14. Firm and Market Supply

Outline

1. Profit and Prices with CRS Production
2. Profit Maximization with Decreasing Returns
3. Homogenization of Production
4. Aggregate Production Sets
5. Market Supply

There are still several outstanding issues concerning supply. Although we defined the profit function, we have not considered whether profit can be maximized—whether supply exists. In fact, we don't know if profit is finite! After all, the value $+\infty$ is allowed. Finally, even if supply is defined for an individual firm, what properties will market supply have? Can we think of it as the supply of a representative firm? Or will new possibilities emerge, as happened with demand?

These issues hinge on the properties of the production set. To answer them, we again turn to duality. We construct two types of dual set. One, the polar cone, only applies to constant returns to scale production. The other, the polar set, applies to any convex production set and coincides with the polar cone under constant returns to scale. Both play a key role in addressing these issues.

Section one considers whether it is possible to maximize profit and shows that the answer depends on the price vector. For any production set, only a subset of possible prices allows profit maximization. Under constant returns to scale, the set of possible prices is the polar cone.

In section two we allow general convex production sets. We investigate the strictly weaker question of whether the profit function is even finite. Again it depends on prices, and the polar set can be used to characterize the prices where profit is finite. We also show by example that there are cases where profit is finite, but profit maximization is impossible.

Section three shows how any convex production technology can be converted to a constant returns to scale technology by adding an additional factor of production, McKenzie's "entrepreneurial factor". This allows us to use constant returns production without loss of generality, and ensures that profit can be maximized whenever it is finite. The price of the entrepreneurial factor is the profit of the original firm. We also examine the cases where profit could not be maximized in the original problem.

Section four turns to aggregate production. The fundamental question are whether the sum of the production sets is itself a production set. This can be handled in a couple of ways, using either irreducibility of aggregate production or semi-independence of asymptotic cones.

Finally, in section five, we examine market supply and ask whether there is a common price where all firms can maximize profit. The polar plays a key role in answering this.

14.1 Can Profit be Maximized?

One of our first priorities after introducing the utility maximization and expenditure and cost minimization problems was to establish that they had solutions, that the various demand correspondences existed. We didn't do that for profit maximization. There's a reason for that—profit cannot always be maximized. The production set is typically not bounded, raising the possibility that profit can always be increased by increasing output. This can occur even if profit is bounded.

This difficulty in maximizing profit will lead to restrictions on prices. For some prices, profit can be maximized, for others, it cannot. For example, it is easy to see that free disposal and inaction imply that profit cannot be maximized if any price is negative. If $p_\ell < 0$, free disposal and inaction imply that $-n\mathbf{e}^\ell$ is feasible, and $\mathbf{p}\cdot(-n\mathbf{e}^\ell) = n|p_\ell| \to +\infty$. This immediately restricts our attention to the positive orthant. Production may put further restrictions on prices.

Negative prices may be allowed when a production technology does not admit free disposal, as occurs with waste disposal technologies. Indeed, trash haulers charge to take stuff away rather than paying for it.

14.1.1 Profit Maximization: CRS Technology I

The relation between price and profit maximizability is especially sharp when using a constant returns technology, as the following example shows.

Example 14.1.1: Consider the production set given by $Y = \{(y_1, y_2) : y_1 \le 0, 3y_2 \le -y_1\}$. Due to free disposal, profit cannot be maximized it either price is negative, so we confine our attention to the case where $\mathbf{p} \ge \mathbf{0}$.

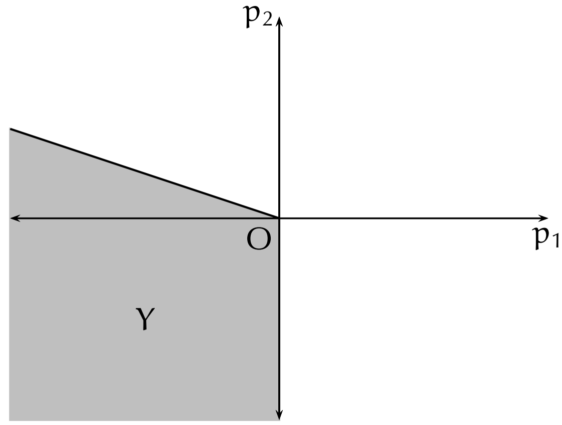

Figure 14.1.2A: The production set $Y = \{(y_1, y_2) : y_1 \le 0, 3y_2 \le -y_1\}$.

Profit is $\mathbf{p}\cdot\mathbf{y} = p_1y_1 + p_2y_2 \le p_1y_1 - p_2y_1/3$. Because $p_2 \ge 0$, we must set $y_2 = -y_1/3$ to maximize profit. Then profit is $p_1y_1 - p_2y_1/3 = (p_1 - p_2/3)y_1$.

There are three cases to consider. If $p_1 > p_2/3 \ge 0$, profit will be negative for $y_1 < 0$. In that case profit is maximized when both $y_1 = 0$ and $y_2 = -y_1/3 = 0$. If we also have $p_2 = 0$, any $y_2 \le 0$ is allowed. If $0 \le p_1 < p_2/3$, maximum profit will grow without bound as y_1 becomes more negative. Here profit cannot be maximized. Finally, if $p_1 = p_2/3 \ge 0$, the maximum profit is always zero and occurs anywhere that $y_1 \le 0$ and $y_2 = -y_1/3$, except in the case $\mathbf{p} = \mathbf{0}$, when any $\mathbf{y} \in Y$ maximizes profit.

14.1.2 Profit Maximization: CRS Technology II 2/10/22

In other words, profit maximization is only possible when $\mathbf{p} \geq \mathbf{0}$ and $p_2/3 \leq p_1$. Let $\Pi = \{\mathbf{p} \in \mathbb{R}^2 : 0 \leq p_1 \leq p_2/3\}$ is the set of prices where profit can be maximized.

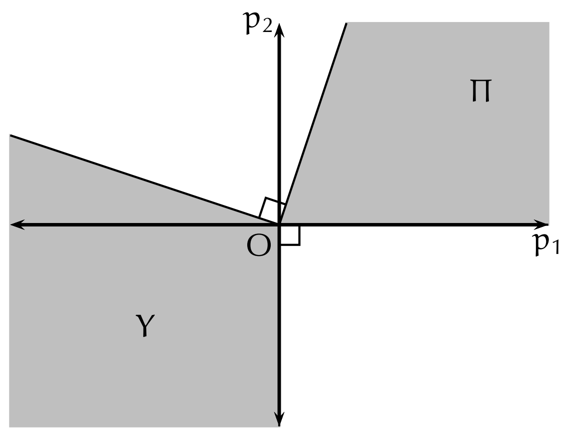

Figure 14.1.2B: Here $\Pi = \{\mathbf{p} \in \mathbb{R}^2 : 0 \leq p_1 \leq p_2/3\}$ is the set of prices where profit can be maximized. Note how the sides of P are perpendicular to the corresponding sides of Y.

The profit function is

$$\pi(\mathbf{p}) = \begin{cases} 0 & \text{if } p_1 \geq p_2/3 \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

and the net output (supply) correspondence is given by

$$\mathbf{y}(\mathbf{p}) = \begin{cases} \mathbf{0} & \text{if } p_1 > p_2/3 > 0 \\ \{\mathbf{y} : y_2 = -y_1/3\} & \text{if } p_1 = p_2/3 > 0 \\ \{\mathbf{y} : \mathbf{y} = (0, y_2), y_2 \leq 0\} & \text{if } p_1 > 0 \text{ and } p_2 = 0 \\ Y & \text{if } \mathbf{p} = \mathbf{0} \\ \text{undefined} & \text{if } p_1 < p_2/3 \text{ or } p_2 < 0. \end{cases}$$

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14.1.3 When Can CRS Profit be Maximized?

Example 14.1.1 shows how profit maximization is only possible for certain price vectors. Profit can only be maximized when $p_1 \geq p_2/3$ and $\mathbf{p} \geq \mathbf{0}$. When either $p_1 < p_2/3$ profits can be increased without bound by increasing input (and so output), while if $\mathbf{p} \ngeq \mathbf{0}$, profits can be increased without bound by increasing inputs (without output).

This is the way profit maximization works under constant returns to scale. The profit function becomes infinite whenever positive profit is possible. There is no maximum. If profit is non-positive, the maximum profit is zero and will be realized at $\mathbf{y} = \mathbf{0}$, and possibly elsewhere. Moreover, Example 14.1.1 has a unique price ray where supply is non-trivial. The mere act of production determines the relative prices.

These features are shared by **all** constant returns to scale technologies.

Proposition 14.1.3. *If a non-empty set Y obeys constant returns to scale, then either $\pi(\mathbf{p}) = 0$ or $\pi(\mathbf{p}) = +\infty$. Moreover, if Y also obeys inaction, the supply correspondence is a non-empty cone whenever $\pi(\mathbf{p}) = 0$.*

Proof. If there is a $\mathbf{y} \in Y$ with $\mathbf{p}\cdot\mathbf{y} > 0$, consider $n\mathbf{y} \in Y$ for $n = 1, 2, \ldots.$ Then $\mathbf{p}\cdot(n\mathbf{y}) = n\mathbf{p}\cdot\mathbf{y} \to +\infty$, so $\pi(\mathbf{p}) = +\infty$.

Alternatively, if $\mathbf{p}\cdot\mathbf{y} \leq 0$ for all $\mathbf{y} \in Y$, $\pi(\mathbf{p}) \leq 0$, consider any $\mathbf{y}/n \in Y$ for $n = 1, 2, \ldots.$ Then $\mathbf{p}\cdot(\mathbf{y}/n) \leq 0$. Letting $n \to \infty$, we find that $\pi(\mathbf{p}) = \sup_Y \mathbf{p}\cdot\mathbf{y} = 0$.

Under inaction, $\mathbf{0} \in Y$, so $\mathbf{0} \in \mathbf{y}(\mathbf{p})$ when $\pi(\mathbf{p}) = 0$. Further, if $\mathbf{y} \in \mathbf{y}(\mathbf{p})$ and $t > 0$, $t\mathbf{y} \in Y$ and $\mathbf{p}\cdot(t\mathbf{y}) = t(\mathbf{p}\cdot\mathbf{y}) = 0$, showing that $t\mathbf{y} \in \mathbf{y}(\mathbf{p})$. □

Proposition 14.1.3 implies that π is a convex indicator function, taking the values 0 and $+\infty$. For the moment, we denote the set it indicates by $\Pi = \{\mathbf{p} : \pi(\mathbf{p}) = 0\}$. If the price vector is not in Π, profit cannot be maximized and if $\mathbf{p} \in \Pi$, profit can be maximized.

As Example 14.1.1 made clear, $\mathbf{y}(\mathbf{p})$ may not exist for some values of $\mathbf{p}$. For profit maximization to be possible, profit must be bounded on Y. Under constant returns, bounded profit implies profit is zero and that the set of corresponding profit maximizers, $\mathbf{y}(\mathbf{p})$, is a cone. If profit is not bounded, $\pi(\mathbf{p}) = +\infty$ and profit cannot be maximized.

14.1.4 CRS Supply Correspondence

In general, we will approach profit maximization by focusing on the necessary condition that profit $\pi(\mathbf{p})$ be finite. In the constant returns case, this necessary condition is also sufficient by Proposition 14.1.3.

Recall that a correspondence is a set-valued function whose values are non-empty sets. More precisely, a correspondence from A to B is a function from A to the set of non-empty subsets of B. If there are prices $\mathbf{p}$ where profit has more than one maximizer, supply is necessarily a correspondence.

Moreover, it will even allow us to prove a type of continuity for $\mathbf{y}(\mathbf{p})$, that $\mathbf{y}(\mathbf{p})$ is a closed correspondence. That is, whenever $\mathbf{p}^n \to \mathbf{p}$ and $\mathbf{y}^n \in \mathbf{y}(\mathbf{p}^n)$ with $\mathbf{y}^n \to \mathbf{y}$, then $\mathbf{y} \in \mathbf{y}(\mathbf{p})$. In other words, the correspondence has a closed graph.[1]

Proposition 14.1.4. *Suppose Y is a constant returns production set. Then profit can be maximized at $\mathbf{p}$ if and only if $\pi(\mathbf{p}) = 0$. Let $\Pi = \{\mathbf{p} : \pi(\mathbf{p}) = 0\}$. Then the net output correspondence $\mathbf{y}(\mathbf{p}) = \{\mathbf{y} \in Y : \mathbf{p}\cdot\mathbf{y} = 0\}$ is a closed correspondence from Π into Y.*

Proof. The first portion of the proposition is from Proposition 14.1.3 above. For the second, suppose $\mathbf{y}^n \in \mathbf{y}(\mathbf{p}^n)$ with $\mathbf{p}^n \to \mathbf{p} \in \Pi$ and $\mathbf{y}^n \to \mathbf{y}$. Since the inner product is continuous, $\mathbf{p}^n\cdot\mathbf{y}^n \to \mathbf{p}\cdot\mathbf{y}$. Now $\mathbf{p}^n\cdot\mathbf{y}^n = 0$, so $\mathbf{p}\cdot\mathbf{y} = 0$. Since $\mathbf{p} \in \Pi$, $\pi(\mathbf{p}) = 0$. Thus $\mathbf{y}$ maximizes profit at price $\mathbf{p}$, $\mathbf{y} \in \mathbf{y}(\mathbf{p})$. $\square$

[1] See section 33.2 for more on closed correspondences.

