25. Equilibrium over Time

It is trivial to add time to our models. Just add a t subscript! This does not change much. Equilibrium still requires that firms maximize profits, consumers maximize utility, and markets clear. There's just one thing wrong. It ignores everything that is important about time.

What adding a simple t subscript ignores is the structure imposed by time itself—the arrow of time.[1] Events are temporally ordered. Inputs precede outputs. Once actions are taken, they are fixed in history. The past cannot be changed. Any serious intertemporal model must be based on this temporal structure.

One important fact is that actions planned for the future are still subject to revision. We can ask whether production and consumption plans are *time consistent*. Once we get to time t, do we wish to continue to follow the plan? Or should we revise our plans in view of history?

Outline:

Intertemporal preferences are the subject of section one. Section two examines the consumer's problem. Producers are modeled with inputs preceding outputs in section three. Section four investigates efficient production and supporting prices. The two sides of the market are brought together in section five. Equilibrium is defined in the usual way, and must be Pareto optimal. This allows us to recast determination of market equilibrium as an optimal growth problem, maximizing utility given a production technology.

[1] The term "arrow of time" was introduced by Arthur Eddington in the 1927 Gifford lectures (Eddington, 1928).

25.1 Intertemporal Preferences

Before discussing intertemporal preferences, we have to consider the commodity space. We know that we will be considering **sequences** of consumption vectors, but we haven't settled on the planning horizon, or even whether there is one. How far into the future do we plan?

One easy answer is "until we die", but that still doesn't tell us when. Further, some people plan beyond their deaths—they write wills dictating the disposition of their property. We will finesse the problem of a horizon by adopting an unbounded planning horizon. If we need to, we can explicitly model the chance of death by using a probability distribution.

25.1.1 Paths, Programs, Streams

We will work in a world with infinitely many discrete time periods $t = 0, 1, 2, \ldots$. Production and consumption *plans* can also be referred to as *paths*, *programs*, or *streams*. The term "plans" suggests something that hasn't happened yet, and is most appropriate when focusing on future choices. The other terms suggest something more concrete, that has already occurred. At any moment of time, consumption streams embody both ideas. They include our not yet realized future plans for the future as well as the choices we've made in the past.

No matter how we think of them, consumption streams will be sequences of vectors in $\mathbb{R}^m$. We denote the set of such vectors as $\mathfrak{s}^m$. Throughout this chapter, the commodity space will be a vector subspace of $\mathfrak{s}^m$ and consumption sets will be the positive orthant of that subspace.

A consumption stream has the form $\mathbf{c} = (\mathbf{c}_0, \mathbf{c}_1, \ldots)$ where $\mathbf{c}_t \in \mathbb{R}^m_+$ is the consumption bundle in period t. When necessary, we will use double subscripts, listing the good first and time period second. The consumption of good ℓ at time t in indicated by $c_{\ell t}$.

A production path is denoted $\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \ldots)$ where $\mathbf{y}_t \in \mathbb{R}^m$ is net output at time t. We write the endowment stream as a sequence of endowments in the various periods, $\boldsymbol{\omega} = (\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \ldots)$. The goods $\boldsymbol{\omega}_t$ are only available at time t, not before, and not after. The endowment can't be used before it arrives (although it may be possible to use it as collateral), and cannot be stored except by use of an explicit storage technology (with corresponding production set).

25.1.2 Properties of Intertemporal Preferences

Consumer i's preferences $\succsim_i$ are defined over consumption streams. Because there are an infinite number of time periods, the range of possibilities for preferences is vast. We will make several assumptions that limit the range of possibilities.

We will require that intertemporal preferences be

1. Monotonic
2. Continuous
3. Strongly Separable over Time
4. Time Stationary
5. Impatient

Let's consider these five properties in more detail.

25.1.3 Monotonicity

We start with monotonicity. Since $\mathfrak{s}^m$ has a natural ordering ($\geq$) inherited from the real numbers, the use of monotonicity is no surprise. The definition is the same as for $\mathbb{R}^m$. We extend the order coordinatewise. A preference order is *monotonic* if $\mathfrak{x} \succsim \mathfrak{y}$ when $\mathfrak{x} \geq \mathfrak{y}$ and $\mathfrak{x} \succ \mathfrak{y}$ whenever $\mathfrak{x} \gg \mathfrak{y}$. Here $\mathfrak{x} \geq \mathfrak{y}$ means that $x_{\ell t} \geq y_{\ell t}$ for every ℓ and t and $\mathfrak{x} \gg \mathfrak{y}$ means that $x_{\ell t} > y_{\ell t}$ for every ℓ and t.

25.1.4 Continuity

Continuity is another matter entirely. When dealing with finite dimensional spaces, there is only one reasonable topology that is compatible with the vector operations, the usual topology on $\mathbb{R}^m$.[2]

When we move to infinite dimensional spaces, this is no longer true. This multiplicity of topologies only gets worse when we consider infinite dimensional subspaces of $\mathfrak{s}^m$. A detailed examination of these issues is beyond the scope of this book.[3]

One possibility is to use $\mathfrak{s}^m$ with the product topology. The *product topology* on $\mathfrak{s}^m$ is defined by $\mathbf{x}^n \to \mathbf{x}$ if and only if $\mathbf{x}^n_t \to \mathbf{x}_t$ for every t.

Another possibility is to use a weighted ℓ^∞ space as in Boyd (1990) and Becker and Boyd (1997). For any $\beta > 0$, we define the β-*weighted supremum norm* by

$$\|\mathbf{x}\|_\beta = \sup_{\ell,t} \left\{ \left| \frac{x_{\ell t}}{\beta^t} \right| \right\}$$

and the associated β-weighted ℓ^∞ space by

$$\ell^\infty(\beta) = \{\mathbf{x} \in \mathfrak{s}^m : \|\mathbf{x}\|_\beta < \infty\}.$$

The point of the β-weighted spaces is to tailor commodity streams to the possibilities of growth. If $\beta = 1$, only bounded streams are allowed. This would be appropriate when there is a hard limit on consumption. If sustained economic growth at the rate $r > 0$ is possible, we can set $\beta = 1 + r$.

The Monotone Representation Theorem can be easily adapted to the β-weighted spaces, allowing us to represent preferences by a continuous utility function U. Debreu's Representation Theorem also applies there, as well as to the positive orthant of $\mathfrak{s}^m$ itself. In either case, there is a continuous utility representation.

[2] More precisely, if we have a finite dimensional vector space, the Euclidean topology is the only Hausdorff topology that makes it a topological vector space.

[3] This is touched on in Becker and Boyd (1997).

25.1.5 Strongly Separable over Time

The third condition is separability. We assume that preferences are strongly separable over time. To do this, we define the commodity groups P_t, the goods available at each time t, by $P_t = \{(\ell t) : \ell = 1, \dots, m\}$ and require that preferences be strongly separable relative to the partition into time periods, $\mathcal{P} = \{P_0, P_1, \dots\}$. We will also require that the commodity groups P_t are essential for at least three time periods.

These two assumptions mean that a form of Debreu's Separability Theorem applies.[4] We may then represent preferences using in the time additive form

$$U(\mathbf{c}) = \sum_{t=0}^{\infty} u_t(\mathbf{c}_t).$$

Later, we will introduce a weaker separability condition that allows use of a broader class of preferences.

[4] Streufert (1993, 1995) has investigated various types of time separability in an infinite horizon context, while Gorman (1968) has done so for finite horizon models.

25.1.6 Time Stationarity

Point four is that preferences are time stationary in the sense of Koopmans (1960). By *time stationary*, we mean that the ranking of two paths, $\mathbf{c} \succsim \mathbf{c}'$ is unchanged when both are delayed a period in time, and preceded by the same consumption vector $\mathbf{x}_0$, so $(\mathbf{x}_0, \mathbf{c}) \succsim (\mathbf{x}_0, \mathbf{c}')$. This is also true for any finite delay in consumption.

Time Stationarity. A preference order $\succsim$ is *time stationary* if for for all admissible consumption streams $\mathbf{c}$ and $\mathbf{c}'$, $\mathbf{c} \succsim U(\mathbf{c}')$ if and only if $(\mathbf{x}_0, \mathbf{c}) \succsim (\mathbf{x}_0, \mathbf{c}')$ for every $\mathbf{x}_0 \in \mathbb{R}^m_+$.

Now suppose $U(\mathbf{c}) = \sum_0^\infty u_t(\mathbf{c}_t)$ be a time additive utility representation of time stationary preferences $\succsim$. Let's normalize utility by replacing each $u_t(\mathbf{c}_t)$ by $u_t(\mathbf{c}_t) - u_t(\mathbf{0})$, which is zero at zero. This changes the overall utility level by $\sum_t u_t(\mathbf{0}) = U(\mathbf{0})$, which doesn't have any affect on preferences. Without loss of generality, we can assume $u_t(\mathbf{0}) = 0$ for every time period t.

Given our time additive representation, time stationarity translates to

$$U(\mathbf{c}) = \sum_{t=0}^{\infty} u_t(\mathbf{c}_t) \geq \sum_{t=0} u_t(\mathbf{c}'_t) = U(\mathbf{c}')$$

if and only if

$$u_0(\mathbf{x}_0) + \sum_{t=0}^{\infty} u_{t+1}(\mathbf{c}_t) \geq u_0(\mathbf{x}_0) + \sum_{t=0} u_{t+1}(\mathbf{c}'_t).$$

But then, $U(\mathbf{c}) \geq U(\mathbf{c}')$ if and only if

$$\sum_{t=0}^{\infty} u_{t+1}(\mathbf{c}_t) \geq \sum_{t=0} u_{t+1}(\mathbf{c}'_t).$$

25.1.7 Consequences of Time Stationarity

We now have two additive separable ways of representing preferences,

$$U(\mathbf{c}) = \sum_{t=0}^{\infty} u_t(\mathbf{c}_t) \quad \text{and} \quad V(\mathbf{c}) = \sum_{t=0}^{\infty} u_{t+1}(\mathbf{c}_t).$$

Theorem 3.2.9 can be generalized to tell us that each of these representations is an increasing affine transformation of the other, $V = \delta U + a$. Applying this to the constant stream $(\mathbf{0}, \mathbf{0}, \dots)$ shows that the constant term is zero. To see this, consider that $0 = U(\mathbf{0}) = u_0(\mathbf{0}) + V(\mathbf{0}) = 0 + \delta U(\mathbf{0}) + a = a$. In other words, $V = \delta U$ for some $\delta > 0$.

By considering streams that are equal to $\bar{\mathbf{c}}$ except in one time period, we find $\delta u_t(\mathbf{c}_t) = u_{t+1}(\mathbf{c}_t)$ for all times t. Induction yields $\delta^t u_0(\mathbf{c}_t) = u_t(\mathbf{c}_t)$. Assuming that there are no difficulties with the infinite sums, we can then write the utility of the consumption stream $\mathbf{c}$ as[5]

$$U(\mathbf{c}) = \sum_{t=0}^{\infty} \delta^t u(\mathbf{c}_t). \tag{25.1.1}$$

This is the well-understood *time additive separable (TAS)* utility. Consumers receive utility in each period from their consumption in that period, independently of consumption in other periods. This utility is described by a *period utility function* (or *felicity function*), a function $u\colon \mathbb{R}^m_+ \to \mathbb{R}$. The period utility converts our consumption stream $\mathbf{c}$ to a utility stream $\big(u(\mathbf{c}_0), u(\mathbf{c}_1), \dots\big)$. We will normally require that the felicity function be continuous, increasing, and concave.

