \leq 0$ to find the transformation function $T(\mathbf{y}) = g_1(y_1) + g_2(y_2) + y_3$.

It is easy to verify that this function obeys the properties of a transformation function, so the production set is $Y = \{\mathbf{y} : T(\mathbf{y}) \leq 0\}$. Note that $y_3 \leq 0$ for all points in Y.

We can specialize this example further to take $f_1(z) = z^{1/2}$ and $f_2(z) = z^{1/3}$. Then $f_1^{-1}(q) = q^2$ and $f_2^{-1}(q) = q^3$. The resulting transformation function is $T(\mathbf{y}) = (\max\{y_1, 0\})^2 + (\max\{y_2, 0\})^3 + y_3$. ◄

13.3.4 Two Inputs and Two Outputs I

Example 13.3.4: Multiple inputs and outputs can also be modeled via a transformation function. Suppose two goods are both used as inputs to produce two possible outputs. Let $\mathbf{q} \in \mathbb{R}^2_+$ denote the output vector and $\boldsymbol{z} \in \mathbb{R}^2_+$ be the input vector. Net output is then $\mathbf{y} = (\mathbf{q}, -\boldsymbol{z})$. We have two production functions, $f_\ell(\boldsymbol{z}^\ell)$. We will presume both production functions are $\mathcal{C}^2$, concave, continuous, and increasing functions from $\mathbb{R}^2_+$ to $\mathbb{R}_+$ with $f_\ell(\mathbf{0}) = 0$. Total input is found by adding the inputs used for production of each output, thus $z_k = z_k^1 + z_k^2$ for outputs $k = 1, 2$.

To derive a transformation function, we find the maximum amount of output one we can produce given total input $\boldsymbol{z}$ and output of good two, q_2. Writing this maximum amount as $F(q_2, \boldsymbol{z})$, the transformation function is then $T(\mathbf{q}, -\boldsymbol{z}) = q_1 - F(q_2, \boldsymbol{z})$ or $T(\mathbf{y}) = y_1 - F(y_2, -y_3, -y_4)$.

The maximization problem is

$$\begin{aligned} F(q_2, z_1, z_2) = \max\ & f_1(z_1^1, z_2^1) \\ \text{s.t. } & q_2 \le f_2(z_1^2, z_2^2) \\ & z_k^1 + z_k^2 \le z_k, \text{ for } k = 1, 2. \\ & z_k^\ell \ge 0, \text{ for } k, \ell = 1, 2. \end{aligned}$$

where we choose the z_k^ℓ. Set $F = -\infty$ if the problem has no solution. If $q_2 \le 0$, set $F = f_1(z_1, z_2)$.

We now focus on the case $0 < q_2 < f_2(z_1, z_2)$. If $\partial f_\ell/\partial z_k = +\infty$ when $z_k = 0$, the constraints $z_k^\ell \ge 0$ will not bind. In that case, the Lagrangian becomes

$$\mathcal{L} = f_1(z_1^1, z_2^1) + \lambda(f_2(z_1^2, z_2^2) - q_2) - \mu_1(z_1^1 + z_1^2 - z_1) - \mu_2(z_2^1 + z_2^2 - z_2).$$

The first-order conditions are

$$\frac{\partial f_1}{\partial z_1^1} = \mu_1, \quad \frac{\partial f_1}{\partial z_2^1} = \mu_2, \quad \lambda\frac{\partial f_2}{\partial z_1^2} = \mu_1, \quad \lambda\frac{\partial f_2}{\partial z_2^2} = \mu_2.$$

13.3.5 Two Inputs and Two Outputs II

After eliminating λ, the first-order conditions yield

$$\frac{\mu_1}{\mu_2} = \frac{\partial f_1/\partial z_1^1}{\partial f_1/\partial z_2^1} = \frac{\partial f_2/\partial z_1^2}{\partial f_2/\partial z_2^2}.$$

This says that the marginal rate of technical substitution between inputs 1 and 2 must be the same in both production processes.[9] We can also obtain the derivatives of T, using the Envelope Theorem to find the y_2, y_3, and y_4 derivatives. Thus $\partial T/\partial y_1 = 1$, $\partial T/\partial y_2 = \lambda$, $\partial T/\partial y_3 = \mu_1$, and $\partial T/\partial y_3 = \mu_2$. It follows that $dT \gg \mathbf{0}$. The Implicit Function Theorem can be used to show that $F \in \mathcal{C}^1$. Further, $F(0,0,0) = 0$, so $T(\mathbf{0}) = 0$ and T is a transformation function.