14.1.5 The Polar Cone

For CRS production sets, we have determined exactly when profit maximization is possible—it is possible if and only if $\pi(\mathbf{p}) = 0$. Our next task is to learn more about the set Π of price vectors which permit profit maximization.

To understand the set Π better, we introduce the *polar cone of* Y, denoted Y°.[2]

Polar Cone. If Y is a cone in $\mathbb{R}^L$, the *polar cone of* Y is defined by

$$Y^\circ = \{\mathbf{p} \in \mathbb{R}^L : \mathbf{p}\cdot\mathbf{y} \le 0 \text{ for all } \mathbf{y} \in Y\}.$$

Now $\Pi = Y^\circ$ whenever Y is a constant returns production set because maximum profit is either zero or infinity. This means that for a CRS technology Y, the polar cone Y° consists of all price vectors where profit maximization is possible, where maximum profits are zero.

[2] The polar cone is sometimes called the dual cone, but it is actually the negative of the dual cone, which is defined as $\{\mathbf{p} : \mathbf{p}\cdot\mathbf{y} \ge 0 \text{ for all } \mathbf{y} \in Y\}$.

14.1.6 Some Basic Properties of Polar Cones

Next, we establish some basic properties of Y°, including the fact that the polar cone is a cone.

Proposition 14.1.5. *If $Y \subset \mathbb{R}^L$ is a cone, its polar cone Y° is a closed convex cone containing $\mathbf{0}$. Moreover, if Y is non-empty and obeys free disposal, $Y^\circ \subset \mathbb{R}^L_+$.*

Proof. Suppose $t > 0$ and $\mathbf{p} \in Y^\circ$. Then $\mathbf{p}\cdot\mathbf{y} \leq 0$ for all $\mathbf{y} \in Y$. Multiplying by t, we find $(t\mathbf{p})\cdot\mathbf{y} \leq 0$ for all $\mathbf{y} \in Y$, so $t\mathbf{p} \in Y^\circ$. In other words, Y° is a cone.

Now suppose $\mathbf{p}^n \in Y^\circ$ for every $n = 1, 2, \ldots$ and that $\mathbf{p}^n \to \mathbf{p}$. Let $\mathbf{y} \in Y$ be arbitrary. Since each $\mathbf{p}^n \in Y^\circ$, $\mathbf{p}^n\cdot\mathbf{y} \leq 0$ for every n. Taking the limit, $\mathbf{p}\cdot\mathbf{y} \leq 0$. Since $\mathbf{y}$ was any element of Y, $\mathbf{p} \in Y^\circ$, showing that Y° is closed.

Let $\mathbf{p}, \mathbf{p}' \in Y^\circ$ and $0 \leq \alpha \leq 1$. Denote the convex combination $\alpha\mathbf{p} + (1-\alpha)\mathbf{p}'$ by $\mathbf{p}''$. By convexity of π, $\pi(\mathbf{p}'') \leq \alpha\pi(\mathbf{p}) + (1-\alpha)\pi(\mathbf{p}') = 0$. Since $\mathbf{0} \in Y$, $\pi(\mathbf{p}'') \geq 0$, establishing that $\pi(\mathbf{p}'') = 0$. It follows that $\mathbf{p}'' \in Y^\circ$, so Y° is convex.

Profit $\mathbf{p}\cdot\mathbf{y}$ is always zero when $\mathbf{p} = \mathbf{0}$, showing that $\mathbf{0} \in Y^\circ$.

Now suppose Y obeys free disposal and let $\mathbf{y} \in Y$. Suppose $\mathbf{p} \notin \mathbb{R}^L_+$ and take k with $p_k < 0$. Define $\mathbf{y}_n = \mathbf{y} - n\mathbf{e}^k$ for all positive integers n. By free disposal, $\mathbf{y}_n \in Y$. Now $\mathbf{p}\cdot\mathbf{y}_n = \mathbf{p}\cdot\mathbf{y} - np_k$. Since $p_k < 0$, $\lim_n \mathbf{p}\cdot\mathbf{y}_n = +\infty$. Thus $\pi(\mathbf{p}) = +\infty$ and so $\mathbf{p} \notin Y^\circ$. □

We can now restate Proposition 14.1.4 in terms of the polar cone. When the production set Y obeys constant returns to scale, profit can be maximized if and only the price vector is in the polar cone Y.

Theorem 14.1.6. *Suppose Y is a constant returns production set. Then profit can be maximized at price vector $\mathbf{p}$ if and only if $\mathbf{p} \in Y^\circ$. The net output correspondence $\mathbf{y}(\mathbf{p}) = \{\mathbf{y} \in Y : \mathbf{p}\cdot\mathbf{y} = 0\}$ is a convex-valued and closed correspondence from Y° to Y.*

14.1.7 The Polar's Polar Cone

One useful fact about the polar cone is that we can recover the production set from the polar cone.

Part (2) of the Profit Theorem tells us that the profit function is the support function: $\pi(\mathbf{p}) = (-\mathbb{I}_Y)_*$, obtained by taking the convex conjugate of the convex indicator of Y. By Propositions 14.1.3 and 14.1.4, $\pi(\mathbf{p})$ is also the convex indicator of the polar cone Y°, $\pi(\mathbf{p}) = (-\mathbb{I}_{Y^\circ})$.

Putting the two together tells us that $\pi_*(\mathbf{y}) = \big((-\mathbb{I}_Y)_*\big)_*$. Since the double convex dual is the convex indicator of the closed convex hull of Y, $\overline{\text{co}}\,Y$, we have $\pi_*(\mathbf{y}) = -(\mathbb{I}_Y)_{**} = -\mathbb{I}_{\overline{\text{co}}\,Y}$. If the production set Y is closed and convex, $\overline{\text{co}}\,Y = Y$, implying that $\pi_*(\mathbf{y}) = -\mathbb{I}_Y(\mathbf{y})$. This means that we can recover the original production set from the polar cone itself (as $Y = Y^{\circ\circ}$) as well as from the profit function via $\pi_* = -\mathbb{I}_Y$. This yields the following proposition.

Proposition 14.1.7. *Suppose Y is a non-empty, closed, convex cone. Then* $Y = Y^{\circ\circ}$.

Proof. A proof via the profit function was given above. Here is a second proof using Separation Theorem A.

Suppose $\mathbf{y} \in Y$. Then for every $\mathbf{p} \in Y^\circ$, $\mathbf{p}\cdot\mathbf{y} \le 0$. By the definition of $Y^{\circ\circ}$, this shows $\mathbf{y} \in Y^{\circ\circ}$. Thus $Y \subset Y^{\circ\circ}$.

Now suppose there is a $\hat{\mathbf{y}} \in Y^{\circ\circ}$ that is not in Y. Since Y is closed and convex, Separation Theorem A yields a $\mathbf{p} \ne \mathbf{0}$ and $\alpha \in \mathbb{R}$ with $\mathbf{p}\cdot\mathbf{y} < \alpha < \mathbf{p}\cdot\hat{\mathbf{y}}$ for all $\mathbf{y} \in Y$. Let $t > 0$ be arbitrary. Since Y is a cone, $t(\mathbf{p}\cdot\mathbf{y}) < \alpha$ for all $t > 0$. Dividing by t and letting $t \to \infty$, we find $\mathbf{p}\cdot\mathbf{y} \le 0$. It follows that $\mathbf{p} \in Y^\circ$. Now $\mathbf{0} \in Y$, so $\alpha > 0$. But then there is $\mathbf{p} \in Y^\circ$ with $\mathbf{p}\cdot\hat{\mathbf{y}} > \alpha > 0$. This shows that $\hat{\mathbf{y}} \notin Y^{\circ\circ}$, contradicting the supposition that $\hat{\mathbf{y}} \in Y^{\circ\circ}$. It follows that $Y^{\circ\circ} \subset Y$, and so $Y = Y^{\circ\circ}$. $\square$

14.1.8 Polar Cone with Linear Technology

When $Y \subset \mathbb{R}^2$ obeys constant returns to scale, we can find the polar cone by using the perpendiculars to the sides of Y.

Example 14.1.8: Consider the production set $Y = \{\mathbf{y} \in \mathbb{R}^2 : 2y_1 + y_2 \leq 0$ and $y_1 + 2y_2 \leq 0\}$. Both $(1, -2)$ and $(-2, 1)$ are in Y. It follows that if $\mathbf{p} \in Y^\circ$, $\mathbf{p}\cdot(1, -2) = p_1 - 2p_2 \leq 0$ and $\mathbf{p}\cdot(-2, 1) = -2p_1 + p_2 \leq 0$. These may be combined to find $p_2/2 \leq p_1 \leq 2p_2$. Free disposal implies $\mathbf{p} \geq \mathbf{0}$. Thus $Y^\circ \subset \{\mathbf{p} \in \mathbb{R}^2_+ : p_2/2 \leq p_1 \leq 2p_2\}$.

Conversely, suppose $\mathbf{p} \geq \mathbf{0}$ with $p_2/2 \leq p_1 \leq 2p_2$. It follows that $-2p_2 \leq -p_1$ and $-2p_1 \leq -p_2$. Take any $\mathbf{y} \in Y$. Then $y_1 \leq -2y_2$, so $p_1 y_1 \leq -2p_1 y_2 \leq -p_2 y_2$, and so $\mathbf{p}\cdot\mathbf{y} \leq 0$. It follows that $Y^\circ \supset \{\mathbf{p} \in \mathbb{R}^2_+ : p_2/2 \leq p_1 \leq 2p_2\}$. We then conclude $Y^\circ = \{\mathbf{p} \in \mathbb{R}^2_+ : p_2/2 \leq p_1 \leq 2p_2\}$. This is illustrated in Figure 14.1.9.

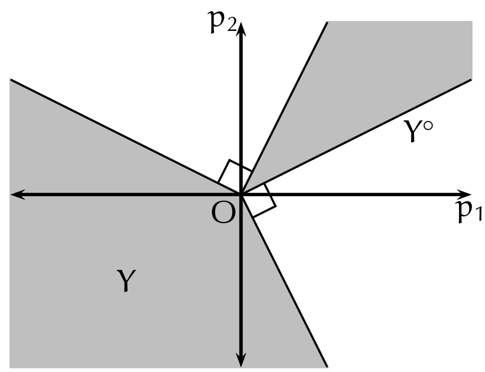

Figure 14.1.9: Note how the sides of the polar of Y are perpendicular to the corresponding sides of Y.

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14.1.9 Polar Cone: Linear Activity Model

The polar cone for a linear activity analysis model is also easily calculated.

Proposition 14.1.10. *If* $Y = \{\mathbf{y} : \mathbf{y} \le \sum_k z_k \mathbf{a}^k\}$*, the polar cone consists of* $\mathbf{p} \in \mathbb{R}^L_+$ *with* $\mathbf{p}\cdot\mathbf{a}^k \le 0$ *for all* $k = 1, \dots, K$.

Proof. First suppose $\mathbf{p} \in Y^\circ$. For all k, $\mathbf{a}^k \in Y$, so $\mathbf{p}\cdot\mathbf{a}^k \le 0$ for all $k = 1, \dots, K$. Free disposal then shows that we also have $\mathbf{p} \ge \mathbf{0}$.

Now suppose $\mathbf{p}$ satisfies $\mathbf{p}\cdot\mathbf{a}^k \le 0$ for all k and $\mathbf{y} \in \mathbb{R}^L_+$. Let $\mathbf{y}$ be an arbitrary vector in Y. Then there is $\mathbf{z} \in \mathbb{R}^K_+$ with $\mathbf{y} \le \sum_k z_k \mathbf{a}^k$. Since $\mathbf{p} \ge 0$, $\mathbf{p}\cdot\mathbf{y} \le \sum_k z_k \mathbf{p}\cdot\mathbf{a}^k \le 0$. As $\mathbf{y}$ was arbitrary, this implies $\mathbf{p} \in Y^\circ$, completing the proof. □

The following example shows how this works in practice.