[5] Our assumption about the infinite sums is not entirely harmless. It is connected with the choice of a commodity space and topology. This gets more complicated if we exclude $\mathbf{0}$ from the commodity, as must be done in cases such as logarithmic utility.

25.1.8 Impatience

We will usually make a fifth assumption—consumers are impatient. All things being equal, they value present consumption more than future consumption. There are several ways to define impatience. One is the notion of *time perspective*, where a given change in felicity has less impact on overall utility when it is farther in the future. We call it time perspective by analogy with visual perspective. Just as the fixed distance between train tracks seems to be smaller and smaller as the tracks recede toward the horizon, a fixed increment in utility gives a smaller and smaller effect as it is moved toward the time horizon.[6]

Here's how it works. Start with a felicity stream $\mathbf{u}$, form $\mathbf{u}'$ by adding an increment Δu to stream $\mathbf{u}$ at time s and form $\mathbf{u}''$ by adding the same increment Δu to stream $\mathbf{u}$ at a later time t. Let $U(\mathbf{u})$ denote the overall utility obtained from the felicity stream $\mathbf{u}$. Then

$$|U(\mathbf{u}') - U(\mathbf{u})| > |U(\mathbf{u}'') - U(\mathbf{u})|.$$

When applied to the time additive separable form, this reduces to $\delta^s|\Delta u| > \delta^t|\Delta u|$ for $s < t$, implying $\delta < 1$.

When $\delta < 1$ we refer to δ as the *discount factor*. Since $0 < \delta < 1$, we can define $\rho > 0$ with $\delta = (1 + \rho)^{-1}$, or equivalently, $\rho = \delta^{-1} - 1$. Here ρ is called the *discount rate*.

Without impatience, it could happen that $\delta > 1$. The future would be *upcounted* instead of discounted.

[6] Although the idea of impatience dates back at least to Rae (1834), it seems to have had little impact until the late 1800's. Böhm-Bawerk (1912) and Fisher (1930) both considered it an important cornerstone of the theory of interest, and it remains a key component of many modern intertemporal models. The idea of discounting utility has not gone unchallenged. Ramsey (1928a) argued (pg. 543) that discounting was a "practice which is ethically indefensible and arises merely from the weakness of the imagination." This did not stop him from investigating the effects of discounting in his models.

25.1.9 Representation of Intertemporal Preference

We can sum up this part of the discussion as follows.

Representation of Intertemporal Preferences. If preferences are (1) monotonic, (2) suitably continuous, (3) strongly separable over time, and (4) stationary, then they have a time additive separable representation as in equation 25.1.1. If the consumer is also (5) impatient in the sense of time perspective, the multiplier δ is a discount factor—it obeys $0 < \delta < 1$.[7]

$$U(\mathbf{c}) = \sum_{t=0}^{\infty} \delta^t u(\mathbf{c}_t). \tag{25.1.1}$$

We haven't stated this as a theorem because of the technical issues previously discussed regarding the commodity space and its topology. See Boyd (1990) and Becker and Boyd (1997) for more.

[7] This is not stated as a theorem because of the technical issues previously discussed regarding the commodity space and its topology that are beyond the scope of this work. See Boyd (1990) and Becker and Boyd (1997).

25.1.10 Homothetic Intertemporal Preferences

Sometimes preferences are made even more specific. When the models are compared to data, it is common to further restrict preferences to be homothetic. When there is only one consumption good in each time period, homotheticity implies that the felicity function u has the constant elasticity of substitution form (equivalently, the constant relative risk aversion form)

$$u(c) = \begin{cases} \frac{1}{1-\sigma}c^{1-\sigma} & \text{when } \sigma > 0,\ \sigma \neq 1 \\ \ln c & \text{when } \sigma = 1 \end{cases} \tag{25.1.2}$$

or any increasing affine transformation of u.[8] The fact that $\sigma > 0$ implies that u and U are strictly concave. You may recognize this as a form of Bergson's Separability Theorem.

[8] See Rader (1981).

25.1.11 The Utility Stream

Another way of thinking about time additive separable utility is to construct overall utility from the utility stream itself. Period utility (felicity) is discounted to the present at a constant rate ρ (discount factor $\delta = (1 + \rho)^{-1}$) and then summed over all future periods. Overall utility can be thought of as the present value of the felicity stream at discount rate ρ.

Although it is not the only way to construct preferences over time. Time additive separable utility is used because it is intuitive; it allows us to build tractable models; it is easy to introduce uncertainty by taking the expectation of time additive separable utility.

25.1.12 Fisher on Intertemporal Utility

For another perspective on time additive utility, we can think in terms of the overall utility we would receive if the consumption stream starting at time t were started today instead. Denote the consumption stream from time t onward by ${}_t\mathbf{c} = (\mathbf{c}_t, \mathbf{c}_{t+1}, \ldots)$. The overall utility at time t is then $U_t = U({}_t\mathbf{c})$. This allows us to write overall utility as a combination of current felicity and future overall utility. In the time additive case we have:

$$U_t = u(\mathbf{c}_t) + \delta U_{t+1}.$$

This defines overall utility recursively as a function of the felicity stream. Stationarity is automatically built-in when we use a recursion like this to define utility.

More generally, one can even think of indifference curves between current felicity and future utility, much as Fisher (1930) drew indifference curves between current consumption and future wealth, which he sometimes seemed to interpret in utility terms.

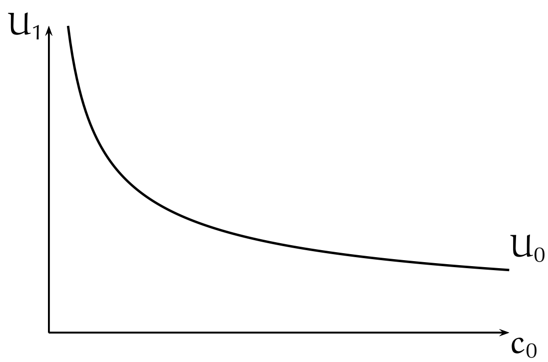

Figure 25.1.1: A Fisherian indifference curve based on Koopmans recursive utility. Current consumption is on the horizontal axis, future utility on the vertical axis, and the curve is $W(c_0, U_1) = U_0$.

25.1.13 Koopmans on Intertemporal Utility

An immediate question is whether the TAS assumption that the indifference curves are linear is too strong. One way to relax that assumption is via Koopmans's formulation of recursive utility.[9]

Uzawa (1968), Epstein and Hynes (1983), and Lucas and Stokey (1984) showed that recursive utility can be successfully applied to a number of economic questions, and that it may have significant effects on long-run outcomes. They all postulate an increasing aggregator function that combines present felicity and future utility to obtain present overall utility.

Koopmans postulated an *aggregator function* W which defines U_t by

$$U_t = W\big(u(\mathbf{c}_t), U_{t+1}\big).$$

Time additive separable utility is the special case $W(u, U) = u + \delta U$.

It is natural to ask which aggregator functions generate utility functions. Answers in varying degrees of generality have been given by Lucas and Stokey (1984), Boyd (1990a), and Streufert (1990, 1993). See also Becker and Boyd (1997), Streufert (1998), and Boyd (2006).

[9] See Koopmans 1960, 1972a, 1972b; Koopmans, Diamond and Williamson, 1964.

25.1.14 Properties of Recursive Utility

Any utility generated by an aggregator is necessarily time stationary. This follows from the fact that $W(u(\mathbf{c}), U) \geq W(u(\mathbf{c}), U')$ for all admissible $\mathbf{c}$ if and only if $U \geq U'$.

The aggregator representation means that preferences are weakly separable relative to each partition of the form

$$\mathcal{P}_t = \left\{ P_0, P_1, \dots, P_t, \bigcup_{s=t+1}^{\infty} P_s \right\}.$$

However, preferences need not be strongly separable over the full temporal partition, unlike time additive separable utility.

The fact of stationarity, that future utility has the same form as present utility plays a key role in showing consumption plans are time consistent. We can relate time 0 and time 1 utilities by $U(\mathbf{c}) = u(\mathbf{c}_0) + \delta U({}_1\mathbf{c})$, or more generally $U_t = u(\mathbf{c}_t) + \delta U_{t+1}$. This is a useful formulation for doing dynamic programming, both for time additive separable utility and for general recursive utility.

One key difference between general recursive utility and time additive separable utility is that the later requires that long-run capital supply be perfectly elastic when the interest rate equals the rate of impatience. We can see that by examining the marginal rate of substitution when $c_t = c_{t+1} = \cdots$. Then $\text{MRS}_\ell^{s,s+1} = 1/(1+\rho)$ for every $s \geq t$. This need not be the case under Koopmans's recursive utility.

This affects the long-run behavior of the economy. It not only affects convergence to steady states or balanced growth paths, but also has an impact on long-run income distribution and incidence and efficiency of taxes on capital.

25.2 The Intertemporal Consumer's Problem

April 6, 2023

Problems Problems 22.4.3, 22.6.3, 22.6.4, 23.1.4, and 23.3.4 are due on Thursday, April 13..

The consumer's budget set is determined by time zero wealth W and a stream of prices $(\mathbf{p}_t)$ with $\mathbf{p}_t \gg \mathbf{0}$. As in the finite dimensional case, we require that prices be strictly positive in order to bound the budget set and make utility maximization possible. The consumer's budget set is then

$$B(\mathbf{p}, W) = \left\{ \mathbf{c} : \mathbf{c}_t \geq 0,\ \sum_{t=0}^{\infty} \mathbf{p}_t \cdot \mathbf{c}_t \leq W \right\}.$$

Since W is time zero wealth, the $\mathbf{p}_t$ are the time zero prices of goods consumed in period t. We refer to these as the *present value prices*. The sum $\sum_t \mathbf{p}_t \cdot \mathbf{c}_t$ is the present value of the consumption stream $\mathbf{c}$ itself.

The consumer's problem is to choose a consumption stream that maximizes utility $U(\mathbf{c})$ over the budget set $B(\mathbf{p}, W)$. Writing out the details, the consumer solves

$$\begin{aligned} v(\mathbf{p}, W) = \max\ & \sum_{t=0}^{\infty} \delta^t u(\mathbf{c}_t) \\ \text{s.t.}\ & \sum_{t=0}^{\infty} \mathbf{p}_t \cdot \mathbf{c}_t \leq W \\ & \mathbf{c}_t \geq \mathbf{0}, \text{for } t = 0, 1, 2, \ldots \end{aligned}$$

The form of the budget constraint emphasizes that the $\mathbf{p}_t$ are the prices of goods when purchased at time zero, not the prices when purchased at time t. In exchange economies, W will be the present value of the consumer's endowment:

$$W = \sum_t \mathbf{p}_t \cdot \boldsymbol{\omega}_t.$$

In production economies we would additionally include the present value of the consumer's shares of the firms.[10]

[10] Becker and Boyd (1997) refer to such an equilibrium as a Fisher competitive equilibrium. In their model this is equivalent to assuming perfect foresight.

25.2.1 Solving The Intertemporal Consumer's Problem

When preferences are homothetic, demand is linear in income, just as in the case of ordinary Marshallian demand.