Here $1/\lambda$ is the marginal rate of transformation MRT_{12} and MRT_{34} is the common value $MRTS^i_{12} = \mu_1/\mu_2$. The other marginal rates of transformation can be expressed in terms of the marginal products: $MRT_{13} = 1/MP^1_1$, $MRT_{14} = 1/MP^1_2$, $MRT_{23} = 1/MP^2_1$, and $MRT_{24} = 1/MP^2_2$.

To see how this works in practice, let's specialize to the identical Cobb-Douglas technology case given by $f_\ell(\boldsymbol{z}) = (z_1)^\alpha (z_2)^{\beta\ell}$ with $\alpha + \beta \le 1$ and $\alpha, \beta \ge 0$. Then the first-order conditions become $z^1_2/z^1_1 = z^2_2/z^2_1$. Since $\mu_1, \mu_2 > 0$, all of each input must be used. That implies $z^\ell_1/z^\ell_2 = z_1/z_2$. Thus some fraction γ of the inputs are used to produce good 2, and the fraction $1-\gamma$ are used to produce good 1. It follows that $q_2 = \gamma^{\alpha+\beta} z_1^\alpha z_2^\beta$, so

$$\gamma = \left(\frac{q_2}{z_1^\alpha z_2^\beta}\right)^{1/(\alpha+\beta)}$$

and that the corresponding transformation function is $T(\mathbf{q}, -\boldsymbol{z}) = q_1 - (1-\gamma)^{\alpha+\beta} z_1^\alpha z_2^\beta$. Putting it all together.

$$T(\mathbf{q}, -\boldsymbol{z}) = \begin{cases} q_1 - \left[(z_1^\alpha z_2^\beta)^{\frac{1}{\alpha+\beta}} - q_2^{\frac{1}{\alpha+\beta}}\right]^{\alpha+\beta} & \text{when } \boldsymbol{z} \ge \mathbf{0} \text{ and } 0 \le q_2 \le z_1^\alpha z_2^\beta \\ q_1 - z_1^\alpha z_2^\beta & \text{when } \boldsymbol{z} \ge \mathbf{0} \text{ and } q_2 \le 0 \\ +\infty & \text{otherwise.} \end{cases}$$

In the CRS case where $\alpha + \beta = 1$, the first case reduces to $q_1 + q_2 - z_1^\alpha z_2^\beta$. The total output of both goods is at most $z_1^\alpha z_2^\beta$, and may be split between the goods any way we wish. In the CRS case $MRT_{12} = 1$ whenever $\mathbf{q} \ge \mathbf{0}$. ◄

[9] If we had many inputs and outputs, we would still find the marginal rates of technical substitution must be the same in all production processes that use the inputs in question.

13.4 Profit and Supply Functions

The firm's main problem is to maximize profits. The firm is presumed to be a price-taker, treating prices as parameters it has no control over.

The *profit function* $\pi(\mathbf{p})$ is defined by

$$\begin{aligned} \pi(\mathbf{p}) = \sup\ & \mathbf{p}\cdot\mathbf{y} \\ \text{s.t. } & \mathbf{y} \in Y. \end{aligned}$$

If the supremum is a maximum, we denote the set of maximizers by $\mathbf{y}(\mathbf{p})$. This is referred to as the *net output* or *supply correspondence* (or *supply function* if single-valued). The profit function is a conjugate function. Specifically, it is the **convex** conjugate of the **convex** indicator function of Y. That is, $\pi(\mathbf{p}) = (-\mathbb{I}_Y)_*(\mathbf{p})$.