Example 14.1.11: Consider the cone generated by $\mathbf{a}^1 = (-1, 2)$ and $\mathbf{a}^2 = (1, -3)$. Here $Y = \{\mathbf{y} : \mathbf{y} \le z_1(-1, 2) + z_2(1, -3) \text{ for some } z_1, z_2 \ge 0\}$. We know $Y^\circ \subset \mathbb{R}^2_+$, so $\mathbf{p}\cdot\mathbf{y} \le z_1(-p_1 + 2p_2) + z_2(p_1 - 3p_2)$. The polar cone consists of those non-negative vectors obeying $z_1(-p_1 + 2p_2) + z_2(p_1 - 3p_2) \le 0$ for all non-negative z_1 and z_2. In particular, the inequality must hold for $z_1 = 0$, implying $p_1 \le 3p_2$ and for $z_2 = 0$ which implies $2p_2 \le p_1$. It is also clear that any vector $\mathbf{p}$ that obeys those two inequalities is in the polar cone. Thus $Y^\circ = \{\mathbf{p} \in \mathbb{R}^2_+ : p_1 \le 3p_2 \text{ and } 2p_2 \le p_1\}$. ◄

To reiterate, when Y is a constant returns production set, the only prices where profit maximization is possible are those in the polar cone Y°. If $\mathbf{p} \notin Y^\circ$, there is some $\mathbf{y} \in Y$ with $\mathbf{p}\cdot\mathbf{y} > 0$, which means $\pi(\mathbf{p}) = +\infty$. However, if $\mathbf{p} \in Y^\circ$, we are always guaranteed that $\mathbf{0} \in \mathbf{y}(\mathbf{p})$. Note that $\mathbf{y}(\mathbf{p})$ is also a cone in this case. This also allows us to write $Y^\circ = \{\mathbf{p} : \pi(\mathbf{p}) = 0\}$.

When Y is also convex, we can recover the production set Y from its dual cone Y° by taking the dual of Y°, which we denote $Y^{\circ\circ}$.

For constant returns to scale technologies we are done. We have completely characterized the set of prices where profit maximization is possible. Unfortunately, we can't say that for other types of technology.

14.2 Profit Maximization with Decreasing Returns

If Y is convex, but does not enjoy constant returns to scale, a finite value of the profit function is not sufficient to guarantee that profit can be maximized. The problem that can arise is when the production set is asymptotic to a straight line and prices are perpendicular to that line, as in the following example.

Example 14.2.1: We return to the slowly decreasing returns production function of Example 13.2.11

$$f(z) = 1 + z - \frac{1}{1+z}.$$

Recall that the function $f \in \mathcal{C}^\infty$ has the following properties: $f(0) = 0$,

$$f' = 1 + \frac{1}{(1+z)^2} > 0, \text{ and } f'' = \frac{-2}{(1+z)^3} < 0.$$

We have an increasing concave production function which defines the production set.

The associated production set is $Y = \{(q, -z) : q \le f(z), z \ge 0\}$. Given price vector $(p, p) \gg \mathbf{0}$, profit at $\mathbf{y} = (q, -z)$ is $pq - pz$. At the maximum, we must have $q = f(z)$ and profit from input z becomes

$$g(z) = p\left[1 + z - \frac{1}{1+z}\right] - pz = p - \frac{p}{1+z} \le p.$$

Profit is bounded by p. The supremum, which is also the limit as $z \to \infty$, is the profit function $\pi(\mathbf{p}) = p$. The derivative of profit is $g'(z) = p/(1+z)^2 > 0$, so profit continues to increase as z increases. It cannot be maximized even though it is bounded above by p.

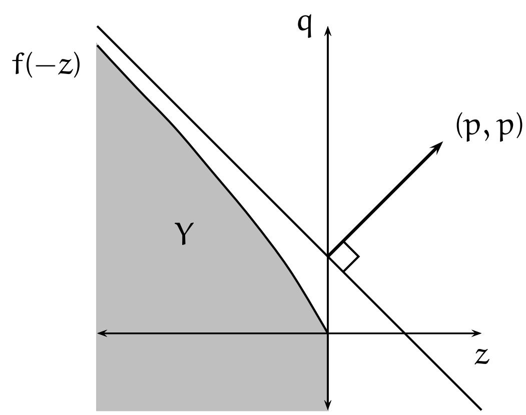

Figure 14.2.2: The production function is asymptotic to an isoprofit line which is perpendicular to $\mathbf{p} = (p, p)$. I've offset the vector $\mathbf{p}$ from the origin to $(0, 1)$ to make this clearer. with slope -1. Steeper isoprofit lines will allow profit maximization, with positive profit, while with flatter isoprofits, profit maximization will be impossible.

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14.2.1 When is Profit Finite?

So where does this leave the existence problem? For constant returns to scale technologies we have a complete solution. It is possible to maximize profit if and only if $\pi(\mathbf{p})$ is finite (which really means $\pi(\mathbf{p}) = 0$). When the production set does not obey constant returns to scale, things are not so simple. There simply is not a useful necessary and sufficient condition for profit maximization.

We already encountered the basic problem in Chapter 7. Not all hyperplanes that support a closed set actually intersect the set. Some support only asymptotically. Here the set in question is the epigraph of a convex function. You can maximize profit at price $\mathbf{p}$ if and only if π has a subgradient at $\mathbf{p}$. In that case, the subgradient is the solution and the supporting hyperplane intersects the epigraph.

Unfortunately, that is not a useful criterion. Finding whether there is a subgradient is all too often equivalent to solving the maximization problem itself. Weaker criteria are not both necessary and sufficient. It is necessary that $\pi(\mathbf{p})$ be finite, but this is not sufficient in the general case as we saw in Example 14.2.1. We could try to mimic the argument in Theorem 5.1.2, and ask whether $\{\mathbf{y} \in Y : \mathbf{p}\cdot\mathbf{y} \geq 0\}$ is compact. This gives us a condition for profit maximization that is sufficient but not necessary. The condition fails in Example 14.1.1 even though profit can be maximized.

14.2.2 Finite Profit on the Polar Set

In order to rule out the possibility of infinite profit we must put some restrictions on prices. In equilibrium, markets will have to clear. For that to happen, profits will have to be finite.

Suppose $\mathbf{p}$ is a price vector where profits are finite and strictly positive. In such cases, we can form the vector $\mathbf{p}' = \mathbf{p}/\pi(\mathbf{p})$. Now $\pi(\mathbf{p}') \leq 1$. Moreover, supply is the same because both supply and demand are homogeneous of degree zero in the price vector. This means we can restrict our attention to prices where $\pi(\mathbf{p})$ is bounded by one. This is the polar set.

Polar Set. The *polar* of a set A, denoted A°, is defined by $A^\circ = \{\mathbf{p} : \mathbf{p}\cdot\mathbf{a} \leq 1 \text{ for all } \mathbf{a} \in A\}$.[3]

[3] Some authors use call our A° the one-sided polar, and reserve the term polar for the absolute polar $\{\mathbf{p} : |\mathbf{p}\cdot\mathbf{a}| \leq 1 \text{ for all } \mathbf{a} \in A\}$. We will not need the absolute polar.

14.2.3 Polar Cone and Polar Set

When A is a cone the polar set and the previously defined polar cone coincide.

Proposition 14.2.1. *Let A be a cone. Then the polar set is $A^\circ = \{\mathbf{p} : \mathbf{p}\cdot\mathbf{a} \le 0$ for all $\mathbf{a} \in A\}$. Moreover, A° is a cone.*

Proof. Now $A^\circ = \{\mathbf{p} : \mathbf{p}\cdot\mathbf{a} \le 1 \text{ for all } \mathbf{a} \in A\}$. If A is a cone and $\mathbf{p}\cdot\mathbf{a} > 0$, then $t\mathbf{a} \in A$ for all $t > 0$. Taking t large enough, we find $\mathbf{p}\cdot(t\mathbf{a}) > 1$, showing that $\mathbf{p} \notin B$. It follows that $A^\circ = \{\mathbf{p} : \mathbf{p}\cdot\mathbf{a} \le 0 \text{ for all } \mathbf{a} \in A\}$.

We previously showed that A° is a cone in Proposition 14.1.5. □

Now that we know that the polar cone and polar set are the same when A is a cone.

As indicated above, the significance of the polar is that profits will be bounded at price vector $\mathbf{p}$ if and only if it is a positive scalar multiple of a vector in the polar. The polar allows us to characterize all prices where the profit function is finite. The prices must be in the *cone generated by* Y°, the smallest cone containing Y°. It is given by $\operatorname{cone}(Y^\circ) = \bigcup_{t>0} tY^\circ$.[4]

The set $\operatorname{cone}(Y^\circ)$ has particular importance for equilibrium analysis. The economy cannot be in equilibrium if infinite profit is possible. Any equilibrium price vector must be in $\operatorname{cone}(Y^\circ)$. The fact that a price vector is in $\operatorname{cone}(Y^\circ)$ does not guarantee that profit can be maximized. We could face a situation as in Example 14.2.1, where profit is bounded, but cannot be maximized. However, if Y enjoys constant returns to scale, profit can be maximized for any $\mathbf{p} \in \operatorname{cone}(Y^\circ)$.

[4] The set $\operatorname{cone}(Y^\circ)$ is clearly a cone. Since any cone containing Y° must also contain every tY° when $t > 0$, this is the smallest cone containing Y°.

14.2.4 Cobb-Douglas Polar

Before proceeding, we give an example of a polar set that is not a cone.

Example 14.2.4: It is easy to see that $Y^\circ = \{\mathbf{p} : \pi(\mathbf{p}) \leq 0\}$. We can find the polar by using the profit function. Suppose $Y = \{\mathbf{y} \in \mathbb{R}^2 : y_1 \leq 0, y_2 \leq \sqrt{-y_1}\}$. Due to free disposal, profit maximization requires $\mathbf{p} \geq \mathbf{0}$. Since $y_2 \leq \sqrt{-y_1}$, we must maximize $p_1 y_1 + p_2\sqrt{-y_1}$ in order to maximize profit.

The first-order condition is $-p_1 y_1 + p_2/(2/\sqrt{-y_1}) = 0$. This yields $y_2 = \sqrt{-y_1} = p_2/2p_1$ and $y_1 = -p_2^2/4p_1^2$. Profit is then $\pi(\mathbf{p}) = p_2^2/4p_1$.

Now $\pi(\mathbf{p}) \leq 1$ if and only if $p_2^2 \leq 4p_1$. It follows that the polar is $Y^\circ = \{\mathbf{p} \in \mathbb{R}^2_+ : p_2^2 \leq 4p_1\}$. Since the slope of the boundary is vertical at the origin, $\text{cone}(Y^\circ) = \mathbb{R}^2_+$.

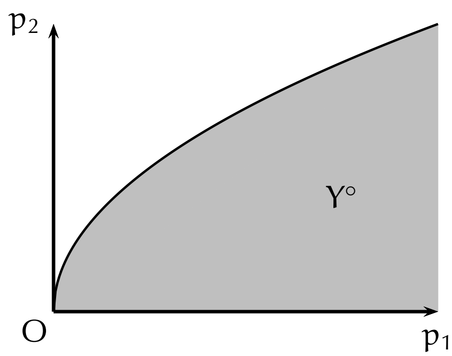

Figure 14.2.5: The polar of Y is bounded by the parabola $p_2^2 = 4p_1$ when the technology is defined by $y_2 \leq \sqrt{-y_1}$. The vertical axis has been compressed to make the shape of the polar more obvious.

◀

14.2.5 Properties of Polars

The remainder of this section gathers together some useful properties of polars.

Although the polar need not be a cone, it will be a non-empty closed and convex set.

Proposition 14.2.6. *Let* A *be a non-empty set. Then* A° *is non-empty, closed, and convex.*

Proof. The polar is always non-empty because $\mathbf{0} \in A^\circ$. Now $A^\circ = \bigcap_{\mathbf{a} \in A}\{\mathbf{p} : \mathbf{p}\cdot\mathbf{a} \leq 1\}$, so A° is closed and convex as the intersection of closed convex sets. □

The next two propositions follow immediately from the definitions.

Proposition 14.2.7. *If* A *and* B *are non-empty sets with* $A \subset B$, *then* $B^\circ \subset A^\circ$.

Proof. Here $B^\circ = \{\mathbf{p} : \mathbf{p}\cdot\mathbf{b} \leq 1 \text{ for all } \mathbf{b} \in B\} \subset \{\mathbf{p} : \mathbf{p}\cdot\mathbf{b} \leq 1 \text{ for all } \mathbf{b} \in A\} = A^\circ$. □

Proposition 14.2.8. *For any non-empty sets* A *and* B, $(A \cup B)^\circ = A^\circ \cap B^\circ$.