We can solve the consumer's problem by setting up the Lagrangian in the usual way.

$$\mathcal{L} = U(\mathbf{c}) - \lambda \left[\sum_{t=0}^{\infty} \mathbf{p}_t \cdot \mathbf{c}_t - W \right] + \sum_{t=0}^{\infty} \sum_{\ell=1}^{m} \mu_{\ell t} c_{\ell t}$$

The first order conditions are then

$$\frac{\partial U}{\partial c_{\ell t}} + \mu_{\ell t} = \lambda p_{\ell t},$$

or equivalently $D_{\mathbf{c}_t} U + \boldsymbol{\mu}_t = \lambda \mathbf{p}_t$.[11]

We will focus on the interior case where $c_{\ell t} > 0$ for all ℓ and t. Then $\mu_{\ell t} = 0$ by complementary slackness. Since we are using time additive separable preferences, we can write this as $\delta^t Du(\mathbf{c}_t) = \lambda \mathbf{p}_t$. Now

$$\frac{\partial U}{\partial c_{\ell t}} = \delta^t \frac{\partial u}{\partial c_{\ell t}}(\mathbf{c}_t) = \lambda p_{\ell t}.$$

[11] There can be considerable mathematical complications here due to the infinite sums, but we will focus on cases where they are not a problem for deriving the first order conditions. Theorem 25.2.1 gives conditions when the first order conditions and budget constraint are sufficient for an optimum.

25.2.2 Marginal Rates of Substitution

We now can eliminate λ from the equations

$$\delta^t \frac{\partial u}{\partial c_{\ell t}}(\mathbf{c}_t) = \lambda p_{\ell t}.$$

by dividing within periods, using the (k, t) term and the (ℓ, t) term, and dividing between adjacent periods by using the (ℓ, t) term and the $(\ell, t+1)$ term.

We then obtain the usual condition that each marginal rate of substitution must equal the corresponding relative price.

The first set of first order conditions apply **within** each period. They use the marginal rate of substitution between good consumed in the same time period. That marginal rate of substitution must equal the relative price of the two goods at time t. The relevant equations are

$$\text{MRS}^t_{k\ell} = \frac{\partial u/\partial c_{k,t}}{\partial u/\partial c_{\ell,t}} = \frac{p_{k,t}}{p_{\ell,t}},$$

where both derivatives are evaluated at $\mathbf{c}_t$

The second set of first order conditions applies **between** time periods. They use the marginal rate of substitution between the same good consumed in adjacent time periods, which must be equal to the relative price of that same good between those two periods. Thus

$$\text{MRS}^{t,t+1}_{\ell} = \delta \frac{\partial u/\partial c_{\ell,t+1}}{\partial u/\partial c_{\ell,t}} = \frac{p_{\ell,t+1}}{p_{\ell,t}},$$

where the derivatives are evaluated at $\mathbf{c}_{t+1}$ and $\mathbf{c}_t$.

One could also consider marginal rates of substitution between different goods in different periods, or the same good in more distantly separated periods, but those conditions can be stated by multiplying together the appropriate within and between period marginal rates of substitution. They would add no further information.

25.2.3 Current Value Prices

Current Value Prices. Alternatively, we can use the discount rate to express prices in current value terms. The *current value prices* $\mathbf{q}_t$ are defined by *upcounting* the present value prices to time t: $\mathbf{q}_t = \delta^{-t}\mathbf{p}_t$. These are the prices that exist at time t. Conversely, we can find the present value prices by discounting the current value prices at rate $\rho = \delta^{-1} - 1$, obtaining $\mathbf{p}_t = \delta^t \mathbf{q}_t$.

In current value terms, the first order conditions become

$$\frac{q_{k,t}}{q_{\ell,t}} = \frac{\partial u/\partial c_{k,t}}{\partial u/\partial c_{\ell,t}} \quad \text{and} \quad \frac{q_{\ell,t+1}}{q_{\ell,t}} = \frac{\partial u/\partial c_{\ell,t+1}}{\partial u/\partial c_{\ell,t}}.$$

The discount factor no longer appears in the first order conditions.

In the one-good case, the first order conditions are now

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{q_{t+1}}{q_t}.$$

When $u'' < 0$, there is a simple relation between the current price and consumption in the one-good case. When $q_{t+1} < q_t$, the first order conditions tell us that $u'(c_{t+1}) < u'(c_t)$. Diminishing marginal utility then implies $c_{t+1} > c_t$. Similarly, if $q_{t+1} > q_t$, we must have $c_{t+1} < c_t$. Of course, if $q_{t+1} = q_t$, then $c_{t+1} = c_t$.

25.2.4 Sufficient Conditions for a Solution

When combined with the budget constraint, the first order conditions are sufficient to solve the consumer's problem.[12]

Theorem 25.2.1. *Suppose* u *is* $\mathcal{C}^1$ *and concave with* $Du \gg 0$. *Let* $\mathbf{c}^* \gg 0$. *If*

1. $U(\mathbf{c}^*)$ *is finite,*
2. $\sum_t \mathbf{p}_t \cdot \mathbf{c}_t^* = W$, *and*
3. *there is a* $\lambda > 0$ *with* $\delta^t Du(\mathbf{c}_t^*) = \lambda \mathbf{p}_t$,

then $\mathbf{c}^*$ *solves the consumer's problem.*

Proof. Suppose $\mathbf{c}'$ is in the budget set. By concavity of u,

$$u(\mathbf{c}_t') \le u(\mathbf{c}_t^*) + Du(\mathbf{c}_t^*) \cdot (\mathbf{c}_t' - \mathbf{c}_t^*)$$

by the Support Property Theorem. Multiplying by δ^t and using $\delta^t Du(\mathbf{c}_t) = \lambda \mathbf{p}_t$, we find

$$\delta^t u(\mathbf{c}_t') \le \delta^t u(\mathbf{c}_t^*) + \lambda \mathbf{p}_t \cdot (\mathbf{c}_t' - \mathbf{c}_t^*).$$

Now sum over t and use assumption (2) to obtain

$$U(\mathbf{c}') \le U(\mathbf{c}^*) + \lambda \left[\sum_t \mathbf{p}_t \cdot \mathbf{c}_t' - W \right].$$

Because $\mathbf{c}'$ is in the budget set, the last term is non-positive, implying $U(\mathbf{c}') \le U(\mathbf{c}^*)$. Since $U(\mathbf{c}^*)$ is finite, so is the left-hand side, and $\mathbf{c}^*$ maximizes utility over the budget set. □

[12] Those familiar with optimal growth problems may wonder why there is no transversality condition. For the this problem, the budget constraint plays the role of the transversality condition. The fact that the present value of consumption is finite implies the transversality condition $\mathbf{p}_t \cdot \mathbf{c}_t \to 0$.

25.2.5 Solving the Single-good Case

We focus on the intertemporal aspect by considering the case of a single good in each time period. We then have the simpler form:

$$\delta \frac{u'(c_{t+1})}{u'(c_t)} = \frac{p_{t+1}}{p_t}.$$

Notice how the marginal utility from period $(t + 1)$ must be discounted when forming the marginal rate of substitution $MRS_{t,t+1}$.

These first order conditions are necessary for an interior optimum. With finitely many goods we only need concavity to make the first order conditions together with the budget constraint sufficient for optimality. That happens here too. We normally think of infinite horizon problems as needing an extra condition, the transversality condition. However, when dealing with the consumer's problem, that condition is already implied by the budget constraint itself.

25.2.6 Consumer's Problem: Constant Interest Rate I

Consider the problem of a consumer facing a constant interest rate $r > 0$.

Example 25.2.2: Consumer's Problem with Constant Interest Rate. Since there is only one good, we can think of the current value price as being \$1 in each period. Discounted to the present at interest rate r, the present value price in period t is $p_t = (1+r)^{-t}$. We assume that felicity obeys $u' > 0$ and $u'' < 0$.

The first order conditions are

$$\delta \frac{u'(c_{t+1})}{u'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{(1+r)^{-t-1}}{(1+r)^{-t}} = \frac{1}{1+r}.$$

We can rewrite these conditions as

$$(1+r)\delta u'(c_{t+1}) = u'(c_t). \tag{25.2.3}$$

Equation (25.2.3) makes sense. We must balance the value of a little more consumption today and the extra value we would get from consumption tomorrow by instead saving that amount and earning interest. The right hand side gives today's marginal value, while the left hand side includes the growth in consumption due to interest $(1+r)$ and that fact that future marginal utility must be discounted by δ.

It will be helpful to write the first order conditions in terms of the discount rate ρ defined by

$$\delta = \frac{1}{1+\rho}.$$

We have

$$\left(\frac{1+r}{1+\rho}\right) u'(c_{t+1}) = u'(c_t).$$

25.2.7 Consumer's Problem: Constant Interest Rate II

There are three possibilities: $\rho = r$, $\rho > r$, $\rho < r$. We consider each in turn.

Case I: If $\rho = r$, the first order conditions become $u'(c_{t+1}) = u'(c_t)$. Because marginal utility is diminishing ($u'' < 0$), $c_t = c_{t+1}$ for all $t = 0, 1, \dots$. Thus $c_t = c_0$ for every t. We now use the budget constraint to solve the model.

$$\begin{aligned} W &= \sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} p_t c_0 \\ &= c_0 \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} = c_0 \left(\frac{1}{1 - \frac{1}{1+r}} \right) \\ &= c_0 \frac{1+r}{r} \end{aligned}$$

Thus $c_t = c_0 = rW/(1+r)$. As $\rho = r$, this can also be written $c_t = \rho W/(1+\rho)$ or even $c_t = (1-\delta)W$.

25.2.8 Consumer's Problem: Constant Interest Rate III

Case 2: If $\rho > r$, we use the first order conditions

$$\left(\frac{1+r}{1+\rho}\right) u'(c_{t+1}) = u'(c_t)$$

to find $u'(c_{t+1}) > u'(c_t)$. By decreasing marginal utility, it follows that $c_{t+1} < c_t$.

Case 3: Finally, if $\rho < r$, a similar argument shows that $c_{t+1} > c_t$.

In cases 1 and 2, consumption is bounded and $U(\mathbf{c})$ is finite as required by Theorem 25.2.1. There might be a problem in case 3 as $U(\mathbf{c})$ may not be finite. In many applications, this is not a problem. We will see this in our logarithmic example. In general,

$$\sum_t \delta^t u\left(\frac{W}{p_t}\right) = \sum_t \delta^t u\big(W(1+r)^t\big) < +\infty$$

is quite sufficient to show utility is finite.

25.2.9 An Example with Logarithmic Felicity I

The results above hold for any felicity function. To get more detail out of the model requires using a specific felicity function. We set $u(c) = \ln c$. The first order conditions then become

$$\left(\frac{1+r}{1+\rho}\right)\frac{1}{c_{t+1}} = \frac{1}{c_t}$$

which can be rewritten as

$$\left(\frac{1+r}{1+\rho}\right)c_t = c_{t+1}.$$

Defining $\beta = (1+r)/(1+\rho)$ we obtain $c_t = \beta^t c_0$ and $p_t \beta^t = (1+\rho)^{-t}$.

25.2.10 An Example with Logarithmic Felicity II

The budget constraint is $\sum_t p_t c_t \le W$, or

$$W \ge \sum_t p_t\left(\beta^t c_0\right) = \sum_t (1+\rho)^{-t} c_0 = \frac{1}{\rho}(1+\rho)c_0.$$

The budget constraint yields $c_0 = \rho W/(1+\rho)$ giving us the optimal consumption stream

$$c_t = \beta^t \frac{\rho W}{1+\rho}.$$

Of course when $r = \rho$, consumption remains constant, consumption converges to zero when $r < \rho$, and consumption grows at a constant rate when $r > \rho$.