13.4.1 Production with a Single Output

Example 13.4.1: We return to the model of Example 13.1.1. Suppose we have a production function $f\colon \mathbb{R}_+^{L-1} \to \mathbb{R}$ with the production set written as $Y = \{(-\boldsymbol{z}, q) : q \le f(\boldsymbol{z}), \boldsymbol{z} \in \mathbb{R}_+^{L-1}\}$. Letting $w_\ell = p_\ell$ for $\ell = 1, \dots, L-1$, we can write $\mathbf{p} = (\boldsymbol{w}, p_L)$. Profit from $(-\boldsymbol{z}, q)$ is then $p_L q - \boldsymbol{w}\cdot\boldsymbol{z}$. We maximize profit under the constraint that $q \le f(\boldsymbol{z})$. The Lagrangian is

$$\mathcal{L} = p_L q - \boldsymbol{w}\cdot\boldsymbol{z} + \lambda\,(f(\boldsymbol{z}) - q)$$

yielding first-order conditions $p_L = \lambda$ and $w_\ell = \lambda\,\partial f/\partial z_\ell$. Combining them we obtain the usual condition that the factor price (w_ℓ) must equal the value of the marginal product ($VMP_\ell = p_L\,\partial f/\partial z_\ell$) for each input.

We can get additional information by using Euler's Theorem. We start with the first-order condition $p_L\,\partial f/\partial z_\ell = w_\ell$. Multiply by z_ℓ and sum to obtain $p_L \sum_\ell z_\ell\,\partial f/\partial z_\ell = \sum_\ell w_\ell z_\ell$. Suppose that f is homogeneous of degree α. Euler's Theorem tells us that

$$\alpha f(\boldsymbol{z}) = \sum_\ell z_\ell \frac{\partial f}{\partial z_\ell}.$$

We multiply by p_L and use the fact that the value of the marginal product equals the factor price at the optimum ($p_L \partial f/\partial z_\ell = w_\ell$) to obtain

$$\alpha p_L f(\boldsymbol{z}) = \sum_\ell z_\ell \left(p_L \frac{\partial f}{\partial z_\ell}\right) = \boldsymbol{w}\cdot\boldsymbol{z}.$$

When f exhibits constant returns to scale, $\alpha = 1$ and the value of output is exactly equal to the cost of production.[10] The factor payments exhaust the revenue. If $\alpha < 1$, the firm will earn positive profits, and if $\alpha > 1$, the factor payments are larger than the value of output. In that case it is better to produce nothing.[11] ◀

[10] This was first noted by Wicksteed (1894), but his proof was incorrect. The first correct proof is that of Flux (1894).

[11] The first-order conditions typically lead to minimum profit or a saddlepoint in this case, a reason to always check the second-order conditions.

13.4.2 Profit Max with Cobb-Douglas Production

Example 13.4.2: Let

$$Y = \{(y_1, y_2, y_3) : y_1, y_2 \leq 0, y_3 \leq (-y_1)^{\alpha}(-y_2)^{\beta}\}$$

where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. We consider the firm's profit maximization problem. Clearly, $y_1 = 0$ or $y_2 = 0$ yield zero profit, so we will focus our attention on interior solutions. Production should be as large as possible (for given inputs) to maximize profit, thus $y_3 = (-y_1)^{\alpha}(-y_2)^{\beta}$. Profit can now be written $p_1 y_1 + p_2 y_2 + p_3(-y_1)^{\alpha}(-y_2)^{\beta}$. The first-order conditions are $p_1 = \alpha p_3(-y_1)^{\alpha-1}(-y_2)^{\beta}$ and $p_2 = \beta p_3(-y_1)^{\alpha}(-y_2)^{\beta-1}$. We can divide them to find $p_1/p_2 = (\beta/\alpha)(y_2/y_1)$. We then substitute back in the first equation to find $p_1 = p_3\alpha(\beta/\alpha)^{\beta}(p_1/p_2)^{\beta}(-y1)^{\alpha+\beta-1}$.