Proof. Here

$$\begin{aligned}
(A \cup B)^\circ &= \bigcap_{\mathbf{a} \in A \cup B} \{\mathbf{p} : \mathbf{p}\cdot\mathbf{a} \leq 1\} \\
&= \left(\bigcap_{\mathbf{a} \in A}\{\mathbf{p} : \mathbf{p}\cdot\mathbf{a} \leq 1\}\right) \bigcap \left(\bigcap_{\mathbf{a} \in B}\{\mathbf{p} : \mathbf{p}\cdot\mathbf{a} \leq 1\}\right) \\
&= A^\circ \cap B^\circ.
\end{aligned}$$

□

14.2.6 Polars of Production Sets

In many cases we can ensure the polar is non-trivial.

Theorem 14.2.9. *Suppose* Y *is a convex production set. Then there is a* $\mathbf{p} \in Y^\circ$ *with* $\mathbf{p} \neq \mathbf{0}$.

Proof. By definition Y is non-empty, closed, and convex. The no free lunch condition ensures $\boldsymbol{e} \notin Y$. Use Separation Theorem A to find $\mathbf{p}' \neq \mathbf{0}$ and $\alpha \in \mathbb{R}$ with $\mathbf{p}' \cdot \mathbf{y} \leq \alpha$ and $\alpha < \mathbf{p} \cdot \boldsymbol{e}$. By inaction, $\mathbf{0} \in Y$, which implies $\alpha \geq 0$. Take α' with $\alpha < \alpha' < \mathbf{p} \cdot \boldsymbol{e}$. Set $\mathbf{p} = \mathbf{p}'/\alpha'$. Now $\mathbf{p} \cdot \mathbf{y} \leq \mathbf{p}' \cdot \mathbf{y}/\alpha' \leq \alpha/\alpha' < 1$ for all $\mathbf{y} \in Y$, so $\mathbf{p} \in Y^\circ$. □

Free disposal also puts some restrictions on Y°.

Proposition 14.2.10. *Suppose* $Y \subset \mathbb{R}^L$ *obeys free disposal. Then* $Y^\circ \subset \mathbb{R}^L_+$.

Proof. Take $\mathbf{a}_0 \in Y$. If $\mathbf{p} \in Y^\circ$, free disposal implies $\mathbf{a}_0 - n\boldsymbol{e}^\ell \in Y$, so $\mathbf{p} \cdot \mathbf{a} - np_\ell = \mathbf{p} \cdot (\mathbf{a}_0 - n\boldsymbol{e}^\ell) \leq 1$ for all $n = 1, 2, \ldots$. Divide by n and let $n \to \infty$ to find $-p_\ell \leq 0$, establishing the result. □

It follows that the polar of a production set is a subset of the positive orthant. Free disposal forces prices to be non-negative.

14.2.7 Second Polars

When a set is closed and convex, it is its own second polar.

Proposition 14.2.11. *Suppose* A *is a closed convex set with* $\mathbf{0} \in A$. *Then* $(A^\circ)^\circ = A$.

Proof. If $\mathbf{a} \in A$, then $\mathbf{p}\cdot\mathbf{a} \leq 1$ for all $\mathbf{p} \in A^\circ$. This can be written $\mathbf{a}\cdot\mathbf{p} \leq 1$ for all $\mathbf{p} \in A^\circ$, which implies $\mathbf{a} \in (A^\circ)^\circ$. This implies $A \subset A^{\circ\circ}$.

Now suppose $\mathbf{a}' \in (A^\circ)^\circ$ but $\mathbf{a}' \notin A$. Since A is closed, convex, and non-empty, we can use Separation Theorem A to find a $\mathbf{p} \neq \mathbf{0}$ and $\alpha \in \mathbb{R}$ with $\mathbf{p}\cdot\mathbf{a}' > \alpha$ and $\mathbf{p}\cdot\mathbf{a} < \alpha$ for all $\mathbf{a} \in A$. Since $\mathbf{0} \in A$, $\alpha > 0$. Now set $\mathbf{p}' = \mathbf{p}/\alpha$. Then $\mathbf{p}' \in A^\circ$, but $\mathbf{p}'\cdot\mathbf{a}' > 1$. This implies $\mathbf{a}' \notin A^{\circ\circ}$. This contradiction implies $A^{\circ\circ} \subset A$. Now use the previous paragraph to find $A = (A^\circ)^\circ$. □

Since convex production sets obey the hypotheses of Proposition 14.2.11, it follows that the second polar of any convex production set is the production set itself.

Another consequence of Proposition 14.2.11 is that for closed convex sets containing the origin, if the polar set is a cone, the original set must also be a cone.

Corollary 14.2.12. *If* A *is a closed convex set with* $\mathbf{0} \in A$ *and* A° *is a cone, then* A *is a convex cone.*

Proof. By the previous corollary, $A = (A^\circ)^\circ$. In other words, A is the polar of the cone A°. Then by Proposition 14.2.3, A is itself a cone. □

In Problem 14.2.3 you are asked to show by example that Corollary 14.2.12 fails when A is not convex.

14.2.8 Polars of Sums

Our final result on polars shows that when we have a sum of sets containing $\mathbf{0}$, the cone generated by the polar of the sum is equal to the intersection of cones generated by the polars of the individual sets. When applied to production sets, it means that the price vectors yielding finite profit for the sum of a collection of production sets are the same as the price vectors that simultaneously yield finite profit for all the individual production sets.

Proposition 14.2.13. *Suppose $Y_1, \ldots, Y_F$ are sets satisfying inaction. Then*

$$\operatorname{cone}\left[\left(\sum_f Y_f\right)^\circ\right] = \bigcap_f \operatorname{cone}(Y_f^\circ)$$

Proof. Let $Y = \sum_f Y_f$. Suppose $\mathbf{p} \in \operatorname{cone}(Y^\circ) = \bigcup_{t>0} tY^\circ$. Then there is t_0 with $t_0^{-1}\mathbf{p} \in Y^\circ$. By inaction, $Y \supset \bigcup_f Y_f$, so $Y^\circ \subset \left(\bigcup_f Y_f\right)^\circ = \bigcap_f Y_f^\circ$, which means $t_0^{-1}\mathbf{p} \in Y_f^\circ$ for each f. This implies $\mathbf{p} \in \bigcap_f \left(\bigcup_{t>0} Y_f^\circ\right) = \bigcap_f \operatorname{cone}(Y_f^\circ)$.

Conversely, suppose $\mathbf{p} \in \bigcap_f \operatorname{cone}(Y_f) = \bigcap_f \left(\bigcup_{t>0} Y_f^\circ\right)$. Then there are $t_f > 0$ and $\mathbf{p}_f \in Y_f^\circ$ with $\mathbf{p} = t_f \mathbf{p}_f$. Let $\mathbf{y} = \sum_f \mathbf{y}_f$ be an arbitrary element of Y. Then $\mathbf{p}\cdot\mathbf{y} = \sum_f \mathbf{p}\cdot\mathbf{y}_f = \sum_f t_f \mathbf{p}_f\cdot\mathbf{y}_f \le \sum_f t_f$ because $\mathbf{p}_f \in Y_f^\circ$ for each f. Setting $t = \sum_f t_f$, we find $\mathbf{p} \in tY^\circ \subset \operatorname{cone}(Y^\circ)$. When combined with the previous paragraph, this shows the two sets are equal. □

14.3 Decreasing Returns? Use Constant Returns!

To close our discussion of the existence of supply, we will show that any convex production set has an associated constant returns to scale production set. We can then analyze the existence problem in the simpler constant returns to scale model.

It may seem that constant returns technologies and strictly decreasing returns technologies are different things. Nonetheless, it is generally possible to translate an economy with convex production sets into an economy with a convex cone technology. One advantage of CRS technologies is that when profit is bounded (meaning zero in the CRS case), it can be maximized. There is no possibility of profit being bounded but not maximizable. Moreover, the supply correspondence is a closed correspondence. Finally, the fact that the maximum profit is zero makes CRS technologies easier to use in equilibrium models. The profit does not have to be accounted for because there isn't any.

14.3.1 Homogenizing Production Functions I

Let's consider the case where the technology is defined by a production function. You are probably familiar with the inverse procedure, where a constant returns production function $F(K, L)$ of two variables, capital and labor, is converted to a per capita production function $f(k)$ by defining $f(k) = F(k, 1)$ where $k = K/L$ is the capital/labor ratio. In that case $f(k) = F(k, 1) = F(K/L, 1) = F(K, L)/L$ by homogeneity of degree one of F.

By inverting this procedure, we can convert any production function with decreasing returns into a constant returns production function by adding an extra variable, an extra factor of production. When $L = 1$, we invert the procedure by starting with $f(z)$ and write $F(K, L) = Lf(K/L)$ for $L > 0$. Obviously $L = 0$ requires special handling.

The procedure is similar for $L > 1$ and results in a production function that is homogeneous of degree one, and we refer to this process as *homogenizing* the production function.

Homogenized Production Function. Let $f: \mathbb{R}^L_+ \to \mathbb{R}_+$ be a concave production function obeying $f(\mathbf{0}) = 0$. The *homogenized production function* $F: \mathbb{R}^{L+1}_+ \to \mathbb{R}_+$ is defined by

$$F(\boldsymbol{z}, t) = \begin{cases} tf\big(t^{-1}\boldsymbol{z}\big) & \text{when } t > 0 \\ \lim_{t\downarrow 0} tf\big(t^{-1}\boldsymbol{z}\big) & \text{when } t = 0. \end{cases}$$

We can recover the original production function from the homogenized function F by using the fact that $F(\boldsymbol{z}, 1) = f(\boldsymbol{z})$. In other words, when we use exactly one unit of the $(L + 1)^{st}$ factor, we obtain the original production function.

The $(L + 1)^{st}$ factor is referred to as an *entrepreneurial factor*.[5] Why call the extra factor an "entrepreneurial" factor? The point is that in the absence of resource constraints, there are no physical reasons to think we cannot double production by simply duplicating a firm, or triple it by making three firms. Production should show at least constant returns to scale. If it doesn't it must mean that there is some additional factor that is not being duplicated. This factor or factors is not something being purchased, but is intrinsic to the firm. That makes it natural to attribute the extra factor to the entrepreneur.[6]

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[5] See McKenzie, 1959.

[6] One thing that might prevent constant returns to scale is that firm grows too large for current management to run directly. Delegation involves problems of incentive compatibility (managers incentives may not be perfectly aligned with profit maximization) and information (some managers may be too distant from the production process to have the information to make correct decisions).

One indication of how important the entrepreneurs can be comes from a study of profits of pass-through firms. In such firms, profits are passed-through to the owner/entrepreneur and taxed as individual income. Smith et al. (2019) found that the profits of such firms fell by over 75% upon the death of the owner. They attribute this to loss of the owner's human capital.

14.3.2 Homogenizing Production Functions II

The special definition of F when $z_{L+1} = 0$ is necessary to prevent the division by zero that would otherwise occur in the top formula. Of course, we still need to show that the limit exists. The following lemma accomplishes that. Lemma 14.3.2 implies that the slope of the chord between $\big(0, f(\mathbf{0})\big)$ and $\big(t\boldsymbol{z}, f(t\boldsymbol{z})\big)$ is decreasing in t. This is pretty obvious in $\mathbb{R}_+$ if you graph the function. See Figure 14.3.1.

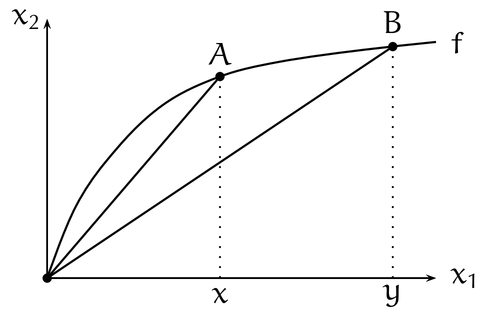

Figure 14.3.1: The argument in Lemma 14.3.2 is clear on the diagram. We use the facts that f is concave, $f(0) = 0$, and x is a convex combination of 0 and y, to show that the chord from the origin to A is steeper than the chord from the origin to B. Since the slope of the chord at x is $f(x)/x$, $f(x)/x > f(y)/y$.

The vector case is similar, but the slope of the chord is then $f(t\boldsymbol{z})/t\|\boldsymbol{z}\|$. Since we are only interested in the effect of changes in the entrepreneurial factor t, we simplify matters slightly by considering $f(t\boldsymbol{z})/t$. The calculations are essentially the same as the extra $\|\boldsymbol{x}\|$ terms will cancel out.