Here $U(\mathbf{c})$ is finite even when consumption grows as

$$U(\mathbf{c}) = \sum_t t\delta^t \ln\beta + \sum_t \delta^t\left(\ln(\rho W) - \ln(1+\rho)\right),$$

which is finite. ◀

25.2.11 Example: Arbitrary Prices I SKIPPED

Interestingly, some utility functions allow us to completely solve the consumer's problem.

Example 25.2.3: Arbitrary Prices. Suppose there is one consumption good in each time period and the felicity function is $u(c) = \ln c$. We will solve the consumer's problem for arbitrary non-zero prices. The first order conditions are

$$\frac{p_{t+1}}{p_t} = \delta \frac{u'(c_{t+1})}{u'(c_t)} = \delta \frac{c_t}{c_{t+1}}.$$

That implies $p_{t+1}c_{t+1} = \delta p_t c_t$. By induction, $p_t c_t = \delta^t p_0 c_0$. Substituting in the budget constraint, we find

$$p_0 c_0 \sum_{t=0}^{\infty} \delta^t = W, \quad \text{or} \quad \frac{p_0 c_0}{1-\delta} = W.$$

It follows that $p_0 c_0 = (1-\delta)W$.

25.2.12 Example: Arbitrary Prices II SKIPPED

But then spending at time t is $p_t c_t = \delta^t p_0 c_0 = \delta^t(1-\delta)W$. From this, we obtain the demand functions

$$c_t = \frac{\delta^t}{p_t}(1-\delta)W.$$

This is essentially a Cobb-Douglas demand function. In each time period t, the consumer spends a share $\delta^t(1-\delta)$ of initial wealth. These are shares because they are non-negative and sum to one: $\sum_{t=0}^{\infty}\delta^t(1-\delta) = 1$.

Another way to look at this is to use current value prices $q_t = \delta^{-t}p_t$. In that case, demand takes the exact same form in every period and is linear in wealth W.

$$c_t = \frac{1}{q_t}(1-\delta)W.$$

25.2.13 Example: Arbitrary Prices III SKIPPED

We can also compute indirect utility. Here

$$u(c_t) = \ln c_t = t \ln \delta - \ln p_t + \ln \big[(1-\delta)W\big],$$

so

$$\begin{aligned} v(\mathbf{p}, W) &= (\ln \delta) \sum_{t=0}^{\infty} t\delta^t - \sum_{t=0}^{\infty} \delta^t \ln p_t + \frac{\ln \big[(1-\delta)W\big]}{1-\delta} \\ &= \frac{\delta \ln \delta}{(1-\delta)^2} + \frac{\ln \big[(1-\delta)W\big]}{1-\delta} - \sum_{t=0}^{\infty} \delta^t \ln p_t \\ &= \frac{\delta \ln \delta}{(1-\delta)^2} + \frac{\ln(1-\delta)}{1-\delta} + \sum_{t=0}^{\infty} \delta^t \ln \big(W/p_t\big) \end{aligned}$$

where we have used the fact that

$$\frac{d}{d\delta} \sum_{t=0}^{\infty} \delta^t = \sum_{t=0}^{\infty} t\delta^{t-1}$$

to compute $\sum t\delta^t$. In the last line we have rewritten indirect utility in terms of W/p_t to emphasize its homogeneity of degree zero in $(\mathbf{p}, W)$.

25.2.14 Example: Arbitrary Prices IV SKIPPED

The analogy with ordinary demand is pretty complete. If we have the indirect utility function, we can even use Roy's Identity to find demand at each time period. The present value version is

$$-\frac{\partial v/\partial p_t}{\partial v/\partial W} = -\frac{-\delta^t/p_t}{1/(1-\delta)W} = \frac{\delta^t}{p_t}(1-\delta)W = c_t.$$

In terms of current value prices, indirect utility has the simpler form

$$v(\mathbf{q}, W) = \frac{\ln(1-\delta)}{1-\delta} + \sum_{t=0}^{\infty} \delta^t \ln\left(W/q_t\right).$$

Whenever utility is homothetic, meaning that felicity has the form of equation 25.1.2, demand will be linear in income and indirect utility will have the same form as felicity (up to an affine transformation). ◀

25.3 Intertemporal Production

We now turn to the production side. We will focus on cases where the technology operates from one period to the next. Inputs at time t result in outputs at time $(t+1)$. This simple input-output technology can described by a technology set $T_t \subset \mathbb{R}^{2m}_+$. We interpret the first m coordinates as the input to production (in time t) and the last m coordinates as the output in period $(t+1)$. Let $\mathbf{a}_t \in \mathbb{R}^m_+$ denote the inputs at t and $\mathbf{b}_t \in \mathbb{R}^m_+$ denote the outputs at t. Then the pair $(\mathbf{a}_t, \mathbf{b}_{t+1})$ will be *feasible* if $(\mathbf{a}_t, \mathbf{b}_{t+1}) \in T_t$ for every $t = 0, \ldots$.

Input-Output Technology Set. A set $T_t \subset \mathbb{R}^{2m}_+$ is an *input-output technology set* if it obeys

(T1) Non-emptiness: $T_t \neq \emptyset$.
(T2) Closure: T_t is closed.
(T3) No free lunch: $\mathbf{b} = \mathbf{0}$ whenever $(\mathbf{0}, \mathbf{b}) \in T_t$.
(T4) Inaction: $(\mathbf{0}, \mathbf{0}) \in T_t$.
(T5) Free disposal: If $(\mathbf{a}, \mathbf{b}) \in T_t$ and both $\mathbf{a}' \geq \mathbf{a}$ and $\mathbf{0} \leq \mathbf{b}' \leq \mathbf{b}$, then $(\mathbf{a}', \mathbf{b}') \in T_t$.
(T6) Productivity: There is $(\mathbf{a}, \mathbf{b}) \in T_t$ with $\mathbf{b} \gg \mathbf{0}$.
(T7) Convexity: T is a convex set.

25.3.1 Intertemporal Technology and Production Set

A collection of input-output sets is called an *intertemporal technology*.

Intertemporal Technology. An *intertemporal technology* is a collection of input-output technology sets $\mathbf{T} = \{T_t : t = 0, 1, \ldots\}$. An intertemporal technology is *stationary* if $T_t = T_{t+1}$ for all t. In that case we denote the common technology set by T.

We use the intertemporal technology to build the production set. Given an intertemporal technology $\mathbf{T} = \{T_t : t = 0, 1, \ldots\}$, we define the corresponding production set by

$$Y = \Big\{\mathbf{b} - \mathbf{a} : \mathbf{b}_0 = \mathbf{0}, (\mathbf{a}_t, \mathbf{b}_{t+1}) \in T_t \text{ for all } t = 0, 1, 2, \ldots \Big\}.$$

In other words, $\mathbf{y} \in Y$ if there are $\mathbf{a}$, $\mathbf{b}$ with

$$\mathbf{y}_t = \begin{cases} -\mathbf{a}_0 & \text{when } t = 0 \\ \mathbf{b}_t - \mathbf{a}_t & \text{otherwise} \end{cases}$$

One consequence is that $\mathbf{0} \in Y$ if $\mathbf{T}$ obeys inaction. In other words, Y also obeys inaction. Except for closure, it is fairly straightforward to show that Y obeys all five properties of a production set. Moreover, it obeys convexity and productivity.

The following examples illustrate how the construction of Y from $\mathbf{T}$ works.

25.3.2 Storage Technology I

We start with a simple storage technology. Goods may be stored in period t for use in period $(t+1)$. One option at time $(t+1)$ is to put the goods back in storage.

Example 25.3.1: Storage Technology. The storage technology is a stationary technology defined by the input-output technology set

$$T = \Big\{(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2m}_{+} : \mathbf{b} \leq \mathbf{a}\Big\}.$$

The set T obeys all 7 technology conditions, (T1)–(T7). This technology allows us to store goods for one period. We can put a bundle of goods in at time t and get them back as "output" in period $(t+1)$. If we wish to store goods for several periods, we can repeatedly use the storage technology.

Here $\mathbf{y}_t = \mathbf{b}_t - \mathbf{a}_t \leq \mathbf{a}_{t-1} - \mathbf{a}_t$ for $t = 1, 2, \ldots$ and $\mathbf{y}_0 = -\mathbf{a}_0 \leq \mathbf{0}$. The vector $\mathbf{y}_t$ is the net amount removed from storage in period t. Now

$$\begin{aligned}
\sum_{s=0}^{t} \mathbf{y}_s &= \sum_{s=0}^{t} \big(\mathbf{b}_s - \mathbf{a}_s\big) \\
&\leq \sum_{s=1}^{t} \mathbf{a}_{s-1} - \sum_{s=0}^{t} \mathbf{a}_s \\
&\leq -\mathbf{a}_t \leq \mathbf{0}.
\end{aligned}$$

since $\mathbf{b}_0 = 0$ and $\mathbf{b}_s \leq \mathbf{a}_{s-1}$ for $s = 1, 2, \ldots$.

25.3.3 Storage Technology II

This shows that

$$Y \subset \left\{ y : \sum_{s=0}^{t} y_s \leq 0 \text{ for all } t \right\}.$$

In other words, the net amount removed in every time period must be less than zero. You take out no more than you've put in.

Conversely, if $\sum_{s=0}^{t} y_s \leq 0$ for all t, we can set

$$a_t = -\sum_{s=0}^{t} y_s \geq 0 \quad \text{and} \quad b_t = a_{t-1}.$$

Then $b_{t+1} - a_t \leq a_t - a_t = 0$ which implies $(a_t, b_{t+1}) \in T$. Moreover,

$$\begin{aligned} b_t - a_t &= a_{t-1} - a_t \\ &= -\sum_{s=0}^{t-1} y_s + \sum_{s=0}^{t} y_s \\ &= y_t \end{aligned}$$

implying $y \in Y$. It follows that

$$Y = \left\{ y : \sum_{s=0}^{t} y_s \leq 0 \text{ for all } t \right\}.$$

◀

25.3.4 One-sector Technology I

The one-sector model gives us the classic optimal growth problem. There is a diminishing returns production function that is available in each period. At each time, there is one good available which may be consumed or invested in production for future use.

Example 25.3.2: One-sector Technology. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a $\mathcal{C}^2$ production function with $f(0) = 0$, $f' > 0$ and $f'' < 0$. Define a stationary technology using the input-output set

$$T = \left\{(a, b) \in \mathbb{R}^2_+ : b \leq f(a)\right\}.$$

as illustrated in Figure 25.3.3

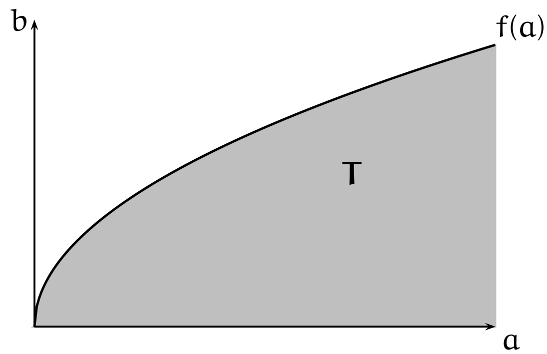

Figure 25.3.3: Unlike production sets, input-output sets are drawn in conventional fashion, with both inputs and outputs in the positive orthant. This input-output technology set uses a production function f obeying f(0), $f' > 0$ and $f'' < 0$. It satisfies properties (T1)–(T7).