Since $\alpha + \beta < 1$, the second-order conditions indicate a maximum. The solution is:

$$\begin{aligned}
y_1 &= -\left[\left(\frac{\alpha^{\beta-1}}{\beta^{\beta}}\right)\left(\frac{p_1^{1-\beta}p_2^{\beta}}{p_3}\right)\right]^{1/(\alpha+\beta-1)} \\
y_2 &= -\left[\left(\frac{\beta^{\alpha-1}}{\alpha^{\alpha}}\right)\left(\frac{p_1^{\alpha}p_2^{1-\alpha}}{p_3}\right)\right]^{1/(\alpha+\beta-1)} \\
y_3 &= \left[\frac{1}{\alpha^{\alpha}\beta^{\beta}}\frac{p_1^{\alpha}p_2^{\beta}}{p_3^{\alpha+\beta}}\right]^{1/(\alpha+\beta-1)}.
\end{aligned}$$

The profit function is then

$$\pi(\mathbf{p}) = \mathbf{p}\cdot\mathbf{y}(\mathbf{p}) = (1-\alpha-\beta)\left[\left(\frac{1}{\alpha^{\alpha}\beta^{\beta}}\right)\left(\frac{p_1^{\alpha}p_2^{\beta}}{p_3}\right)\right]^{1/(\alpha+\beta-1)}.$$

◀

13.4.3 Profit and the Transformation Function

When we have a transformation function T, we can use it to solve the profit maximization problem, which can be written as

$$\begin{aligned} \pi(\mathbf{p}) = \sup\ & \mathbf{p}\cdot\mathbf{y} \\ \text{s.t. } & T(\mathbf{y}) \le 0. \end{aligned}$$

The Lagrangian is

$$\mathcal{L} = \mathbf{p}\cdot\mathbf{y} - \lambda T(\mathbf{y})$$

and the first-order conditions are

$$p_\ell = \lambda \frac{\partial T}{\partial y_\ell}.$$

We can divide to eliminate the multiplier, obtaining

$$\frac{p_k}{p_\ell} = \frac{\partial T/\partial y_k}{\partial T/\partial y_\ell} = MRT_{k\ell}\,.$$

The relative price must equal the marginal rate of transformation.

13.4.4 Prices Determine Which Good is Produced

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In some cases, the price vector will determine which goods are inputs and which are outputs. The following example using a transformation function shows this.

Example 13.4.3: Let the transformation function be given by

$$T(y_1, y_2) = \begin{cases} \dfrac{y_1}{1-y_1} + y_2 & \text{for } y_1 < 1 \\ +\infty & \text{for } y_1 \geq 1. \end{cases}$$

When $y_1 < 1$, $\partial T/\partial y_1 = (1-y_1)^{-2}$ and $\partial T/\partial y_2 = 1$. We can define $Y = \{(y_1, y_2) : T(y_1, y_2) \leq 0\}$. Because the transformation function T is increasing in both goods when $y_1 < 1$, The boundary of Y will be precisely those points where $T(\mathbf{y}) = 0$.

Profit maximization requires that the marginal rate of transformation equal the relative price, while the fact that $dT \gg \mathbf{0}$ requires that we be on the boundary of the production set, $T(\mathbf{y}) = 0$. Here $MRT_{12} = (1-y_1)^{-2}$, so $p_1/p_2 = (1-y_1)^{-2}$. This implies $y_1 = 1 - \sqrt{p_2/p_1}$ as the other solution $y_1 = 1 + \sqrt{p_2/p_1}$ has $y_1 > 1$. It follows that $y_2 = 1 - \sqrt{p_1/p_2}$ and the supply function is

$$\mathbf{y}(\mathbf{p}) = \begin{pmatrix} 1 - \sqrt{p_2/p_1} \\ 1 - \sqrt{p_1/p_2} \end{pmatrix}.$$

If $p_2 > p_1$, good 1 is the input and good 2 is the output while if $p_1 > p_2$, good 2 is the input and good 1 is the output. Which good is the input and which is the output depends on prices.