Lemma 14.3.2. *Suppose $f\colon \mathbb{R}^L_+ \to \mathbb{R}$ is a concave function and $f(\mathbf{0}) = 0$. Then $f(t\boldsymbol{z})/t$ is weakly decreasing in $t \in \mathbb{R}_{++}$ for any $\boldsymbol{z} \in \mathbb{R}^L_+$.*

Proof. Let $0 < t < s$. We can write $t\boldsymbol{z}$ as the convex combination

$$t\boldsymbol{z} = \left(1 - \frac{t}{s}\right)\mathbf{0} + \left(\frac{t}{s}\right)s\boldsymbol{z}.$$

Since f is concave

$$f(t\boldsymbol{z}) \ge \left(1 - \frac{t}{s}\right)f(\mathbf{0}) + \left(\frac{t}{s}\right)f(s\boldsymbol{z}) = \left(\frac{t}{s}\right)f(s\boldsymbol{z}).$$

It follows that

$$\frac{f(t\boldsymbol{z})}{t} \ge \frac{f(s\boldsymbol{z})}{s}$$

whenever $0 < t < s$. $\square$

14.3.3 Homogenizing Production Functions III

Corollary 14.3.3. *Suppose* $f\colon \mathbb{R}^L_+ \to \mathbb{R}_+$ *is a concave increasing function with* $f(\mathbf{0}) = 0$. *Let* F *be the homogenized production function for* $t > 0$. *Then* $\lim_{t\downarrow 0} F(\boldsymbol{z}, t)$ *is either finite or* $+\infty$. *Either way, it is non-negative.*

Proof. We replace t by $1/t$ in Lemma 14.3.2, to find that $F(\boldsymbol{z}, t) = tf(t^{-1}\boldsymbol{z})$ is now increasing in t. For the limiting case, t decreases to 0, so $\lim_{t\to 0} F(\boldsymbol{z}, t) = \inf_{t>0} F(\boldsymbol{z}, t)$, which always exists in $\mathbb{R}^*$, either as a finite number or $+\infty$. As f is non-negative, the infimum F is also non-negative. □

By construction, the new production function F obeys constant returns to scale. Further, if we are endowed with one unit of z_{L+1}, we can use it in conjunction with $\boldsymbol{z}$, to produce $F(\boldsymbol{z}, 1) = f(\boldsymbol{z})$ units of output.

Example 14.3.4: Suppose $f(z) = z^\rho$ with $0 < \rho < 1$. Then $F(z_1, z_2) = z_2(z_1/z_2)^\rho = z_1^\rho z_2^{1-\rho}$. Since this expression is continuous at zero, we don't need to use the limit to define $F(z_1, 0)$. Moreover, $F(z, 1) = z^\rho = f(z)$. ◀

If f is already constant returns to scale, the homogenization process just gives us f back: $F(\boldsymbol{z}, z_{L+1}) = f(\boldsymbol{z})$.[7]

[7] See problems 14.3.2 and 14.3.3.

14.3.4 The Augmented Production Set

We are now ready to apply the idea of homogenization to the production set itself. The resulting set is the augmented or homogenized production set. We do this in the obvious way for positive amounts of the entrepreneurial factor. Then we take the closure of the resulting set rather than the limit of the homogenized function.

Augmented Production Set. Let Y be a convex production set in $\mathbb{R}^L$. The *augmented* or *homogenized production set* $\hat{Y} \subset \mathbb{R}^{L+1}$ is defined by $\hat{Y} = \mathrm{cl}\{(t\mathbf{y}, -t) : \mathbf{y} \in Y,\, t > 0\}$.

This definition is equivalent to $\hat{Y} = \mathrm{cl}\{(\mathbf{y}, -t) : \mathbf{y} \in tY,\, t > 0\}$. To see this set

$$Y_0 = \big\{(t\mathbf{y}, -t) : \mathbf{y} \in Y, t > 0\big\},$$

so that $\hat{Y} = \overline{Y_0}$. Writing $\mathbf{y}' = t\mathbf{y}$, we can rewrite Y_0 as

$$Y_0 = \big\{(\mathbf{y}', -t) : \mathbf{y}' \in tY, t > 0\big\},$$

which is the other form. The set Y_0 will be useful for determining the properties of $\hat{Y}$.

We should wonder whether taking the closure only makes a difference when the entrepreneurial input is zero, or whether it can also affect net outputs with a positive entrepreneurial input. The following lemma tells us that the closure is only relevant when the entrepreneurial input is zero.

Lemma 14.3.5. *Let Y be a closed set in $\mathbb{R}^L$ and $\hat{Y}$ the corresponding augmented production set. If $(\mathbf{y}, -z) \in \hat{Y}$ with $z > 0$, then $(\mathbf{y}, -z) \in Y_0$.*

Proof. Take a convergent sequence $(\mathbf{y}_n, -z_n) \in Y_0$ with $(\mathbf{y}_n, -z_n) \to (\mathbf{y}, -z)$ where $z > 0$. This means that $\mathbf{y}_n/z_n \in Y$ for large n. Since $\mathbf{y}_n/z_n \to \mathbf{y}/z$ and Y is closed, we can conclude that $\mathbf{y}/z \in Y$. It follows that $(\mathbf{y}, -z) \in Y_0$. □

14.3.5 No Entrepreneurial Factor 2/15/22

So what happens if the entrepreneurial input is zero? The kinds of sequences considered should remind you of the defining sequences for the asymptotic cone, the set of CRS vectors. In fact, the following lemma shows that $\mathbf{y}$ is in the asymptotic cone, the set of CRS vectors in Y, if and only if $(\mathbf{y}, 0) \in \hat{Y}$. This result doesn't just apply to production sets, but to any set that is closed and divisible. Recall that convex production sets are always divisible because convexity and inaction implies divisibility.

Lemma 14.3.6. *Suppose Y is a closed set that obeys divisibility. A vector of the form $(\mathbf{y}, 0)$ is in the augmented production set $\hat{Y}$ if and only if $\mathbf{y}$ in the asymptotic cone $\mathbf{A}(Y)$, or equivalently, if and only if $\mathbf{y}$ is a CRS element of Y. When $(\mathbf{y}, 0) \in \hat{Y}$, we also have $(\mathbf{y}, -z) \in \hat{Y}$ for all $z \geq 0$.*

Proof. By the definition of $\hat{Y}$, a vector $(\mathbf{y}, 0) \in \hat{Y}$ if and only if there is a sequence $(t_n\mathbf{y}_n, -t_n) \in Y_0$ with $t_n \to 0$ and $t_n\mathbf{y}_n \to \mathbf{y}$. That is, $\mathbf{y}_n \in Y$ and $t_n > 0$ with $t_n\mathbf{y}_n \to \mathbf{y}$. But this is equivalent to saying that $(t_n, \mathbf{y}_n)$ is a defining sequence for $\mathbf{y}$ for the asymptotic cone. Proposition 13.2.8 showed this is equivalent to $\mathbf{y}$ being a CRS element of Y.

Now suppose $\mathbf{y}$ is a CRS element of Y and let $z > 0$. Since $\mathbf{y}$ is a CRS element of Y, $\mathbf{y}/z \in Y$. It follows that $\big(z(\mathbf{y}/z), -z\big) = (\mathbf{y}, -z) \in \hat{Y}$ for every $z > 0$. Letting $z \to 0$ we also find $(\mathbf{y}, 0) \in \hat{Y}$. $\square$

14.3.6 Characterizing Augmented Production Sets

Define Y_1 by

$$Y_1 = \{(\mathbf{y}, 0) : \mathbf{y} \in CRS(Y)\} = \{(\mathbf{y}, 0) : t\mathbf{y} \in Y \text{ for all } t > 0\}.$$

We can now characterize the augmented production set by combining Lemmas 14.3.5 and 14.3.6.

Proposition 14.3.7. *Let Y be a closed set that obeys divisibility. Then $\hat{Y} = Y_0 \cup Y_1 = \{(z\mathbf{y}, -z) : \mathbf{y} \in Y, z > 0\} \cup \{(\mathbf{y}, 0) : t\mathbf{y} \in Y$ for all $t > 0\}$.*

In other words, the homogenized production set $\hat{Y}$ consists of the set of CRS elements Y_1, together with Y_0, the set of net outputs that use positive amounts of the entrepreneurial factor.

14.3.7 The Augmented Production Set is CRS I

We can now show that the augmented production set is a constant returns to scale production set in $\mathbb{R}^{L+1}$.

Proposition 14.3.8. *Suppose $Y \subset \mathbb{R}^L$ is a convex production set (i.e., Y is also non-empty, closed, and satisfies the no free lunch, inaction, and free disposal conditions). Then $\hat{Y} = \text{cl}\{(z\mathbf{y}, -z) : \mathbf{y} \in Y,\ z > 0\}$ is a non-empty closed convex cone that satisfies the no free lunch, inaction, and free disposal conditions.*

Proof. Since $\mathbf{0} \in \hat{Y}$, the set $\hat{Y}$ is a **non-empty set** satisfying **inaction**. It is **closed** by definition and is a **cone** since it is the closure of a cone (Proposition 30.3.2).

Now consider the no free lunch condition. Suppose $(\mathbf{y}, -z) \in \hat{Y}$ with $(\mathbf{y}, -z) > \mathbf{0}$. Clearly z must be zero. By Lemma 14.3.6, $\mathbf{y} \in Y$. But then $\mathbf{y} > \mathbf{0}$ which is impossible by the no free lunch condition for Y. This shows that $\hat{Y}$ obeys the **no free lunch condition**.

14.3.8 The Augmented Production Set is CRS II

Proof continues. We now turn to free disposal. Suppose $(y, -z) \in \hat{Y}$ and $(y', -z') \leq (y, -z)$. There are two cases: $z > 0$ and $z = 0$.

In case one, $z > 0$. Since $z' \geq z > 0$ we can define $y_0 = y/z$ and $y_0' = y'/z'$. Then $y_0 \in Y$. Since $z' \geq z > 0$, $(z/z')y_0 \geq y_0'$. By inaction and convexity, $(z/z')y_0 \in Y$ and by free disposal $y_0' \in Y$. This shows that $(y', -z') \in \hat{Y}$.

Now consider the second case, $z = 0$. By Lemma 14.3.6, y is a CRS element of Y. Since $y' \leq y$, $y' \in Y$ by free disposal in Y. Moreover, for any $t > 0$, $ty' \leq ty \in Y$, so ty' is also in Y by free disposal. This means that y' is also a CRS element of Y. Lemma 14.3.6 then implies $(y', z') \in \hat{Y}$ since $z' \leq 0$ and establishes **free disposal**.

The last thing to show is that $\hat{Y}$ is a convex set. Since the closure of a convex set is convex, it is enough to show $Y_0 = \{(zy, -z) : y \in Y,\ z > 0\}$ is convex. Let $(zy, z), (z'y', z') \in Y_0$ and let $0 < \alpha < 1$. Then set $z'' = \alpha z + (1-\alpha)z'$ and

$$y'' = \frac{\alpha z}{\alpha z + (1-\alpha)z'}y + \frac{(1-\alpha)z'}{\alpha z + (1-\alpha)z'}y'.$$

Now

$$\frac{\alpha z y + (1-\alpha)z'y'}{\alpha z + (1-\alpha)z'} = \frac{\alpha z}{\alpha z + (1-\alpha)z'}y + \frac{(1-\alpha)z'}{\alpha z + (1-\alpha)z'}y' = y'' \in Y$$

by convexity of Y. But then $z''y'' = \alpha z y + (1-\alpha)y'$. It follows that $(z''y'', -z'') \in Y_0$, showing that Y_0 is convex. It follows that $\hat{Y} = \overline{Y}_0$ is **convex**. □

Proposition 14.3.8 showed that the augmented production set is a constant returns production set. Further, since $y \in Y$ if and only if $(y, -1) \in \hat{Y}$, we get exactly the same production possibilities for the first L goods as Y if we are endowed with exactly one unit of good $L + 1$.

We can handle the case of many firms by introducing an entrepreneurial factor for each firm.[8]

[8] The entrepreneurial factor is not needed for firms with CRS technologies. We will see that it will make no difference to production possibilities whether or not we add entrepreneurial factors to CRS production sets.

14.3.9 Homogenized IS Augmented

There is one loose end to tie up. Earlier, we motivated the definition of the homogenized production set by considering homogenized production functions. You should not be surprised to hear that when we construct the production set from a production function in the usual way, homogenization of the production set corresponds to homogenization of the production function. This is shown in the following theorem.

Theorem 14.3.9. *Let* $f\colon \mathbb{R}^L_+ \to \mathbb{R}_+$ *be an increasing and continuous production function obeying* $f(0) = 0$. *Let* F *be the corresponding homogenized production function and set* $Y = \{(q, -z) : z \geq 0, q \leq f(z)\}$. *Then the homogenized production set* $\hat{Y}$ *is the production set obtained from the homogenized production function* F.

Proof. Define $F_0(z, -t) = tf(z/t)$. As before, set $Y_0 = \{(ty, -t) : y \in Y,\ t > 0\}$, so that $\hat{Y} = \overline{Y_0}$. By Lemma 14.3.5, $(q, z, -t) \in Y_0$ with $t > 0$ if and only if $q \leq F_0(z, t)$.