25.3.5 One-sector Technology II

The continuity of f ensures T is closed while concavity ($f'' < 0$) means T is convex. Both inaction and the no free lunch condition follow from $f(0) = 0$ while $f' > 0$ implies the productivity condition is satisfied.

A stream $\mathbf{y}$ is in Y if there are sequences a_t, b_t with $b_{t+1} \leq f(a_t)$ for all $t = 0, 1, 2, \ldots$ and $y_t \leq b_t - a_t$ where $b_0 = 0$. We can remove b_t from the equation by restating the condition on y_t: $y_{t+1} \leq f(a_t) - a_{t+1}$ for $t = 1, 2, \ldots$ and $y_0 = -a_0$. This allows us to write the production set as

$$Y = \Big\{\mathbf{y} : y_t \leq f(a_{t-1}) - a_t, t = 1, 2, \ldots \text{ and } y_0 = -a_0\Big\}.$$

We interpret a_t as the *capital stock* at time t. It denotes the goods set aside for use as inputs in time t that will yield output in period $(t + 1)$. ◄

25.3.6 Time-Varying Technology

A variant of the one-sector model allows the production to change over time. We let $f_t : \mathbb{R}_+ \to \mathbb{R}_+$ be a $\mathcal{C}^2$ production function with $f_t(0) = 0$, $f'_t > 0$ and $f''_t < 0$. Set $T_t = \{(a, b) \in \mathbb{R}^2_+ : b \leq f_t(a)\}$ for $t = 0, 1, 2, \ldots$. Of course, the production set is

$$Y = \Big\{ \mathbf{y} : y_t \leq f_{t-1}(a_{t-1}) - a_t, t = 1, 2, \ldots \text{ and } y_0 = -a_0 \Big\}.$$

One traditional application is the case of exogenous technical progress.

Example 25.3.4: Exogenous Technical Progress. One time-varying technology is the case of exogenous technical progress, where production possibilities grow at rate $\alpha > 0$. This is described by $f_t(a) = e^{\alpha t} f(a)$. ◀

25.4 Efficiency of Intertemporal Production

The definition of production efficiency is the same for intertemporal production sets as for ordinary production sets. A net output stream $\mathbf{y} \in Y$ is *efficient* if there is no $\mathbf{y}' \in Y$ with $\mathbf{y}' > \mathbf{y}$. When Y is generated by an intertemporal technology $\mathbf{T} = \{T_t\}$, each technology set is used independently. As a result, efficiency in Y will imply a type of efficiency in each period.

Lemma 25.4.1. *Let* Y *be a production set generated by an intertemporal technology* $\mathbf{T} = \{T_t\}$*. Suppose* $\mathbf{y}^* = \mathbf{b}^* - \mathbf{a}^* \in Y$ *is efficient. Then there is no time* t *when there is a* $(\mathbf{a}', \mathbf{b}') \in T_t$*,* $(\mathbf{a}', \mathbf{b}') \neq (\mathbf{a}^*_t, \mathbf{b}^*_{t+1})$ *with* $\mathbf{a}' \leq \mathbf{a}^*_t$ *and* $\mathbf{b}' \geq \mathbf{b}^*_{t+1}$*. In other words,* $(\mathbf{a}^*_t, \mathbf{b}^*_{t+1})$ *is efficient in* T_t*.*

Proof. If such $(\mathbf{a}', \mathbf{b}') \in T_r$ were available, we could use the new value to increase $\mathbf{y}$ by using $\mathbf{a}_r = \mathbf{a}'$ and $\mathbf{b}_{r+1} = \mathbf{b}'$ with $\mathbf{a}_s = \mathbf{a}^*_s$ and $\mathbf{b}_{s+1} = \mathbf{b}^*_{s+1}$ for $s \neq r$. Then $\mathbf{y}_t = \mathbf{b}_t - \mathbf{a}_t$ will be the same as $\mathbf{y}^*_t$ in periods other than r and $r+1$, and it will increase in one or both of the periods r and $r+1$. This means $\mathbf{y} > \mathbf{y}^*$, which is impossible because $\mathbf{y}^*$ is efficient. This shows there is no time period r where such $(\mathbf{a}', \mathbf{b}')$ are available. $\square$

25.4.1 Malinvaud Prices

With slight modifications, Theorem 13.5.4 applies to obtain prices $(\mathbf{p}_t^*, \mathbf{p}_{t+1}^*) > \mathbf{0}$ so that

$$\mathbf{p}_{t+1}^* \cdot \mathbf{b}_{t+1}^* - \mathbf{p}_t^* \cdot \mathbf{a}_t^* \geq \mathbf{p}_{t+1}^* \cdot \mathbf{b}_{t+1} - \mathbf{p}_t^* \cdot \mathbf{a}_t$$

for all $(\mathbf{a}_t, \mathbf{b}_{t+1}) \in T_t$. In other words, the efficient choice $(\mathbf{a}_t^*, \mathbf{b}_{t+1}^*)$ maximizes profit over T_t. Of course, this profit is obtained in period $t+1$, when the output is sold, not in period t when the firm purchases inputs.

This naturally leads to the question of whether there is a price stream $\mathbf{p}^*$ with $(\mathbf{a}_t^*, \mathbf{b}_{t+1}^*)$ maximizing profit at every time t whenever $\mathbf{y}^* = \mathbf{b}^* - \mathbf{a}^*$ is efficient. Malinvaud (1953, 1962) addressed a slightly weaker question. Given an efficient stream $\mathbf{y}^* = \mathbf{a}^* - \mathbf{b}^*$, is there a set of prices so that $\mathbf{y}^*$ maximizes profit in each period? Or equivalently, is there a set of prices $\mathbf{p}^*$ with

$$\sum_{t=0}^{\infty} \mathbf{p}_t^* \cdot (\mathbf{y}_t^* - \mathbf{y}_t) \geq 0$$

for every $\mathbf{y} \in Y$ that differs from $\mathbf{y}^*$ at only **finitely many periods**. Such prices are called *Malinvaud prices*.

Malinvaud Prices. A price stream $\mathbf{p}^* = (\mathbf{p}_t^*)$ is a system of *Malinvaud prices* at $\mathbf{y}^* = \mathbf{b}^* - \mathbf{a}^*$ if for each time t, $\mathbf{p}_{t+1}^* \cdot \mathbf{b}_{t+1}^* - \mathbf{p}_t^* \cdot \mathbf{a}_t^* \geq \mathbf{p}_{t+1}^* \cdot \mathbf{b}_{t+1} - \mathbf{p}_t^* \cdot \mathbf{a}_t$ for all $(\mathbf{a}, \mathbf{b}) \in T_t$.

25.4.2 Do Malinvaud Prices Exist?

It is easy enough to use a separation argument to find a system of prices where profit is being maximized at a given efficient output stream over the first T periods. However, if the price ray is that supports T_t is not unique, different time horizons may yield different prices. Even if it is unique, there may still be issues picking the appropriate levels in each time period. As a result, it takes something more than a simple limiting argument to put those prices together over an infinite time horizon.

In fact, Malinvaud's original argument was incorrect (Malinvaud, 1953). Malinvaud (1962) introduced the concept now known as tightness and showed that if $\mathbf{y}$ was not tight in every pair of adjacent time periods, then a system of Malinvaud prices would exist. Kurz (1969) further explored the notion of tightness providing an equivalent definition (Kurz 1969, Theorem 1) and connected it to McFadden's (1967) notion of reachability. McFadden (1975) provided an example where production was tight and Malinvaud prices do not exist.

25.4.3 Tight Streams SKIPPED

To define tightness, we first partition goods into two groups, the goods that cannot be produced, and the goods that can be produced. The same division will apply in every time period. We denote the producible goods by a subscript p and the non-producible goods by a subscript n. Thus we can write $\mathbf{a}_t = (\mathbf{a}_{n,t}, \mathbf{a}_{p,t})$ and $\mathbf{b}_t = (\mathbf{b}_{n,t}, \mathbf{b}_{p,t})$.

The basic idea is that a stream is not tight if we can reduce the inputs of non-producible goods at time t and still increase the output of produced goods in the next period $t + 1$. This will typically mean that the produced inputs in period t must be larger, allowing them to not only substitute for the reduction in non-producible inputs, but allow greater output than before.

Tight and Non-tight Streams. An input output pair $(\mathbf{a}^*, \mathbf{b}^*) \in T_t$ is *not-tight* if there is an input output pair $(\mathbf{a}, \mathbf{b}) \in T_t$ obeying $\mathbf{a}_{n,t} \ll \mathbf{a}^*_{n,t}$, and $\mathbf{b}_{p,t+1} \gg \mathbf{b}^*_{p,t+1}$.

A stream $\mathbf{y}^* = \mathbf{b}^* - \mathbf{a}^*$ is *not-tight* if each input output pair $(\mathbf{a}^*_t, \mathbf{b}^*_{t+1}) \in T_t$ is not tight. A stream is $\mathbf{y}* = \mathbf{b}^* - \mathbf{a}^*$ is *tight* if it fails to be non-tight.

25.4.4 Malinvaud Prices Exist

When an efficient stream is non-tight, Malinvaud prices exist (Malinvaud, 1962).

Theorem 25.4.2. *Suppose* Y *is generated from an intertemporal technology* $\mathbf{T} = \{T_t\}$. *If* y^* *is efficient and non-tight for every* t*, there is a price stream* $p^* > 0$ *with*

$$\sum_{t=0}^{\infty} p_t^* \cdot (y_t^* - y_t) \geq 0$$

for every $y \in Y$ *that differs from* y^* *at only finitely many periods.*

The optimal output stream maximizes profits in each time period when the prices are Malinvaud prices.

Corollary 25.4.3. *Let* $y^* \in Y$ *be as in Theorem 25.4.2 and write* $y^* = b^* - a^*$. *If* p^* *is a corresponding Malinvaud price stream, then*

$$p_{t+1}^* \cdot b_{t+1}^* - p_t^* \cdot a_t^* \geq p_{t+1}^* \cdot b' - p_t^* \cdot a'$$

for all $(b', a') \in T_t$.

Proof. Consider the stream y' obtained from $y^* = b^* - a^*$ by replacing (a_t^*, b_{t+1}^*) with any $(a', b') \in T_t$. This differs from y^* in only the time periods t and $t+1$, so by Theorem 25.4.2

$$\begin{aligned}
0 &\leq \sum_{t=0}^{\infty} p_t^* \cdot (y_t^* - y_t') \\
&= p_{t+1}^* \cdot \big(b_{t+1}^* - a_{t+1}^* - (b' - a_{t+1}^*)\big) + p_t^* \cdot \big(b_t^* - a_t^* - (b_t^* - a')\big) \\
&= p_{t+1}^* \cdot (b_{t+1}^* - b') - p_t^* \cdot (a_t^* - a') \\
&= \big[p_{t+1}^* \cdot b_{t+1}^* - p_t^* \cdot a_t^*\big] - \big[p_{t+1}^* \cdot b' - p_t^* \cdot a'\big],
\end{aligned}$$

establishing the result. □

25.4.5 Discounted Profit and Period Profits

So far, we've focused on profits for each period's input-output technology, but we've haven't examined how these profits relate to value of the profit stream at time zero. To that end, let $\pi_t = \mathbf{p}_{t+1}\cdot\mathbf{b}^*_{t+1} - \mathbf{p}_t\cdot\mathbf{a}^*_t$ denote the maximum profit earned at time $(t+1)$ from inputs at time t. We will refer to this as (discounted) time t profit.