The profit function is $\pi(\mathbf{p}) = p_1 - 2\sqrt{p_1 p_2} + p_2 = \left(\sqrt{p_1} - \sqrt{p_2}\right)^2$. Profit is positive unless $p_1 = p_2$, in which case nothing is produced, $\mathbf{y} = (0, 0)$.

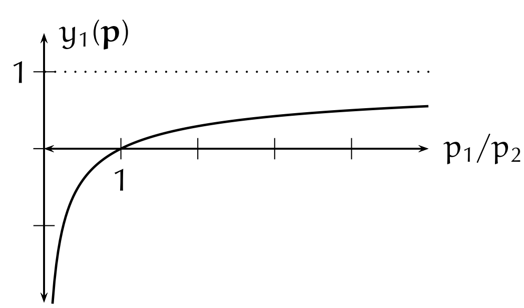

Figure 13.4.4: Net output of good one is shown as a function of the relative price p_1/p_2. It is demanded when $p_1 < p_2$ and supplied when $p_2 < p_1$. The supply is asymptotic to $y_1 = 1$.

◀

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13.4.5 Profit and Supply: Basic Properties

Given production set Y and price vector $\mathbf{p}$, the profit function is $\pi(\mathbf{p}) = \sup\{\mathbf{p}\cdot\boldsymbol{z} : \boldsymbol{z} \in Y\} = (-\mathbb{I}_Y)_*(\mathbf{p})$. Suppose $y(\mathbf{p})$ solves the profit maximization problem. The positive components of $y(\mathbf{p})$ are the output supply and the negative components are (unconditional) factor demand.

Now π is a support function, the conjugate of an indicator function. A number of properties immediately follow from the Support Function Theorem.

Profit Theorem. *Let Y be a production set. Then:*

(1) *The profit function $\pi(\mathbf{p})$ is convex, lower semicontinuous and homogeneous of degree one. In addition, π is continuous on* $\text{int}(\text{dom}\,\pi)$ *and weakly increasing in* $\mathbf{p}$.

(2) *If Y is closed and convex, $\pi_*(\mathbf{y}) = -\mathbb{I}_Y(\mathbf{y})$. Thus $Y = \{\mathbf{y} : \pi_*(\mathbf{y}) = 0\}$. Moreover,* $Y = \{\mathbf{y} : \mathbf{p}\cdot\mathbf{y} \le \pi(\mathbf{p})$ *for all* $\mathbf{p}\}$.

(3) *The net outputs obey $\mathbf{y}(\mathbf{p}) = \partial_*\pi(\mathbf{p})$ (Hotelling's Lemma). Moreover, if there is a unique maximizer, π is Gâteaux differentiable at $\mathbf{p}$ and its Gâteaux derivative is* $\mathbf{y}(\mathbf{p})$.

(4) *If $\mathbf{y}_i \in \mathbf{y}(\mathbf{p}_i)$ for $i = 0, 1$, then $(\mathbf{y}_1 - \mathbf{y}_0)\cdot(\mathbf{p}_1 - \mathbf{p}_0) \ge 0$ (Law of Supply).*

(5) *The net output $y(\mathbf{p})$ is homogeneous of degree zero.*

(6) *If $\pi(\mathbf{p})$ is $\mathcal{C}^2$, then $y(\mathbf{p})$ is differentiable and the matrix $d^2\pi = d\mathbf{y}(\mathbf{p})$ is symmetric and positive semi-definite. Moreover,* $\big[d\mathbf{y}(\mathbf{p})\big]\mathbf{p} = \big[d^2\pi(\mathbf{p})\big]\mathbf{p} = 0$.

(7) *If Y is convex then $\mathbf{y}(\mathbf{p})$ is a convex set. If Y is strictly convex, then $\mathbf{y}(\mathbf{p})$ is single-valued.*

Proof. As with the Expenditure and Cost Theorems, much of the Profit Theorem can be proven using traditional methods as in Chapter 4. We shortcut this by appealing to the Support Function Theorem. Properties (1)–(6) follow from the Support Function Theorem, slightly modified here because π is a convex conjugate of the convex indicator function $-\mathbb{I}_Y$, not a concave conjugate of a concave indicator function.