That leaves vectors of the form $(q, z, 0)$. These are in $\hat{Y}$ if and only if there is a sequence $y_n = (q_n, -z_n, -t_n)$ with $y_n \to y$, $t_n \downarrow 0$ and $q_n \leq F_0(z_n, t_n)$. Taking the limit we find $q \leq F(z, 0)$, in other words, y is in the production set generated by F itself. $\square$

14.3.10 Augmented Profit Function

The augmented production set obeys constant returns to scale. As a result, the maximum profit is zero. This zero profit condition allows us to price the entrepreneurial factor. The following two propositions show the entrepreneurial factor must have a price at least equal to the profit of the original production set. Further, if the entrepreneurial factor is actually used, its price is exactly the profit from the original production set.

Proposition 14.3.10. *Let Y be a convex production set. Denote its augmented production set by $\hat{Y}$, and the corresponding profit functions by π and $\hat{\pi}$. For any $\mathbf{p}$, $\pi(\mathbf{p})$ is finite if and only if $\hat{\pi}(\mathbf{p}, q) = 0$ for all $q \geq \pi(\mathbf{p})$.*

Proof. Part I (only if): Suppose $\pi(\mathbf{p})$ is finite and $q \geq \pi(\mathbf{p})$. Then $\mathbf{p}\cdot\mathbf{y} \leq \pi(\mathbf{p}) \leq q$ for all $\mathbf{y} \in Y$. It follows that for $z > 0$,

$$(\mathbf{p}, q)\cdot(z\mathbf{y}, -z) = z(\mathbf{p}\cdot\mathbf{y} - q) \leq 0$$

for all $\mathbf{y} \in Y$ and $z > 0$.

Since $\hat{Y} = \mathrm{cl}\{(z\mathbf{y}, -z) : \mathbf{y} \in Y, z > 0\}$, $(\mathbf{p}, q)\cdot\hat{\mathbf{y}} \leq 0$ for all $\hat{\mathbf{y}} \in \hat{Y}$. Taking the supremum over $\hat{Y}$, we obtain $\hat{\pi}(\mathbf{p}, q) \leq 0$. By inaction, $\hat{\pi}(\mathbf{p}, q) \geq 0$, and so $\hat{\pi}(\mathbf{p}, q) = 0$.

Part II (if): Now suppose $\hat{\pi}(\mathbf{p}, q) = 0$. Since $(\mathbf{y}, -1) \in \hat{Y}$ for all $\mathbf{y} \in Y$, $\mathbf{p}\cdot\mathbf{y} \leq q$ for all $\mathbf{y} \in Y$. It follows that $\pi(\mathbf{p}) \leq q$, implying that $\pi(\mathbf{p})$ is finite. □

14.3.11 Pricing the Entrepreneurial Factor

Since $\hat{Y}$ is a constant returns to scale technology, the supply correspondence is non-empty whenever $\hat{\pi}(\mathbf{p}, q) = 0$. The constraint that $q \geq \pi(\mathbf{p})$ has an associated complementary slackness condition, that $z\big(q - \pi(\mathbf{p})\big) = 0$ where z is the entrepreneurial input. If there is a vector in the supply correspondence with non-zero input of the entrepreneurial factor, this implies that $q = \pi(\mathbf{p})$.

Proposition 14.3.11. *Suppose Y is a convex production set in $\mathbb{R}^L$ and $\hat{Y}$ its augmented production set. Let $(\mathbf{p}, q)$ be a price vector in the polar cone $(\hat{Y})^\circ$. If $\hat{\mathbf{y}} = (\mathbf{y}, -z) \in \hat{Y}$ maximizes profit, then $z\big(q - \pi(\mathbf{p})\big) = 0$. In particular, if $z > 0$, then $q = \pi(\mathbf{p})$.*

Proof. We only need consider the case $z > 0$. Then there is $\mathbf{y}' \in Y$ with $\mathbf{y} = z\mathbf{y}'$. Now $0 = (\mathbf{p}, q)\cdot(\mathbf{y}, -z) = (\mathbf{p}, q)\cdot(z\mathbf{y}', -z) = z\mathbf{p}\cdot\mathbf{y}' - zq$. Dividing by z yields $\mathbf{p}\cdot\mathbf{y} = q$, and so $\pi(\mathbf{p}) \geq \mathbf{p}\cdot\mathbf{y} = q$. But by Proposition 14.3.10, $q \geq \pi(\mathbf{p})$. It follows that $q = \pi(\mathbf{p})$. $\square$

One consequence of this is that if profit can be maximized at a CRS element of Y (so $(\mathbf{y}, 0) \in \hat{Y}$), the entrepreneurial factor will not be used if its price is non-zero. Its use would add cost without any compensating gain in revenue. Keep in mind that if $(\mathbf{y}, 0) \in \hat{Y}$ maximizes profit, that profit must be zero, so $\mathbf{p}\cdot\mathbf{y} = 0$.

14.3.12 Revisiting the No-Max Case

We return to Example 14.2.1 to illustrate how this works in a case where profit cannot be maximized with the original decreasing returns production function.

Example 14.3.12: We reconsider the production function from Example 14.2.1: $f(z) = 1 + z - 1/(1 + z)$ with associated production set $Y = \{(q, -z) : q \leq f(z), z \geq 0\}$. We convert this to a CRS production function. The result is

$$F(z_1, z_2) = z_1 + z_2 - \frac{z_2^2}{z_1 + z_2}$$

for $\mathbf{z} > \mathbf{0}$ with $F(z_1, 0) = z_1$.

As in Example 14.2.1, we set the output price and non-entrepreneurial factor price to p. This yields price vector $\mathbf{p} = (p, p, p_3)$ where p_3 is the price of the entrepreneurial factor. Profit is

$$\begin{aligned} g(\mathbf{z}) &= pz_1 + pz_2 - p\frac{z_2^2}{z_1 + z_2} - pz_1 - p_3 z_2 \\ &= (p - p_3)z_2 - p\frac{z_2^2}{z_1 + z_2} \end{aligned}$$

for $z_2 > 0$.

There are two cases to consider. First let $p > p_3$. Setting $z_1 = z_2^2$ and letting $z_2 \to +\infty$ shows that profit is unbounded. There is no maximum profit and price vectors (p, p, p_3) with $p > p_3$ are outside the polar cone of $\hat{Y}$.

The other case is $p \leq p_3$. In that case, $g(\mathbf{z}) \leq 0$, the maximum profit is zero and can be obtained with $\mathbf{z} = \mathbf{0}$. Even setting $p_3 = \pi(p, p) = p$ ensures that profit is less than zero for any $(z_1, z_2) \gg \mathbf{0}$ and approaches zero as $z_1 \to \infty$ for fixed z_2 (e.g. $z_2 = 1$, which is our original case).

When $z_2 = 1$ the entrepreneurial factor soaks up what had been the maximum possible profit in the original model, leaving the net profit less than zero–very slightly less for large inputs.

Although zero profit cannot be obtained from large values of $(q, -\mathbf{z})$, it is obtained at $\mathbf{0}$. This option was not available in the original problem.

When $p_3 > p$, the entrepreneurial factor gets more than the maximum profit and the $z_2 = 1$ case (or any case with fixed z_2) has negative profit as its upper bound. The only way to escape is via inaction, where the entrepreneurial factor is not used, yielding no output and no cost.

Profit can be maximized if and only if $p \leq p_3$ and the maximum occurs at $\mathbf{0}$. The case that did not have a solution in Example 14.2.1 has been converted to a case where inaction maximizes profit. ◄

14.3.13 Profit Maximization: Summing Up

The addition of an entrepreneurial factor has converted an example where profit is bounded, but not maximizable to one where profit can be maximized. However, there is a cost. The solution that was added is inaction. This might create problems if we actually ended up with these prices in equilibrium in the constant returns model. The original model and the constant returns versions would not really correspond.

In fact, such possibilities cannot arise in general equilibrium. Equilibrium will impose an additional condition that rule them outs. Suppose $q > 0$. Consumers will supply the corresponding entrepreneurial factor. Since it is not demanded, there will be excess factor supply. We will see in Chapter 15, (Corollary 15.2.5) that goods in excess supply must have zero price. In other words, $q = 0$. But if $q = 0$, $\pi(\mathbf{p}) = 0$ also since $q \geq \pi(\mathbf{p}) \geq 0$. The equilibrium prices must be such that profit can be maximized. Although the use of an augmented production set means we have eliminated the troublesome case, it doesn't really matter because that case could not occur in equilibrium.

The upshot of all this is that replacing Y with its augmented production set $\hat{Y}$ allows us to maximize profit for any $\hat{\mathbf{p}} \in (\hat{Y})^\circ$ even if profit maximization was not possible in the original production set Y. It eliminates the cases where profit was bounded but not maximizable. Since those cases can never arise in any market equilibrium, we don't lose anything of importance.[9] What we gain is a well-behaved supply correspondence—one that exists and is closed on the polar cone.

[9] See Theorem 16.7.2.

14.4 Aggregate Production Sets

The remaining questions have to do with market supply rather than the supply of an individual firm. Aggregating supply is much simpler than aggregating demand. We start by aggregating the production sets.

Suppose there are F firms, labeled $f = 1, 2, \dots, F$ with corresponding production sets Y_f. We assume there are no production externalities. Each production set is expressed in terms of that firm's net outputs, independently of what the other firms are doing. Keep in mind that the production set states only technological possibilities. It does not address whether the necessary resources are actually available.

We attempt to construct an aggregate production set by simply summing the individual firms' production sets:

$$Y = Y_1 + Y_2 + \cdots + Y_F.$$

Is the aggregate production set a production set? Does it obey the five conditions: non-emptiness, closure, no free lunch, inaction, and free disposal?

Three of these requirements, non-emptiness, inaction, and free disposal, follow easily from the corresponding properties of the Y_f. The other two conditions are more of a problem. The no free lunch condition can easily fail as the following example shows.

Example 14.4.1: One simple example is $Y_1 = \{(y_1, y_2) : y_2 \le -2y_1, y_1 \le 0\}$ and $Y_2 = \{(y_1, y_2) : y_1 \le -2y_2, y_2 \le 0\}$. Here $Y_1 + Y_2 = \mathbb{R}^2$! To see this observe that when $x > 0$, $(x, x) = (2x, -x) + (-x, 2x) \in Y_1 + Y_2$. But then any vector in $\mathbb{R}^2$ is in $Y_1 + Y_2$ by free disposal.

We have seen this sort of problem before in linear activity models. In fact, this **is** a linear activity model. The production set Y_1 is generated by $\mathbf{a}^1 = (-1, 2)$ and Y_2 is generated by $\mathbf{a}^2 = (2, -1)$. The aggregate production set is the linear activity model generated by both activities. Since there area positive linear combinations of the activities that provide a free lunch (e.g., $\mathbf{a}^1 + \mathbf{a}^2$), the no free lunch condition is violated.

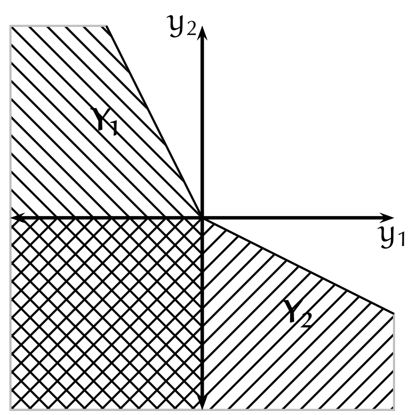

Figure 14.4.1: Let $Y_1 = \{(y_1, y_2) : y_2 \le -2y_1, y_1 \le 0\}$ and $Y_2 = \{(y_1, y_2) : y_1 \le -2y_2, y_2 \le 0\}$. The sum, which is also the convex hull, includes all of $\mathbb{R}^2$.

◀

14.4.1 Irreversibility and No Free Lunch

It is easy enough to separately require that Y itself obeys the no free lunch condition. However, there is another way.

Rather than requiring that $Y = Y_1 + Y_2$ obeys the no free lunch condition, we can require that Y is irreversible. When combined with free disposal, this implies there is no free lunch and will be useful for other purposes later on.

Proposition 14.4.2. *Let Y obey inaction, free disposal, and irreversibility. Then Y obeys the no free lunch condition.*

Proof. Suppose $\mathbf{y} > \mathbf{0}$. Then inaction and free disposal show $-\mathbf{y} \in Y$ since $-\mathbf{y} < \mathbf{0}$. If $\mathbf{y} \in Y$, irreversibility implies $\mathbf{y} = \mathbf{0}$, contradicting $\mathbf{y} > 0$. Thus $\mathbf{y} \notin Y$, establishing the no free lunch condition. $\square$

14.4.2 The Sum of Closed Sets is Closed, when One is Compact

The remaining issue is whether the aggregate production set Y is closed. This amounts to asking whether the sum of closed sets is closed, or whether the sum of closed production sets is closed.