Theorem 25.4.4. *Let $\mathbf{p}_t$ be a Malinvaud price stream for an intertemporal technology $\mathbf{T}$ and suppose $(\mathbf{a}^*_t, \mathbf{b}^*_{t+1})$ maximizes profit from inputs at time t. If $\sum_t \mathbf{p}_t\cdot\mathbf{a}^*_t$ is finite, then $\sum_t \mathbf{p}_t\cdot\mathbf{y}^*_t = \sum_t \pi_t$ is the maximum discounted profit at time zero, where $\mathbf{y}^* = \mathbf{b}^* - \mathbf{a}^*$.*

Proof. When the value of input is finite, $\sum_t \mathbf{p}_t\cdot\mathbf{b}^*_t - \mathbf{p}_t\cdot\mathbf{a}^*_t$ is either finite or positively infinite because the negative terms (the $\mathbf{a}^*$ terms) are summable and the other terms are non-negative. Moreover, the value of the sum does not depend on the order of summation.

We now compute

$$\begin{aligned}\sum_{t=0}^{\infty} \mathbf{p}_t\cdot\mathbf{y}^*_t &= \sum_{t=0}^{\infty}(\mathbf{p}_t\cdot\mathbf{b}^*_t - \mathbf{p}_t\cdot\mathbf{a}^*_t)\\ &= \sum_{t=0}^{\infty}(\mathbf{p}_{t+1}\cdot\mathbf{b}^*_{t+1} - \mathbf{p}_t\cdot\mathbf{a}^*_t)\\ &= \sum_{t=0}^{\infty}\pi_t.\end{aligned}$$

Since $b_0 = 0$, we can rewrite overall profit as shown in the second line, which is the sum of the period profits.

Suppose $\mathbf{y} = \mathbf{b} - \mathbf{a}$ is feasible. Then

$$\begin{aligned}\sum_{t=0}^{\infty} \mathbf{p}_t\cdot(\mathbf{y}^*_t - \mathbf{y}_t) &= \sum_{t=0}^{\infty}\left[\pi_t - \mathbf{p}_t\cdot\mathbf{y}_t\right]\\ &= \sum_{t=0}^{\infty}\left[\pi_t - \mathbf{p}_t\cdot\mathbf{b}_t - \mathbf{p}_t\cdot\mathbf{a}_t\right]\\ &= \sum_{t=0}^{\infty}\left[\pi_t - \mathbf{p}_t\cdot\mathbf{b}_{t+1} - \mathbf{p}_t\cdot\mathbf{a}_t\right] \qquad \text{Since } \mathbf{b}_0 = \mathbf{0}\\ &\geq 0,\end{aligned}$$

Showing that $\mathbf{y}^*$ maximizes discounted profit. □

25.4.6 Profitable but Inefficient Production

Maximizing each period's profit π_t is clearly necessary for maximizing overall profit. But in fact, it may not be sufficient. As we've seen, it can only fail to be sufficient when the discounted value of input is infinite. Let's explore this further.

Example 25.4.5: Profitable but Inefficient Production. Consider the one-sector storage technology $f(a) = a$. The first order conditions imply $p_{t+1} = p_t$. We can normalize the Malinvaud prices to $\mathbf{p} = (1, 1, 1, \dots)$. Now $(a_t, b_{t+1}) = (1, 1)$ maximizes period profits. This yields

$$\begin{aligned} \mathbf{a} &= (1, 1, 1, \dots), \\ \mathbf{b} &= (0, 1, 1, \dots), \text{ and so} \\ \mathbf{y} &= \mathbf{b} - \mathbf{a} = (-1, 0, 0, \dots) \end{aligned}$$

But $\mathbf{y} < \mathbf{0} \in Y$, so $\mathbf{y}$ is not efficient!

A non-trivial way to see that $\mathbf{y}$ is not efficient is to replace (a_t, b_{t+1}) with $(a'_t, b'_{t+1}) = (0, 0)$ for $t = 3, 4, \dots$. This yields

$$\mathbf{y}' = (-1, 0, 0, 1, 0, 0, \dots).$$

When we use $\mathbf{y}'$, we store one unit at time zero and remove it from storage in period three. In contrast, $\mathbf{y}$ stores one unit at time zero and never permanently removes it from storage.

25.4.7 Why Did the First Order Conditions Fail?

How did we go wrong with $\mathbf{y}$? We satisfied all of the first order conditions, but that wasn't enough. One indication that something has gone wrong is the fact that neither $\sum_t p_t b_t$ nor $\sum_t p_t a_t$ converge. Here

$$\sum_t p_t a_t = \sum_t p_t b_t = \sum_t 1 = +\infty.$$

The lack of convergence is only a symptom of the problem. The root of the problem is, even though we've satisfied the first order conditions, our net output stream is not efficient.

Look at the case with endowment $\boldsymbol{\omega} = (1, 0, 0, \ldots)$. If we used $\mathbf{y} = (-1, 0, 0, \ldots)$, we would never consume our endowment, or anything else! We would keep saving our time zero endowment for the future, one period at a time. But we'd never actually use it. We would store it forever! The problem is that by never consuming our capital stock, we make it forever useless. We have **too much capital** in the long run. ◄

25.4.8 The Transversality Condition

That is clearly not optimal. To ensure optimality, we need to prevent this type of *capital over-accumulation*. Using our wealth optimally requires using it while we can still get utility from it. We don't want value left idle at infinity.

So how do we detect capital over-accumulation? The present value of goods used as inputs at time t is $\mathbf{p}_t \cdot \mathbf{a}_t$. In Example 25.4.5, this value is always one, indicating that goods are always left unconsumed. Moreover, the value of the unconsumed goods does not disappear as t increases. In Example 25.4.5, that value remains one.

This suggests that we can eliminate capital over-accumulation by imposing the *transversality condition*:[13]

$$\lim_{t\to\infty} \mathbf{p}_t \cdot \mathbf{a}_t = 0 \quad \text{or} \quad \lim_{t\to\infty} \delta^t \mathbf{q}_t \cdot \mathbf{a}_t = 0,$$

depending on whether we use present or current value prices. To see that they are the same equation, recall that $\mathbf{p}_t = \delta^t \mathbf{q}_t$.

The transversality condition requires that the discounted value of capital remaining at time t converge to zero. It ensures we don't leave any unused value at infinity.

Since the terms $\mathbf{p}_t \cdot \mathbf{a}_t$ are non-negative, their summability implies transversality. Although the converse may hold under additional assumptions, it generally fails.

In Example 25.4.5, prices were constant and $a_t = 1$, so the discounted value of capital at each time is the same. The better, efficient path $\mathbf{y}'$ had $b_{t+1} = a_t = 0$ for all $t = 3, 4, \ldots$, so $\lim p_t a_t = 0$, satisfying the transversality condition.

[13] We will sometimes refer to this as the capital transversality condition.

25.4.9 Consumption Transversality Condition

Interestingly, **two** types of transversality condition hold in Example 25.2.2, is a consumer's problem with a constant interest rate r and logarithmic felicity. The prices are $p_t = (1+r)^{-t}$. We define the *consumption transversality condition* as

$$\lim_{t\to\infty} \mathbf{p}_t\cdot\mathbf{c}_t = 0 \quad \text{or} \quad \lim_{t\to\infty} \delta^t \mathbf{q}_t\cdot\mathbf{c}_t = 0,$$

again depending on whether we use present or current value prices.

In the case $u(c) = \ln c$, we find

$$c_t = \beta^t \frac{\rho W}{1+\rho}$$

regardless of how rate of impatience ρ and interest rate r relate. Here $\beta = (1+r)/(1+\rho)$. Now

$$p_t c_t = \frac{1}{(1+r)^t}\left(\frac{1+r}{1+\rho}\right)^t \frac{\rho W}{1+\rho} = \frac{\rho W}{(1+\rho)^{t+1}}$$

so $p_t c_t \to 0$, satisfying the consumption transversality condition.

Using current value prices, consumption at time t would be worth

$$q_t c_t = \delta^{-t} p_t c_t = (1+\rho)^t \frac{\rho W}{(1+\rho)^{t+1}} = \frac{\rho W}{1+\rho}.$$

For the transversality condition, we would have to multiply by δ^t, and the limit would again be zero. The advantage of using current value prices is that it reveals the stationarity behind the solution to the consumer's problem. Future consumption has the same current value in every time period.

25.4.10 Capital Transversality Condition

The second type of transversality condition comes from thinking about this problem in terms of capital. A natural measure of capital for the consumer's problem is the present value of remaining consumption, the present value of wealth yet unspent. We'll denote the wealth at the beginning of time t by W_t. It is spent on present and future consumption, which has present (i.e., time zero) value

$$\begin{aligned} W_t &= \sum_{s=t}^{\infty} p_s c_s \\ &= \sum_{s=t}^{\infty} \frac{\rho W}{(1+\rho)^{s+1}} \\ &= \frac{\rho W}{(1+\rho)^{t+1}} \sum_{s=0}^{\infty} \frac{1}{(1+\rho)^s} \\ &= \frac{W}{(1+\rho)^t}. \end{aligned}$$

Since W_t converges to zero, we have a second type of transversality condition in terms of financial capital. Since it is already in present value terms, W_t is analogous to $p_t a_t$. The fact that $W_t \to 0$ corresponds to the capital transversality condition $p_t a_t \to 0$. Moreover, capital transversality implies the partial sums of $\sum p_t c_t$ converge, that the infinite sum is finite.

25.4.11 Intertemporal Profit Maximization: Sufficiency

The first order conditions and the (capital) transversality condition are sufficient for production efficiency.

Theorem 25.4.6. *Let Y be a production set derived from an intertemporal technology* $\mathbf{T}$*. Suppose that for every time period* t*,* $\mathbf{p}_t \gg \mathbf{0}$*, that* $(\mathbf{a}_t^*, \mathbf{b}_{t+1}^*)$ *maximizes* $\mathbf{p}_{t+1}\cdot\mathbf{b} - \mathbf{p}_t\cdot\mathbf{a}$ *for* $(\mathbf{a}, \mathbf{b}) \in T_t$ *for each* $t = 0, 1, 2, \ldots$*, and that* $\mathbf{b}_0^* = \mathbf{0}$*. If the capital transversality condition* $\mathbf{p}_t\cdot\mathbf{a}_t^* \to 0$ *is satisfied, then* $\mathbf{y}^* = \mathbf{b}^* - \mathbf{a}^*$ *is efficient.*

Proof. Take a feasible $\mathbf{y} = \mathbf{b} - \mathbf{a}$. This obeys $(\mathbf{a}_t, \mathbf{b}_{t+1}) \in T_t$ for $t = 0, 1, \ldots$ and $\mathbf{b}_0 = \mathbf{0}$. Suppose $\mathbf{y} \geq \mathbf{y}^*$. We will show $\mathbf{y} = \mathbf{y}^*$, establishing efficiency of $\mathbf{y}^*$.