Property (7) is an easy consequence of the (strict) convexity of the production set and the linearity of the objective function $\mathbf{p}\cdot\mathbf{y}$. $\square$

13.4.6 About the Profit Theorem

Since we have used the supremum to define the profit function, we have sidestepped the issue of whether profits can be maximized. The profit function exists regardless of whether profits can actually be maximized. We will take up this question further in the next section.

Because π_* is the negative of an indicator function, it takes the values 0 and $+\infty$. That means we can write $Y = \{\mathbf{y} : \pi_*(\mathbf{y}) \le 0\}$. Although π_* looks superficially like a transformation function, it is not. One big problem is that it is not increasing on dom T. A symptom of this is the fact that $T^{-1}(0) \neq \partial Y$. Rather $T^{-1}(0) = Y$.

Property (3) is sometimes referred to as *Hotelling's Lemma*. The function π is convex and lower semicontinuous rather and concave and upper semicontinuous because it is a **convex** conjugate. This also flips the sign in condition (4), the Law of Supply.[12]

Since both supply and factor demand are included in the net output vector, condition (4) combines the Law of Supply and Law of Demand in one package. Suppose only the price of good ℓ changes. Then $(p'_\ell - p_\ell)(y'_\ell - y_\ell) \ge 0$. If $p'_\ell > p_\ell$, we find $y'_\ell \ge y_\ell$. If good ℓ is being supplied, this indicates quantity supplied has (weakly) increased. If ℓ is a demanded factor, $y_\ell < 0$. In that case, either y'_ℓ is less negative (quantity demanded decreases), or $y'_\ell > 0$, in which case a demanded good has changed to a supplied good.

[12] Hotelling seems to have been the first to use of duality in economics (Hotelling, 1932).

13.5 Efficient Production

Profit-maximizing vectors have an important welfare property. They induce firms to utilize resources efficiently. For an individual firm, efficiency means that there is is no way for the firm to reallocate existing resources so as to increase production of one good without reducing production of another. It follows that overall production cannot be increased without using more of some resource. This describes production efficiency for a firm.

As for the entire economy, we allow reallocation of resources and production between firms, but still require that it be impossible to increase aggregate net output. Efficiency requires that production of one good cannot be increased without either decreasing production of another good or increasing utilization of some input.

The aggregation results of section 14.4 will allow us to treat the productive portion of the economy as a single firm, and we will find that profit maximizing net outputs at the aggregate level are always the aggregate of profit maximizing nets outputs for each firm.

Production Efficiency. We say $\mathfrak{y} \in Y$ is *efficient* if there is no $\mathfrak{y}' \in Y$ with $\mathfrak{y}' > \mathfrak{y}$.

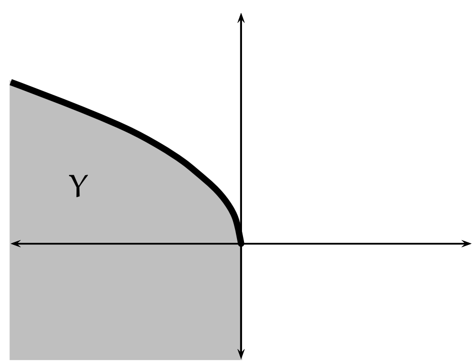

Figure 13.5.1: The heavy curve illustrates the set of efficient net output vectors for the production set Y. The negative portion of the vertical axis is not efficient because such points are dominated by $\mathbf{0}$.

13.5.1 Necessary Conditions for Efficiency

Of course, efficient production vectors must be in the boundary of the production set. Figure 13.5.1 shows that the converse is not true. When production is described by a transformation function T, efficient net outputs necessarily obey $T(\mathbf{y}) = 0$, even though this condidition is not always sufficient for efficiency.

