27. Equilibrium in Complete Asset Markets

When we studied equilibrium over time in Chapter 25, we considered models where everyone had perfect foresight. At time zero they knew the economic conditions they would face in all future periods. When we solved the consumer's problem, the consumer knew the prices of all future goods. The producers also knew all future prices when choosing input and output streams.

What if there is uncertainty about future conditions? That is what this chapter is about. We will focus on a simple two period model. In the initial period, period zero, there is uncertainty about the future, but no uncertainty about the present. We refer to this as the *ex ante* period. Agents will make some decisions and transactions *ex ante*, in the face of this uncertainty. All the uncertainty is resolved at the beginning of period one. The state of the world is revealed. *Ex post*, after all uncertainty is resolved, contracts made in period zero are executed, and further trades may be made

Although economists often consider complex multiperiod models where uncertainty is resolved over time, or even models where uncertainty is continuously resolved, we will eschew such complication. The basic elements of the problem can be seen in a two-period model, and that is what we shall use.

The two period model has its limitations. A fuller study of financial markets would lead to multiperiod models where uncertainty is gradually resolved over time, and to infinite-horizon models where phenomena such as asset bubbles may occur.[1] Further, not all problems are best handled in discrete time. In particular, the analysis of option pricing is more easily handled in continuous time in spite of the extra mathematical complexity due to the use of stochastic calculus.[2]

[1] These extensions of the two period model are beyond the scope of this work. They are considered in the latter part of LeRoy and Werner (2014).

[2] LeRoy and Werner (2014) comment further on this in their preface.

27.0.1 Chapter Outline

Section one reformulates our representation of the economy to incorporate uncertainty. Section two looks at the case of complete forward markets, the Arrow-Debreu equilibrium. We then take a look at insurance in the Arrow-Debreu model in section three. The Arrovian securities model, with goods being traded in spot markets and Arrovian securities connecting the time periods is the subject of section four. Next we show that the Arrow-Debreu equilibria and Arrovian securities equilibria are generally equivalent in section five. Section six looks at the boundaries of the Arrovian securities models, where the equivalence with the Arrow-Debreu model breaks down, and also takes a look at a model where there are fewer securities than states.

Outline:

27.1 A World of Uncertainty: Contingent Commodities

Our approach to uncertainty is based on that used in Chapters 22–24. We model uncertainty by considering a world with S possible states, $s = 1, \ldots, S$. We use this to reformulate our economy with I consumers, F firms, and m goods in a way that allows for uncertainty. The key step is the concept of a *contingent commodity*.

A unit of the *contingent commodity* (ℓ, s) is the right to receive one unit of good ℓ if and only if state s occurs. The states can represent either economic events (recession, boom) or natural events (rainy or sunny weather). One could consider a contingent umbrella which you only receive in the event that it rains. A *contingent commodity vector* is a vector listing the amounts of all possible state contingent commodities. The actual vector of goods received or provided is contingent on which state occurs, and we list what would happen in all possible states.

27.1.1 Contingent Commodity Vectors

We will write the contingent commodity vector in several formats

$$\begin{aligned}\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_S) &= \left(\begin{pmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{m,1} \end{pmatrix}, \begin{pmatrix} x_{1,2} \\ x_{2,2} \\ \vdots \\ x_{m,2} \end{pmatrix}, \dots, \begin{pmatrix} x_{1,S} \\ x_{2,S} \\ \vdots \\ x_{m,S} \end{pmatrix} \right) \\ &= \begin{pmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{m,1} \\ x_{1,2} \\ \vdots \\ x_{m,S} \end{pmatrix} \in \mathbb{R}^{mS}_{+} \end{aligned} \tag{27.1.1}$$

In the first format we list the goods in state one first, then the state two goods, etc. Here $\mathbf{x}_s = (x_{1,s}, \dots, x_{m,s})^{\mathsf{T}}$ denotes the vector of goods in state s. This is made more explicit in the next expression, where write the contingent commodity vectors as a row vector (by state) of vertical commodity vectors for each state (i.e., in $\mathbb{R}^m_+$). The last version writes everything as a column vector in $\mathbb{R}^{mS}_+$, with each commodity in state one listed first, then state two goods, etc.[3]

Technically, the contingent commodity vectors must be treated as column vectors, although we will sometimes write them as rows, just as we sometimes do with ordinary commodity vectors. The endowment is also treated as contingent, with $\boldsymbol{\omega}^i \in \mathbb{R}^{mS}_+$.