$$
\begin{aligned}
0 &\leq \sum_{t=0}^{T+1} \mathbf{p}_t\cdot(\mathbf{y}_t - \mathbf{y}_t^*) = \sum_{0}^{T+1} \mathbf{p}_t\cdot(\mathbf{b}_t - \mathbf{a}_t - \mathbf{b}_t^* + \mathbf{a}_t^*) \\
&= \sum_{0}^{T} (\mathbf{p}_{t+1}\cdot\mathbf{b}_{t+1} - \mathbf{p}_{t+1}\cdot\mathbf{b}_{t+1}^*) - \sum_{0}^{T} (\mathbf{p}_t\cdot\mathbf{a}_t - \mathbf{p}_t\cdot\mathbf{a}_t^*) && \text{Rearrangment} \\
&\quad - \mathbf{p}_{T+1}\cdot\mathbf{a}_{T+1} + \mathbf{p}_{T+1}\cdot\mathbf{a}_{T+1}^* && \text{Rearrangment} \\
&= \sum_{0}^{T} \left[(\mathbf{p}_{t+1}\cdot\mathbf{b}_{t+1} - \mathbf{p}_t\cdot\mathbf{a}_t) - (\mathbf{p}_{t+1}\cdot\mathbf{b}_{t+1}^* - \mathbf{p}_t\cdot\mathbf{a}_t^*)\right] && \text{Profit maximization} \\
&\quad - \mathbf{p}_{T+1}\cdot\mathbf{a}_{T+1} + \mathbf{p}_{T+1}\cdot\mathbf{a}_{T+1}^* && \text{Dropped negative term} \\
&\leq -\mathbf{p}_{T+1}\cdot\mathbf{a}_{T+1} + \mathbf{p}_{T+1}\cdot\mathbf{a}_{T+1}^* \\
&\leq \mathbf{p}_{T+1}\cdot\mathbf{a}_{T+1}^* \to 0.
\end{aligned}
$$

Taking the limit and using transversality shows $0 \leq \sum_0^\infty \mathbf{p}_t\cdot(\mathbf{y}_t - \mathbf{y}_t^*) \leq 0$. Since $\mathbf{p}_t \gg \mathbf{0}$ and $\mathbf{y}_t \geq \mathbf{y}_t^*$, each term is non-negative, and so each term must be zero. We conclude $\mathbf{p}_t\cdot\mathbf{y}_t = \mathbf{p}_t\cdot\mathbf{y}_t^*$ for all t. Moreover, since $\mathbf{p}_t \gg \mathbf{0}$, it follows that each $\mathbf{y}_t = \mathbf{y}_t^*$. In other words, there are no feasible streams in Y that are larger than $\mathbf{y}^*$. The stream $\mathbf{y}^*$ is efficient. □

In the next section, we will use a very similar argument to show that when utility is concave, the first order conditions plus the transversality condition are sufficient for optimality.

25.5 The Optimal Growth Problem

The basic structure of equilibrium is the same as before. Firms maximize profits, consumers maximize utility, and markets clear. There is a substantial literature devoted to equilibrium in infinite-horizon models.[14]

We will only consider the simplest case, where there is a representative consumer and a representative firm. We approach equilibrium indirectly, by first considering Pareto optimality. In the optimal growth problem a single agent (consumer, planner) maximizes time additive separable utility over all feasible consumption paths, all non-negative paths in $\boldsymbol{\omega} + Y$ with the production set Y derived from an intertemporal technology $\mathbf{T}$.[15]

Optimal Growth Problem. Given an intertemporal technology $\mathbf{T}$, felicity function u, and discount factor δ, $0 < \delta < 1$, the *optimal growth problem* is

$$\begin{aligned} \max \sum_{t=0}^{\infty} \delta^t u(\mathbf{c}_t) & \\ \text{s.t. } \mathbf{c}_t \leq \boldsymbol{\omega}_t + \mathbf{b}_t - \mathbf{a}_t, & \; t = 1, 2, \ldots \\ \mathbf{c}_0 + \mathbf{a}_0 \leq \boldsymbol{\omega}_0 & \\ (\mathbf{a}_t, \mathbf{b}_{t+1}) \in T_t, & \; t = 0, 1, \ldots \\ \mathbf{c}_t \geq 0, & \; t = 0, 1, \ldots . \end{aligned} \tag{25.5.4}$$

[14] E.g., Peleg and Yaari (1970), Bewley (1972, 1982), Stigum (1973), Boyd and McKenzie (1993), Becker and Boyd (1997).

[15] Optimal growth problems often restrict the endowment to time zero, so $\boldsymbol{\omega} = (\boldsymbol{\omega}_0, 0, 0, \ldots)$.

25.5.1 Characterizing the Solution I

We will assume the optimal growth problem has a solution. In that case, the solutions can be characterized by finding an appropriate price system.

Theorem 25.5.1. *In the optimal growth problem (25.5.4), suppose* Y *is derived from an intertemporal technology* $\mathbf{T}$*, with discount factor* $0 < \delta < 1$*, concave felicity* $u \in \mathcal{C}^2$ *with* $Du \gg \mathbf{0}$*, and finite utility* $U(\mathbf{c})$ *for all feasible consumption streams* $\mathbf{c}$*. If* $\mathbf{c}^* \gg \mathbf{0}$ *is an optimal consumption stream, then* $\mathbf{p}_t = \delta^t Du(\mathbf{c}_t^*)$ *are Malinvaud prices, and* $\sum_t \mathbf{p}_t \cdot \mathbf{c}_t^*$ *converges. Moreover, the consumption transversality condition* $\lim \mathbf{p}_t \cdot \mathbf{c}_t^* = 0$ *is satisfied.*

Proof. We show the $\mathbf{p}_t$ are Malinvaud prices by contradiction. Suppose not. Then we can increase profit at some time s. There is $(\mathbf{a}', \mathbf{b}') \in T_s$ with $\mathbf{p}_{s+1} \cdot \mathbf{b}' - \mathbf{p}_s \cdot \mathbf{a}' > \mathbf{p}_{s+1} \cdot \mathbf{b}_{s+1}^* - \mathbf{p}_s \cdot \mathbf{a}_s^*$. For $0 < \varepsilon < 1$, use this to define new input and output streams by

$$\mathbf{a}_t = \begin{cases} \mathbf{a}_t^* & \text{for } t \neq s \\ \varepsilon \mathbf{a}' + (1-\varepsilon)\mathbf{a}_s^* & \text{for } t = s \end{cases}$$

and define

$$\mathbf{b}_t = \begin{cases} \mathbf{b}_t^* & \text{for } t \neq s+1 \\ \varepsilon \mathbf{b}' + (1-\varepsilon)\mathbf{b}_{s+1}^* & \text{for } t = s+1. \end{cases}.$$

Because T_t is convex, $(\mathbf{a}_s, \mathbf{b}_{s+1}) \in T_s$.

Then

$$\mathbf{c}_t - \mathbf{c}_t^* = \begin{cases} \varepsilon(\mathbf{a}_s^* - \mathbf{a}') & \text{for } t = s \\ \varepsilon(\mathbf{b}' - \mathbf{b}_{s+1}^*) & \text{for } t = s+1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbf{0} \ll \mathbf{c}^* \leq \boldsymbol{\omega} + \mathbf{b}^* - \mathbf{a}^*$, $\mathbf{c} = \boldsymbol{\omega} + \mathbf{b} - \mathbf{a} \geq \mathbf{0}$ for $\varepsilon > 0$ small enough.

25.5.2 Characterizing the Solution II

Proof continues. Now consider

$$\Delta U = U(\mathbf{c}) - U(\mathbf{c}^*) = \delta^{s+1}\left[u(\mathbf{c}_{s+1}) - u(\mathbf{c}^*_{s+1})\right] + \delta^s\left[u(\mathbf{c}_s) - u(\mathbf{c}^*_s)\right].$$

Because $\mathbf{c}^*$ is optimal, $\Delta U \le 0$.

Using Taylor's Theorem and the fact that $\mathbf{p}_s = \delta^s Du(\mathbf{c}^*_s)$, we find

$$\delta^s u(\mathbf{c}_s) = \delta^s u(\mathbf{c}^*_s) + \mathbf{p}_s\cdot\big(\varepsilon(\mathbf{a}^*_s - \mathbf{a}')\big) + o_1(\varepsilon)$$

and

$$\delta^{s+1}u(\mathbf{c}_{s+1}) = \delta^{s+1}u(\mathbf{c}^*_{s+1}) + \mathbf{p}_{s+1}\cdot\big(\varepsilon(\mathbf{b}' - \mathbf{b}^*_{s+1})\big) + o_2(\varepsilon)$$

where o_j are functions with $o_j(x)/x \to 0$ as $x \to 0$.

Thus

$$0 \ge \Delta U = \varepsilon\mathbf{p}_{s+1}\cdot(\mathbf{b}' - \mathbf{b}^*_{s+1}) - \varepsilon\mathbf{p}_s\cdot(\mathbf{a}' - \mathbf{a}^*_s) + o_1(\varepsilon) + o_2(\varepsilon).$$

Dividing by ε and letting $\varepsilon \to 0$, we obtain

$$0 \ge \lim_{\varepsilon\to 0}\Delta U/\varepsilon = \mathbf{p}_{s+1}\cdot(\mathbf{b}' - \mathbf{b}^*_{s+1}) - \mathbf{p}_s\cdot(\mathbf{a}' - \mathbf{a}^*_s).$$

This contradicts our supposition that $\mathbf{p}_{s+1}\cdot\mathbf{b}' - \mathbf{p}_s\cdot\mathbf{a}' > \mathbf{p}_{s+1}\cdot\mathbf{b}^*_{s+1} - \mathbf{p}_s\cdot\mathbf{a}^*_s$. This contradiction shows that profit in period every period t is maximized at $(\mathbf{a}^*_t, \mathbf{b}^*_{t+1})$, and thus that $\mathbf{p}_t$ are Malinvaud prices.

The concavity of u implies

$$\mathbf{p}_t\cdot\mathbf{c}^*_t \le \delta^t u(\mathbf{c}^*_t) - \delta^t u(0).$$

Summing over all t shows

$$\sum_t \mathbf{p}_t\cdot\mathbf{c}^*_t \le U(\mathbf{c}^*) - U(\mathbf{0}).$$

Since the sum converges, the terms must converge to zero, giving the consumption transversality condition $\lim \mathbf{p}_t\cdot\mathbf{c}^*_t = 0$. □

25.5.3 Sufficient Conditions for Optimality

The previous theorem showed that the consumption transversality condition appears as a necessary condition for a Pareto optimum.[16]

We had used the transversality condition earlier when considering efficient production. The following theorem shows that the first order conditions and capital transversality condition are sufficient for a Pareto optimum.

Theorem 25.5.2. *Suppose* $\mathbf{T} = \{T_t\}$ *is an intertemporal technology, the period utility* u *is* $\mathcal{C}^1$ *and concave, the discount factor obeys* $\delta < 1$*, and that* $U(\mathbf{c})$ *is finite for every feasible consumption stream* $\mathbf{c}$*. If* $\mathbf{p}_t = \delta^t Du(\mathbf{c}_t^*)$ *are Malinvaud prices when* $\mathbf{c}^* = \boldsymbol{\omega} + \mathbf{b}^* - \mathbf{a}^*$ *is feasible, and the capital transversality condition* $\lim_{t\to\infty} \mathbf{p}_t \cdot \mathbf{a}_t^* = 0$ *is satisfied, then* $\mathbf{c}^*$ *maximizes utility over all feasible consumption streams.*

Proof. Let $\mathbf{c}'$ be a feasible consumption stream and consider the partial sums $\sum_{t=0}^{T} \delta^t \big[u(\mathbf{c}_t') - u(\mathbf{c}_t^*)\big]$. Since u is concave and $\delta^t Du(\mathbf{c}_t^*) = \mathbf{p}_t$, we have

$$\delta^t u(\mathbf{c}_t') \le \delta^t u(\mathbf{c}_t^*) + \mathbf{p}_t \cdot [\mathbf{c}_t' - \mathbf{c}_t^*]$$

Summing, we find

$$\begin{aligned}\sum_{t=0}^{T} \delta^t \big[u(\mathbf{c}_t') - u(\mathbf{c}_t^*)\big] &\le \sum_{t=0}^{T} \mathbf{p}_t \cdot [\mathbf{c}_t' - \mathbf{c}_t^*] \\ &= \sum_{t=0}^{T} \mathbf{p}_t \cdot [\mathbf{b}_t' - \mathbf{a}_t' - \mathbf{b}_t^* + \mathbf{a}_t^*] \\ &\le -\mathbf{p}_T \cdot \mathbf{a}_T' + \mathbf{p}_T \cdot \mathbf{a}_T^* \\ &\le \mathbf{p}_T \cdot \mathbf{a}_T^*,\end{aligned}$$

using the same calculations as in 25.4.6. Letting $T \to \infty$, we find $U(\mathbf{c}') - U(\mathbf{c}^*) \le 0$ for any feasible $\mathbf{c}'$. That shows $\mathbf{c}^*$ is optimal. □

[16] Recall that the consumption transversality condition is $\lim \mathbf{p}_t \cdot \mathbf{c}_t = 0$.