We start by considering the case where one of the sets is bounded, and hence compact. Then the sum of two closed sets is closed. Unfortunately, production sets are often unbounded.

Proposition 14.4.3. *Let A and B be closed convex sets. If either A or B is compact, then $A + B$ is closed.*

Proof. Label the sets so that A is compact. Suppose $\mathbf{x}^n \in A + B$ with $\mathbf{x}^n \to \mathbf{x}$. We can write $\mathbf{x}^n = \mathbf{a}^n + \mathbf{b}^n$ with $\mathbf{a}^n \in A$ and $\mathbf{b}^n \in B$. There is a subsequence where $\{\mathbf{a}^{n_j}\}_{j=1}^{\infty}$ converges to a point in A, $\mathbf{a}^{n_j} \to \mathbf{a} \in A$. Then $\mathbf{b}^{n_j} = \mathbf{x}^{n_j} - \mathbf{a}^{n_j} \to \mathbf{x} - \mathbf{a} = \mathbf{b}$. It follows that $\mathbf{b} = \mathbf{x} - \mathbf{a} \in B$ because B is closed. This shows $\mathbf{x} \in (A + B)$, implying $A + B$ is closed. $\square$

14.4.3 The Sum of Closed Sets need NOT be Closed

Add diagrams

Now that we know that problems summing sets involve unbounded sets, we are ready for an example where the sum of closed convex sets is not closed.

Example 14.4.4: The following example is based on Ky Fan (1965). Let $A = \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } xy \ge 1\}$ and $B = \{(x,y) \in \mathbb{R}^2 : y = 0\}$. Both are closed convex sets.

I claim $A + B = \{(x,y) : y > 0\}$. To see this, let $y > 0$ and $x \in \mathbb{R}$. Then write $(x,y) = (1/y, y) + (x - 1/y, 0)$. Now $(1/y, y) \in A$ since the terms are positive and multiply to one while $(x - 1/y, 0) \in B$. This establishes the claim.

Now the sum $A + B$ is not closed because the horizontal axis in not in $A + B$, but is in its closure. In fact, the set $A + B$ is an open half-space. ◀

14.4.4 The Sum of Closed Convex Cones need NOT be Closed

The sum can fail to be closed even if the sets are convex cones.

Example 14.4.5: In $\mathbb{R}^3$, let $A = \{(0,0,z) : z \leq 0\}$ and $B = \{(x,y,z) \geq 0 : (x-z)^2 + y^2 \leq z^2\}$. Both sets are closed convex cones. As such, their sum is also their convex hull.

Now consider the point $(0,1,0)$. It is not in $A + B$ since its x and y coordinates of must both come from B, and any point in B with a non-zero y co-ordinate must also have a non-zero x co-ordinate.

However, $(0,1,0)$ is the limit of points in $A + B$. For any $\varepsilon > 0$, the point $(\varepsilon, 1, 0)$ is in $A + B$ since it is the sum of $(0,0,-(1+\varepsilon^2)/2\varepsilon) \in A$ and $(\varepsilon, 1, (1+\varepsilon^2)/2\varepsilon) \in B$. Thus $(0,1,0)$ is in the closure of $A + B$, and $A + B$ is not closed. ◀

14.4.5 When is the Sum of Closed Convex Sets Closed?

One sufficient condition for the sum of closed convex sets to be closed in $\mathbb{R}^L$ is that they are contained in a closed convex set that contains no straight lines (Choquet, 1962).

Choquet's Theorem. *If Z is convex, weakly complete, and contains no straight lines, then for any weakly closed sets $X, Y \subset Z$, $X + Y$ is also weakly closed.*

The term "weakly" in the theorem refer to the weak topology. In $\mathbb{R}^L$, the weak topology and the norm topology are the same, and Z being complete in $\mathbb{R}^L$ is equivalent to Z being closed.

Both Examples 14.4.4 and 14.4.5 fail to satisfy this criterion. In Example 14.4.4, any convex set containing A and B must include $A + B$, which contains the straight line $y = \alpha$ for any $\alpha > 1$, while entire z-axis is contained in $A \cup B$ in Example 14.4.5.[10]

[10] There are numerous sufficient conditions in the literature for the sum of closed convex sets to be closed. For example, Debreu (1959) required that the asymptotic cones of the sets be positively semi-independent. See also Fan (1965), Dieudonné (1966), Jameson (1972), and Khan and Vohra (1987).

14.4.6 Irreversibility and Closed Sums

Another sufficient condition for the sum of closed sets in normed vector spaces to be closed is irreversibility.

Proposition 14.4.6. *Let* A *and* B *be closed convex sets containing zero in a normed vector space* $(V, \|\cdot\|)$. *If their sum* $C = A + B$ *is irreversible, it is closed.*

Proof. Let $\mathbf{x}$ be in the closure of $A + B$. Then there are $\mathbf{a}^n \in A, \mathbf{b}^n \in B$ with $\mathbf{x}^n = \mathbf{a}^n + \mathbf{b}^n \to \mathbf{x}$. We have two cases to consider depending on whether the $\|\mathbf{a}^n\|$ and $\|\mathbf{b}^n\|$ have convergent subsequences.

Case I: At least one of $\{\|\mathbf{a}^n\|\}, \{\|\mathbf{b}^n\|\}$ has a convergent subsequence. Without loss of generality, we may assume there is a subsequence with $\|\mathbf{a}^{n_j}\|$ convergent. But then $\{\mathbf{a}^{n_j}\}$ is bounded, and also has a convergent subsequence. We retain notation and also denote this new subsequence $\mathbf{a}^{n_j}$. But then $\mathbf{b}^{n_j} = \mathbf{x}^{n_j} - \mathbf{a}^{n_j}$ is also convergent. As the limits of $\mathbf{a}^{n_j}$ and $\mathbf{b}^{n_j}$ are in A and B, respectively, $\mathbf{x} \in C$. In this case, C is closed.

Case II: Otherwise, no subsequence of $\|\mathbf{a}^n\|$ or $\|\mathbf{b}^n\|$ is convergent, and by passing to a subsequence, we may assume that $\|\mathbf{a}^n\|, \|\mathbf{b}^n\| \to \infty$. Then $\mathbf{x}^n/\|\mathbf{a}^n\| = \mathbf{a}^n/\|\mathbf{a}^n\| + \mathbf{b}^n/\|\mathbf{a}^n\| \to 0$. Since $\mathbf{a}^n/\|\mathbf{a}^n\|$ is bounded, it has a convergent subsequence $\{\mathbf{a}^{n_j}/\|\mathbf{a}^{n_j}\|\}$. Moreover, its limit $\bar{\mathbf{a}}$ must be non-zero as it has norm one. Now take J large enough that $1/\|\mathbf{a}^{n_j}\| < 1$ for $j > J$. The convexity of A and B together with the fact that $\mathbf{0} \in A, B$ imply $\mathbf{a}^{n_j}/\|\mathbf{a}^{n_j}\| \in A$ and $\mathbf{b}^{n_j}/\|\mathbf{a}^{n_j}\| \in B$ for $j > J$.

As A is a closed, $\bar{\mathbf{a}} \in A$. Clearly $\mathbf{b}^{n_j}/\|\mathbf{a}^{n_j}\| \to -\bar{\mathbf{a}}$ and $-\bar{\mathbf{a}} \in B$. Now $A, B \subset C$, so both $\bar{\mathbf{a}} \in C$ and $-\bar{\mathbf{a}} \in C$. Since $\bar{\mathbf{a}} \neq \mathbf{0}$, this contradicts the assumption that C is irreversible. Only case I is possible and so C is closed. $\square$

14.4.7 Closing the Sum of Production Sets

By induction, the same is true of finitely many closed convex sets. Proposition 14.4.6 lets us conclude that the sum of production sets is closed when the sum is irreversible.

Theorem 14.4.7. *Suppose that each Y_f is closed, convex, and obeys inaction, and that $Y = \sum_{f=1}^{F} Y_f$ obeys irreversibility. Then Y is closed.*

Proof. Clearly Y is non-empty since $0 \in Y$ (by inaction). The set Y is also convex as the sum of convex sets. By Proposition 14.4.6, Y is is closed. $\square$

14.4.8 Summing Production Sets

We can use this to show the needed result, that the sum of production sets is a production set when aggregate production is irreversible.

Theorem 14.4.8. *Suppose the each* Y_f *is a production set and that* $Y = \sum_{f=1}^{F} Y_f$ *obeys irreversibility. Then* Y *is a production set.*

Proof. Non-emptiness, inaction, and free disposal are obvious. Corollary 14.4.7 showed that Y is closed, while Proposition 14.4.2 established shows there is no free lunch. $\square$

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14.4.9 Positive Semi-Independence

Irreversibility is not the only way to show that the sum of closed sets is closed. As you may gather from Proposition 14.4.3, the only obstacles to the sum being closed involve sequences diverging to infinity. That hints that the asymptotic cone might be useful, and it is. We start with a definition.

Positively Semi-Independent Cones. A collection $A_1, \dots, A_n$ of non-empty cones in $\mathbb{R}^L$ are *positively semi-independent cones* if for every collection of $\mathbf{x}_i$ with $\mathbf{x}_i \in A_i$ and $\sum_i \mathbf{x}_i = 0$, we must have $\mathbf{x}_i = 0$ for all $i = 1, \dots, n$.

14.4.10* Irreversibility and Positive Semi-Independence I

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In a bit we will use a condition involving positive semi-independence instead of irreversibility of their sum to ensure that the sum of closed sets is closed. We need to take a moment to see how the two concepts are related. First, we look at an example showing they are not the same.

Example 14.4.9: In $\mathbb{R}^2$, let $A_1 = \{t\boldsymbol{e}^1\}$ be the horizontal axis and $A_2 = \{t\boldsymbol{e}^2\}$ be the vertical axis. These cones are convex and obey inaction. Now $A = A_1 + A_2$ is reversible. The sets A_1 and A_2 are also positively semi-independent since if $s\boldsymbol{e}^1 + t\boldsymbol{e}^2 = \mathbf{0}$, then $s = t = 0$. ◀

In Example 14.4.9, we sum cones that are themselves reversible. The actual condition will be applied to asymptotic cones, which automatically implies they obey inaction. But then, each $A_i \subset A = A_1 + \cdots + A_n$. So if A is irreversible, so is each A_i, unlike the example above.

14.4.11* Irreversibility and Positive Semi-Independence II

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If the sum of cones obeying inaction is irreversible, then the cones are positively semi-independent.

Proposition 14.4.10. *Let a collection $A_1, \dots, A_n$ of cones in $\mathbb{R}^L$ obey inaction. If $A = A_1 + \cdots + A_n$ is irreversible, the A_i are positively semi-independent and each A_i is irreversible.*

Proof. Let $x_i \in A_i$ with $\sum_i x_i = 0$ with at least one $x_i \neq 0$. Without loss of generality, we can assume $x_1 \neq 0$. By inaction, both $x_1 \in A$ and $x = \sum_{i=2}^n x_i \in A$. Since both are non-zero and $x_1 + x = 0$, irreversibility of A implies $x_1 = x = 0$. We then consider $\sum_{i=2}^n x_i = 0$. Repeating the argument, we find $x_2 = \sum_{i=3}^n x_i = 0$. We continue until until we run out of vectors, showing that each $x_i = 0$. Therefore the A_i are positively semi-independent.

Inaction implies each $A_i \subset A$, and each A_i inherits irreversibility from A. □

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14.4.12* Irreversibility and Positive Semi-Independence III

The following example shows that the converse fails. Irreversibility of each cone A_i together with positive semi-independence, does not imply the sum of the A_i is irreversible, even when every A_i obeys inaction.

Example 14.4.11: Let $A_1 = \{t(2,-1) : t \geq 0\} \cup \{t(-2,-1) : t \geq 0\}$ and $A_2 = \{t(0,1) : t \geq 0\}$. These cones are positively semi-independent because both the pairs $\{(0,1),(2,-1)\}$ and $\{(0,1),(-2,-1)\}$ are both linerly independent. Moreover, both A_1 and A_2 are irreversible. Nonetheless, the sum $A = A_1 + A_2$ is reversible. To see this, observe that both $(2,0) = (0,1) + (2,-1)$ and $(-2,0) = (0,1) + (-2,-1)$ are in A. ◀

14.4.13* Irreversibility and Positive Semi-Independence IV

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We need something more than irreversibility of the A_i to get irreversibility of the sum from positive semi-independence. That something more is additivity. Don't forget that for cones, additivity is equivalent to convexity. When we do have additivity, positive semi-independence of the A_i together with irreversibility of each A_i implies irreversibility of the sum.