[3] One more way to do this is to think of each x_ℓ as a random variable $x_\ell(s)$, or even treat $\mathbf{x} \in \mathbb{R}^{mS}_+$ as a random variable $\mathbf{x}(s)$ with values in $\mathbb{R}^m_+$.

27.1.2 Certain Commodity Vectors

One type of contingent commodity vector is not really contingent. These are the vectors that are certain, that yield the same consumption bundle in each state. We define certainty by saying that a contingent commodity vector $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_S)$ is *certain* if $\mathbf{x}_r = \mathbf{x}_s$ for all states $r, s = 1, \dots, S$. Thus $\mathbf{x}$ can be written $(\mathbf{x}_1, \mathbf{x}_1, \dots, \mathbf{x}_1)$ when $\mathbf{x}$ is certain.

The states create a natural grouping of commodities. Define commodity groups by $P_s = \{(\ell, s) : \ell = 1, \dots, m\}$. Then $\mathcal{P} = \{P_s\}_{s=1}^S$ is a partition of the commodity space that reflects the structure imposed by the states. We refer to $\mathcal{P}$ as the *state partition*.

27.1.3 Consumer Preferences

Consumers have preferences over contingent commodity vectors. Although many types of preferences can be considered, we will make three simplifying assumptions. First, we assume preferences are continuous, implying they can be represented by a utility function. Second, preferences need to reflect the contingent commodity structure. This is accomplished by requiring they be strongly separable over the state partition $\mathcal{P}$. Provided that at least 3 groups are essential, Debreu's Separability Theorem implies utility has an additive separable representation over $\mathcal{P}$. Third, since contingent commodity vectors can be regarded as lotteries, we require that preferences satisfy von Neumann and Morgenstern's Independence Axiom. This yields an expected utility representation. However, we will not require that consumers agree on the probabilities of the various states.

Combining these requirements, we find that $\boldsymbol{x} \succsim_i \boldsymbol{x}'$ if and only if there is a continuous function $u_i \colon \mathbb{R}^m_+ \to \mathbb{R}$ and probabilities π^i_s with

$$\sum_s \pi^i_s \, u_i(x_{1,s}, \dots, x_{m,s}) \ge \sum_s \pi^i_s \, u_i(x'_{1,s}, \dots, x'_{m,s}).$$

where $\pi^i_s \ge 0$ and $\sum_s \pi^i_s = 1$. The π^i_s can be subjective prior probabilities that may differ from person to person, or they can be the actual probabilities of the states. In the latter case, these preferences are ordinary expected utility. Note that the preferences are *ex ante* preferences, telling us what consumers prefer before they know the true state of the world.

What about preferences following the resolution of uncertainty, *ex post* preferences? We make the natural assumption that these depends only on consumption in the state that actually occurs, and that it ex post preferences are represented by u_i. In that case, any choice made *ex post* will be consistent with *ex ante* preferences.[4]

[4] Subjective expected utility is automatically separable over states. If preferences are not separable over states, definition and consistency of *ex post* preferences can be an issue.