25.5.4 Pareto Optimum and Equilibrium

The result of this theorem is that we can treat the Pareto optimum as an equilibrium where the consumer has wealth $W = \sum_t \mathbf{p}_t \cdot \mathbf{c}_t^*$ when W is finite. The consumer's utility maximum occurs at $\mathbf{c}^*$. The firm maximizes profits over each T_t at $(\mathbf{a}^*, \mathbf{b}^*)$, and markets clear, $\boldsymbol{\omega} + \mathbf{b}^* - \mathbf{a}^* = \mathbf{c}^*$.

Theorem 25.5.2 shows when Malinvaud prices and the transversality condition are sufficient for optimality, while Theorem 25.5.1 shows when they are necessary.

The condition $\mathbf{p}_t \cdot \mathbf{c}_t^* \to 0$ is not quite the transversality condition we considered earlier, but as we saw earlier, the fact that $\sum_t \mathbf{p}_t \cdot \mathbf{c}_t^* = W < \infty$ performs a similar function here.

25.5.5 Ramsey Model SKIPPED

We now apply these to the classic discounted Ramsey Model.

Example 25.5.3: Ramsey Model. The classic *Ramsey model* is the special case where there is only one good in each period, the endowment is non-zero only at time 0, and the one-sector technology is described by a production function f as in Example 25.3.2.[17]

For convenience, we will make the additional assumption that f obeys the Inada condition: $\lim_{x\to 0} f'(x) = +\infty$.[18] Now let b be initial endowment. In this case we solve

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \delta^t u(c_t) \\ &\text{s.t. } c_t + a_t \le f(a_{t-1}), t = 1, 2, \ldots \\ &\qquad c_0 + a_0 \le b. \end{aligned}$$

Production $f(a_{t-1})$ in period t is divided between consumption c_t and capital a_t. The capital input a_t produces $f(a_t)$ in the next period, which is again divided between consumption and capital, etc.

The Inada condition ensures that $a_t > 0$ whenever $p_{t+1} > 0$. Combining the first order conditions for production and consumption, we obtain

$$\frac{\delta u'(c_{t+1})}{u'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{f'(a_t)}$$

or more simply,

$$\delta f'(a_t) u'(c_{t+1}) = u'(c_t).$$

This set of equations is called the *Euler equations*.

The Euler equations make economic sense. The right-hand side is the marginal value of consumption today. If we postpone a little bit of consumption by one period, we gain output in the next period at rate $f'(a_t)$. This must be discounted and multiplied by marginal utility in the next period to find the utility gain from postponing consumption. At the optimum, we must be indifferent at the margin between moving consumption across periods (strict concavity ensures we are not actually indifferent, but rather any move will make us worse off). ◀

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[17] The Ramsey model is based on Ramsey (1928a), who also considered undiscounted utility.

[18] There's also an Inada condition at infinity, but we don't need it here.

25.5.6 Ramsey Model with Linear technology SKIPPED

Let's consider the case where production is linear.

Example 25.5.4: Ramsey Model with Linear technology. Let's suppose $f(a) = \theta a$ and

$$u(c) = \begin{cases} \frac{c^{1-\sigma}}{1-\sigma} & \text{when } \sigma > 0 \text{ with } \sigma \neq 1 \\ \ln c & \text{when } \sigma = 1 \end{cases}$$

so $u'(c) = c^{-\sigma}$.

We will assume $\delta\theta^{1-\sigma} < 1$. This ensures $U(\mathbf{c}) < +\infty$. The point is that $c_t \leq f^t(a_0) \leq a_0\theta^t$, so $u(c_t) \leq (1-\sigma)^{-1} a_0^{1-\sigma}\theta^{t(1-\sigma)}$. The utility sum will always converge when $\delta\theta^{1-\sigma} < 1$, as required by both Theorem 25.5.1 and Theorem 25.5.2.

Then $p_t = \delta^t u'(c_t) = \delta^t c_t^{-\sigma}$ and the Euler equations can be written $(\delta\theta)c_{t+1}^{-\sigma} = c_t^{-\sigma}$. Thus $c_{t+1} = (\delta\theta)^{1/\sigma} c_t$. Set

$$\beta = (\delta\theta)^{1/\sigma} \quad \text{so that} \quad c_t = \beta^t c_0.$$

Also, the condition $\delta\theta^{1-\sigma} < 1$ can be written using β as $\beta < \theta$.

25.5.7 Linear Ramsey Model: Solving the Difference Equation SKIPPED

Now $a_{t+1} = \theta a_t - c_{t+1} = \theta a_t - \beta^f c_0$ and $a_0 = b - c_0$. We solve the difference equation using the summing factor $\theta^{-(t+1)}$ (see section 29.6). The difference equation becomes

$$\frac{a_{t+1}}{\theta^{t+1}} = \frac{a_t}{\theta^t} - \frac{c_0}{\theta}\left(\frac{\beta}{\theta}\right)^t$$

or

$$z_{t+1} - z_t = -\frac{c_0}{\theta}\left(\frac{\beta}{\theta}\right)^t$$

where $z_t = a_t/\theta^t$. We can now sum over times 0 through $(s-1)$. Most of the terms on the left-hand side cancel, yielding

$$z_s - z_0 = \frac{a_s}{\theta^s} - a_0 = -\sum_{s=0}^{s-1}\frac{c_0}{\theta}\left(\frac{\beta}{\theta}\right)^s$$

This can be written

$$\theta^{-s}a_s = a_0 + \frac{c_0}{\theta}\left[\frac{(\beta/\theta)^s - 1}{(\beta/\theta) - 1}\right]$$

or, switching from s to t and simplifying,

$$a_t = \theta^t(b - c_0) - c_0\left[\frac{\beta^t - \theta^t}{\beta - \theta}\right] = \theta^t\left(b - c_0\frac{1 + \theta - \beta}{\theta - \beta}\right) - c_0\frac{\beta^t}{\beta - \theta}.$$

25.5.8 The Role of Transversality SKIPPED

The first order conditions are not enough to pin down c_0. That is the job of the transversality condition. To use the transversality condition, we start by writing p_t in terms of our unknown c_0 and the parameters of the model, σ and θ.

$$p_t = \delta^t c_t^{-\sigma} = \delta^t \beta^{-\sigma t} c_0^{-\sigma} = \theta^{-t} c_0^{-\sigma}.$$

We require $p_t a_t \to 0$. Now

$$p_t a_t = c_0^{-\sigma}\left(b - c_0 \frac{1+\theta-\beta}{\theta-\beta}\right) - c_0^{1-\sigma}\frac{\beta^t}{\theta^t}(\beta-\theta).$$

Since $\beta < \theta$, the second term converges to zero, and the transversality condition requires that the first term vanish. We must have

$$b - c_0 \frac{1+\theta-\beta}{\theta-\beta} = 0.$$

Solving for c_0, we get

$$c_0 = b\frac{\theta-\beta}{1+\theta-\beta}.$$

Finally, we obtain $a_0 = b - c_0 = b/(1+\theta-\beta)$.

The optimal consumption and capital stocks are

$$c_t = b\theta^t \frac{\theta-\beta}{1+\theta-\beta} \quad \text{and} \quad a_t = \frac{b\theta^t}{1+\theta-\beta}.$$

Both the consumption and capital transversality conditions hold.

This example has the property that consumption will grow over time if $\beta > 1$. In terms of the primitives, this translates to $\delta\theta > 1$. If productivity exceeds discounting, the economy grows, if discounting exceeds productivity, the economy shrinks. Remarkably, this is not an artifact of the utility function, but will happen with a linear technology regardless of the utility function. ◀

25.5.9 Ramsey Model: Diminishing Returns SKIPPED

Now consider the Ramsey problem with $f'' < 0$. If there is a $\hat{a}$ with $f(\hat{a}) < \hat{a}$, larger values of a cannot be sustained due to the diminishing marginal product. The capital stock must shrink. In fact, the capital stock is bounded above by $\max\{b, \hat{a}\}$ where b is the initial capital stock.

If we also have $\delta f'(0) > 1$, there will be a *steady state*, a level $\bar{a}$ where it is optimal to remain. In fact, this steady state is unique.

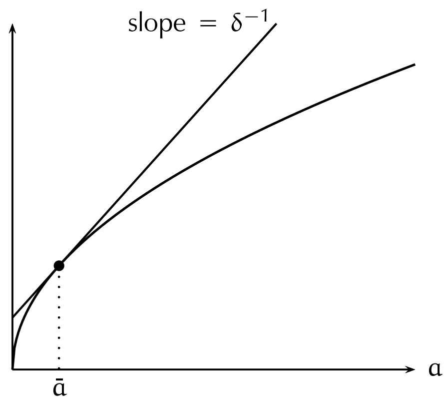

Figure 25.5.1: The steady state is the unique point $\bar{a}$ where $f'(\bar{a}) = \delta^{-1}$.

25.5.10 Steady State and the Turnpike Theorem

We find the steady state by setting $c_t = c_{t+1} = c$ in the Euler equations. Then

$$\delta f'(a_t)u'(c) = u'(c)$$

This simplifies to $\delta f'(a_t) = 1$. We define $\bar{a}$ as the solution to $\delta f'(\bar{a}) = 1$. The Intermediate Value Theorem shows that such a solution exists because the left-head side is continuous, greater than 1 at 0, and less than 1 at $\hat{a}$. Moreover the solution is unique because $f'' < 0$.

To see that this defines a steady state, suppose the initial stock $b = \bar{a}$ and define $\bar{c} = f(\bar{a}) - \bar{a}$. We first show $\bar{c} > 0$. Define $g(a) = f(a) - a$. Now $g(0) = 0$ and $g'(a) = f'(a) - 1 > 0$ for $a \leq \bar{a}$ because $f'(\bar{a}) > 1$ and $f'' < 0$. It follows that $\bar{c} > 0$.

Now consider the path with $a_t = \bar{a}$, $c_t = \bar{c}$ and $b_t = f(\bar{a})$. This path satisfies the Euler equations and transversality condition. It is optimal to remain at $\bar{a}$ if you start at $\bar{a}$.

The celebrated Turnpike Theorem states that the optimal path converges to the steady state under some very mild assumptions on f, u, and δ, moreover, for δ near 1. The full turnpike theorem shows that this is true in multi-sector models, something that is much less obvious.

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