Proposition 14.4.12. *Let $A_1, \dots, A_n$ be a collection of positively semi-independent additive cones in $\mathbb{R}^L$. If each A_i is irreversible, so is $A = A_1 + \cdots + A_n$.*

Proof. Suppose there is $\mathbf{x} \in A$ with $-\mathbf{x} \in A$. We can then write $\mathbf{x} = \sum_i \mathbf{x}_i$ and $-\mathbf{x} = \sum_i \mathbf{x}'_i$ with $\mathbf{x}_i, \mathbf{x}'_i \in A_i$. Then $\mathbf{0} = \mathbf{x} - \mathbf{x} = \sum_i (\mathbf{x}_i + \mathbf{x}'_i)$. Because each A_i is additive, $\mathbf{x}_i + \mathbf{x}'_i \in A_i$. We can now appeal to positive semi-independence, to find that each $\mathbf{x}_i + \mathbf{x}'_i = \mathbf{0}$. The irreversibility of each A_i, then shows that $\mathbf{x}_i = \mathbf{x}'_i = \mathbf{0}$, which establishes irreversibility of A. □

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14.4.14* Sums of Closed Sets, Again

We may now state another result concerning sums of closed sets. This is from Debreu (1959, sec. 1.9) and can also be found in Border (1985, Corollary 2.41), which contains a proof for the case $n = 2$ (Proof 2.38, the rest follows by induction). We omit the proof itself, which is in the same spirit as the proof of Proposition 14.4.6.

Proposition 14.4.13. *Let $A_1, \dots, A_n$ be closed non-empty sets in $\mathbb{R}^L$. If the asymptotic cones $\mathbf{A}(A_i)$ are positively semi-independent, then $A = A_1 + \cdots + A_n$ is closed.*

14.5 Market Supply

Now that we know that irreversibility implies that the aggregate production set Y is in fact a production set, we can consider the profit and supply functions π and $\mathbf{y}$ for aggregate production as well as the profit and supply functions π_f and $\mathbf{y}^f$ for each individual firm.[11]

The following set of results show that the aggregate production set represents the firms in the sense that its profit is the sum of the firms' profits and its net output is the market net output, whenever both concepts make sense.

Provided they all make sense, the profit and supply functions for the aggregate production set are the same as the market profit and supply functions, as shown in Propositions 14.5.1 and 14.5.2.

Proposition 14.5.1. *Let Y_f, $f = 1, \dots, F$, and $Y = Y_1 + \cdots + Y_F$ be production sets. The corresponding profit functions obey $\pi(\mathbf{p}) = \sum_f \pi_f(\mathbf{p})$ for all $\mathbf{p}$.*

Proof. Let $\mathbf{y} \in Y$. Then there are $\mathbf{y}^f \in Y_f$ with $\mathbf{y} = \sum_f \mathbf{y}^f$. Dotting with $\mathbf{p}$, we obtain

$$\mathbf{p}\cdot\mathbf{y} = \sum_f \mathbf{p}\cdot\mathbf{y}^f \le \sum_f \pi_f(\mathbf{p}).$$

Now take the supremum over $\mathbf{y} \in Y$ to obtain $\pi(\mathbf{p}) \le \sum_f \pi_f(\mathbf{p})$.

Now $\pi(\mathbf{p}) = \sup_{\mathbf{y}\in Y} \mathbf{p}\cdot\mathbf{y} \ge \sum_f \mathbf{p}\cdot\mathbf{y}^f$ for any $\mathbf{y}^f \in Y_f$. Since each $\mathbf{y}^f$ can be chosen independently, $\pi(\mathbf{p}) \ge \sum_f \pi_f(\mathbf{p})$. Combining the two inequalities yields the result. $\square$

One consequence of this is that the aggregate profit function is finite if and only if each of the firms has a finite profit function. Keep in mind that finite profit is necessary for profit to be maximizable, but it is not sufficient unless the technology is constant returns to scale.

[11] When dealing with supply functions, we use superscripts to denote the firm and subscripts to denote goods in order to make the notation clearer.

14.5.1 Aggregate Supply

The second proposition shows that aggregate profits can be maximized if and only if each of the firms can maximize profits.

Proposition 14.5.2. *Let Y_f, $f = 1, \dots, F$, and $Y = Y_1 + \cdots + Y_F$ be production sets. Then $\mathbf{y}(\mathbf{p})$ exists if and only if all of the $\mathbf{y}^f(\mathbf{p})$ exist. In that case, $\mathbf{y}(\mathbf{p}) = \sum_f \mathbf{y}^f(\mathbf{p})$.*

Proof. First suppose all the $\mathbf{y}^f(\mathbf{p})$ exist and let $\mathbf{y}^f \in \mathbf{y}^f(\mathbf{p})$ for each f. Then $\pi(\mathbf{p}) = \sum_f \pi_f(\mathbf{p}) = \sum_f \mathbf{p}\cdot\mathbf{y}^f = \mathbf{p}\cdot\big(\sum_f \mathbf{y}^f\big)$. Thus $\sum_f \mathbf{y}^f$ maximizes profit over Y, showing that $\mathbf{y}(\mathbf{p})$ exists and $\mathbf{y}(\mathbf{p}) \supset \sum_f \mathbf{y}^f(\mathbf{p})$.

Now suppose that $\mathbf{y}(\mathbf{p})$ exists. Let $\mathbf{y} \in \mathbf{y}(\mathbf{p})$. There are $\mathbf{y}^f \in Y_f$ with $\mathbf{y} = \sum_f \mathbf{y}^f$. Then $\pi(\mathbf{p}) = \mathbf{p}\cdot\mathbf{y} = \sum_f \mathbf{p}\cdot\mathbf{y}^f$ and each $\mathbf{p}\cdot\mathbf{y}^f \le \pi_f(\mathbf{p})$. By Proposition 14.5.1, $\sum_f \pi_f(\mathbf{p}) = \pi(\mathbf{p})$. It follows that $\mathbf{p}\cdot\mathbf{y}^f = \pi_f(\mathbf{p})$ for each f. It follows that each $\mathbf{y}^f(\mathbf{p})$ exists. Moreover, $\mathbf{y}(\mathbf{p}) \subset \sum_f \mathbf{y}^f(\mathbf{p})$.

Combining the two results completes the proof. □

Corollary 14.5.3. *Under the above conditions, market net output $\mathbf{y}(\mathbf{p})$ obeys the Law of Supply.*

Proof. For each firm, $(\hat{\mathbf{p}} - \mathbf{p})\cdot(\hat{\mathbf{y}}^f - \mathbf{y}^f) \ge 0$ whenever $\hat{\mathbf{y}}^f \in \mathbf{y}^f(\hat{\mathbf{p}})$ and $\mathbf{y}^f \in \mathbf{y}^f(\mathbf{p})$. Sum over all $f = 1, \dots, F$ to and apply Proposition 14.5.2 to prove the result. □

The firms' profits sum to aggregate profit regardless of whether the result is finite or infinite. If profit can be maximized, it must be finite. We now turn our attention to the question of when aggregate profit is finite.

14.5.2 Adding Profit Functions

The aggregate profit will be finite if and only if the polars of the individual production sets have a non-trivial intersection.

Proposition 14.5.4. *Suppose $Y_1, \dots, Y_F$ are production sets and $Y = \sum_f Y_f$. Let π_f and π be the associated profit functions. Then $\pi(\mathbf{p})$ is finite if and only if every $\pi_f(\mathbf{p})$ is finite. Moreover, there is a non-trivial $\mathbf{p}$ with $\pi(\mathbf{p})$ finite if and only if the intersection of the Y_f° includes something other than $\mathbf{0}$.*

Proof. Suppose $\pi(\mathbf{p})$ is finite. Then $\mathbf{p} \in \text{cone}(Y^\circ)$. By Proposition 14.2.13, $\mathbf{p} \in \bigcap_f \text{cone}(Y_f^\circ)$, so every $\pi_f(\mathbf{p})$ is finite.

Conversely, suppose every $\pi_f(\mathbf{p})$ is finite. Then $\mathbf{p} \in \bigcap_f \text{cone}(Y_f^\circ)$. By Proposition 14.2.13, we find $\mathbf{p} \in \text{cone}(Y^0)$, which implies $\pi(\mathbf{p})$ is finite.

If $\bigcap_f Y_f^\circ$ is non-trivial, take a price vector $\mathbf{p} \neq \mathbf{0}$ in the intersection. Then profit $\pi(\mathbf{p})$ is finite.

Conversely, suppose each firms' profit $\pi_f(\mathbf{p})$ is finite. Because $\mathbf{p} \in \text{cone}(Y_f^\circ)$ for each f, we can find multipliers $\mu_f > 0$ with $\mu_f \mathbf{p} \in Y_f^\circ$.

Let $\mu = \max_f \mu_f$. Since each Y_f° is convex and contains $\mathbf{0}$, $\mathbf{p}' = (1/\mu)\mathbf{p} = (\mu_f/\mu)\mathbf{p}_f \in Y_f^\circ$ for each f. Then $\mathbf{p}' \neq \mathbf{0}$ and $\pi(\mathbf{p}')$ is finite. $\square$

14.5.3 What if the Intersection is Zero?

Example 14.5.5: Example 14.4.1 can be used to see how the theorem works. Recall that the production sets were $Y_1 = \{(y_1, y_2) : y_2 \le -2y_1, y_1 \le 0\}$ and $Y_2 = \{(y_1, y_2) : y_1 \le -2y_2, y_2 \le 0\}$, yielding $Y_1 + Y_2 = \mathbb{R}^2$. There is a free lunch and profit cannot be maximized at any non-zero price vector.

By Proposition 14.5.4, the intersection of the polar cones must be trivial. Computing them, we obtain $Y_1^\circ = \{\mathbf{p} \in \mathbb{R}^2_+ : p_1 \ge 2p_2\}$ and $Y_2^\circ = \{\mathbf{p} \in \mathbb{R}^2_+ : p_2 \ge 2p_1\}$. Now if $\mathbf{p} \in Y_1^\circ \cap Y_2^\circ$, $p_1 \ge 2p_2 \ge 4p_1 \ge 0$, implying that $p_1 = p_2 = 0$. This shows $Y_1^\circ \cap Y_2^\circ = \{\mathbf{0}\}$, so $Y^\circ = \{\mathbf{0}\}$, exactly as Proposition 14.5.4 requires. ◀

14.5.4 Is Aggregate Profit Finite?

One last question remains. Are there non-trivial price vectors in cone(Y°)? Irreversibility does the trick here too. It guarantees that are there non-trivial price vectors where aggregate profits are finite. If the aggregate production set is irreversible, we can find a non-zero price in Y° where the profit function is finite for every firm.

Theorem 14.5.6. *Suppose Y_f are convex production sets and $Y = \sum_f Y_f$ is irreversible. Then Y° contains a non-zero element $\mathbf{p}$. Moreover, $\pi(\mathbf{p})$ and $\pi_f(\mathbf{p})$ are all finite.*

Proof. Let $\mathbf{x} \in Y$ with $\mathbf{x} \neq \mathbf{0}$. Such will exist by free disposal. Then $-\mathbf{x} \notin Y$. Now Y and $\{-\mathbf{x}\}$ are disjoint, closed, non-empty convex sets. Separation Theorem A yields a non-zero $\mathbf{p}$ and α with $-\mathbf{p}\cdot\mathbf{x} > \alpha$ and $\mathbf{p}\cdot\mathbf{y} < \alpha$ for all $\mathbf{y} \in Y$. By inaction, $\alpha > 0$. It follows that $\alpha^{-1}\mathbf{p} \in Y^\circ$. $\square$

This does not quite say there is common price where all firms can maximize profit. As we saw in Example 14.2.1, profit can be finite without being maximizable when there are decreasing returns to scale.

14.5.5 CRS Aggregate Profit Maximization

Things are simpler under constant returns. Profit can be maximized whenever the profit function is constant. In that case, Theorem 14.5.6 tells us that aggregate irreversibility guarantees there is at least one non-zero price vector where all firms can simultaneously maximize profit.

Theorem 14.5.7. *Suppose* Y_f *are convex production sets exhibiting constant returns to scale. If* $Y = \sum_f Y_f$ *is irreversible, then* Y° *contains a non-zero element. Moreover, for every non-zero* $\mathbf{p} \in Y^\circ$*, the set of profit-maximizing net outputs for firm* f*,* $\mathbf{y}^f(\mathbf{p})$*, is non-empty and their sum is the set of profit-maximizing net outputs for the aggregate production set* Y*:* $\mathbf{y}(\mathbf{p}) = \sum_f \mathbf{y}^f(\mathbf{p})$*.*

Proof. Combine Theorem 14.5.6 with Theorem 14.1.6. □

This is one of the reasons it is convenient to homogenize production by introducing entrepreneurial factors. We will see that there are other advantages in Chapter 16.

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