Proposition 13.5.2. *If Y is a production set and $\mathbf{y} \in Y$ is efficient, then $\mathbf{y} \in \partial Y$.*

Proof. If $\mathbf{y}$ is efficient, $\mathbf{y}^n = \mathbf{y} + \frac{1}{n}\boldsymbol{e} \notin Y$. Taking the limit, we find that $\lim \mathbf{y}^n = \mathbf{y} \in \overline{Y^c}$. Since $\mathbf{y} \in Y = \bar{Y}$, $\mathbf{y} \in \partial Y$. $\square$

13.5.2 Profit Maximization Implies Efficiency

When prices are strictly positive, any profit-maximizing vector must be efficient.

Theorem 13.5.3. *If $\mathbf{p} \gg 0$ and $\mathbf{y}^*$ maximizes profit over Y, then $\mathbf{y}^*$ is efficient.*

Proof. Suppose $\mathbf{y}^*$ is not efficient. Then there is $\mathbf{y}' > \mathbf{y}^*$ with $\mathbf{y}' \in Y$. Since $\mathbf{p} \gg 0$, $\mathbf{p}\cdot\mathbf{y}' > \mathbf{p}\cdot\mathbf{y}^*$ contradicting the fact that $\mathbf{y}^*$ maximizes profit. This contradiction shows that $\mathbf{y}^*$ is efficient $\square$

13.5.3 Efficiency Prices

A type of converse is also true. When production is convex, any efficient vector maximizes profits for some non-negative prices. The heavy lifting in the proof is done by a separation theorem.

Theorem 13.5.4. *Suppose Y is non-empty, closed and convex and that $\mathbf{y}^* \in Y$ is efficient. There is a $\mathbf{p} > 0$ with $\mathbf{p}\cdot\mathbf{y} \le \mathbf{p}\cdot\mathbf{y}^*$ for all $\mathbf{y} \in Y$.*

Proof. Consider $A = \{\mathbf{y} : \mathbf{y} > \mathbf{y}^*\}$. Then A and Y are disjoint, non-empty convex sets. Moreover, A is anti-comprehensive. Corollary 7.2.6 to Separation Theorem B yields a $\mathbf{p} > 0$ and α with $\mathbf{p}\cdot\mathbf{y} \le \alpha$ for all $\mathbf{y} \in Y$ and $\mathbf{p}\cdot\mathbf{a} \ge \alpha$ for all $\mathbf{a} \in A$.

Consider $\mathbf{y}^* + \frac{1}{n}\boldsymbol{e} \in A$. We know $\mathbf{p}\cdot\mathbf{y}^* + \frac{1}{n}\mathbf{p}\cdot\boldsymbol{e} \ge \alpha$. Letting $n \to \infty$, we find $\mathbf{p}\cdot\mathbf{y}^* \ge \alpha$. But since $\mathbf{y}^* \in Y$, $\mathbf{p}\cdot\mathbf{y}^* \le \alpha$. We conclude $\alpha = \mathbf{p}\cdot\mathbf{y}^*$, establishing profit maximization. $\square$

13.5.4 Efficiency without Supporting Prices

If the production set is not convex, there may be efficient points that are not profit maximizing.

Example 13.5.5: Suppose we have production function $f(z) = z^2$. Let $Y = \{(-z, q) : q \leq z^2, z \geq 0\}$. The net output $y = (-1, 1)$ is efficient due to the fact that either reducing input or increasing output moves us out of the production set.

However, the net output $y = (-1, 1)$ is never profit-maximizing. By free disposal, prices that allow profit maximization must be non-negative. Indeed, the output price must be positive if the firm is going to produce anything. Profit is then $\pi(z) = pz^2 - wz$ where $p > 0$ is the output price and $w \geq 0$ is the input price. The first-order condition is $2pz = w$, so $z = w/2p$. Here $z = 1$, requiring $w = 2p$. But $\pi'' = 2p > 0$, so $y = (-1, 1)$ **minimizes** profit rather than maximizing it. Moreover, profit is $(w, p)\cdot(-1, 1) = -p < 0$. The firm would actually do better by not producing at all. There are no prices where $y = (-1, 1)$ maximizes profit. ◄

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