27.1.4 Contingent Production Plans

That brings us to production. Just a consumers have contingent consumption plans, firms will have contingent production plans. Production in one state should not interfere with production in another state. Otherwise, we would have a situation where things that did not happen could affect production. Accordingly, we define a *contingent production set* to be a set $Y \subset \mathbb{R}^{mS}$ with $Y = Y_1 \times \cdots \times Y_S$ where each Y_s is a production set for commodity group P_s. With such a production technology, $\mathbf{y}^*$ maximizes profit $\mathbf{p}\cdot\mathbf{y}$ over Y if and only if each $\mathbf{y}_s^*$ maximizes profit in each state, $\mathbf{p}_s\cdot\mathbf{y}_s$ over Y_s.

Of course, unless production is constant returns to scale, we will have to worry about the distribution of the profit. We handle this by defining ownership shares of the firm. Since we will not be considering trades in these shares, we restrict our attention to deterministic shares without loss of generality. As usual, θ_f^i denotes i's share of firm f. It would also be possible to consider state contingent shares that apply to the profit generated in each state.

27.1.5 Contingent Goods Economy

We are now ready to define a contingent goods economy.

Contingent Goods Economy. An economy

$$\mathcal{E} = \left(\left(\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i \right)_{i=1}^{I}, \left(\theta_f^i \right)_{i=1, f=1}^{I, F}, \left(Y^f \right)_{f=1}^{F} \right)$$

is a *contingent goods economy* with m goods and S states if

1. Consumer i's consumption set is $\mathfrak{X}_i = \mathbb{R}_+^{mS}$ and the endowment vector $\boldsymbol{\omega}^i \in \mathfrak{X}_i$.
2. Consumer i has preferences $\succsim_i$ defined over $\mathfrak{X}_i$.
3. Firm f's production set obeys $Y^f = Y_1^f \times \cdots \times Y_S^f$ where each Y_s^f is a production set.
4. The firm shares obey $\theta_f^i \geq 0$ and $\sum_{f=1}^{F} \theta_f^i = 1$.

27.1.6 Expected Utility

We will often require that preferences are defined by a subjective expected utility function. We will have a special way to write the economy to reflect that. Results that hold generally will usually be stated for contingent goods economies using preference orders. Results based on expected utility will be stated using contingent goods economies with expected utility.

Contingent Goods Economy with Expected Utility. An economy

$$\mathcal{E} = \Big(\big(\mathfrak{X}_i, (\pi^i_s)_{s=1}^S, u_i, \boldsymbol{\omega}^i \big)_{i=1}^I, \big(\theta^i_f\big)_{i=1,f=1}^{I,F}, \big(Y^f\big)_{f=1}^F \Big)$$

is a *contingent goods economy with expected utility* if

$$\mathcal{E} = \Big(\big(\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i \big)_{i=1}^I, \big(\theta^i_f\big)_{i=1,f=1}^{I,F}, \big(Y^f\big)_{f=1}^F \Big)$$

is a contingent goods economy where $\succsim_i$ is defined by the subjective expected utility function $U_i(\mathbf{x}) = \sum_{s=1}^S \pi^i_s u_i(\mathbf{x})$ with $\pi^i_s \geq 0$, $\sum_{s=1}^S \pi^i_s = 1$, and where each u_i is a continuous function on $\mathbb{R}^m_+$.

We will typically write these types of contingent goods economies in the abbreviated forms $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \theta^i_f, Y^f)$ and $\mathcal{E} = (\mathfrak{X}_i, \pi^i_s, u_i, \boldsymbol{\omega}^i, \theta^i_f, Y^f)$.

27.1.7 Time and the Resolution of Uncertainty

The contingent economy structure implicitly defines two time periods. There is an *ex ante* time period, before uncertainty has been resolved. We call this time zero. There is also an *ex post* time period, when we know what the state is, and all of the uncertainty is gone. This is period one.

In period zero, there may be a market for some contingent commodities. These are traded at time zero before the actual state is known, both by consumers and firms. These markets, where we trade contracts for future delivery, are called *forward markets*.[5]

To properly value the contingent commodities in forward markets, we need to forecast the future in every possible state. We will use the simple option of assuming perfect foresight, but other types of expectations could also be used, such as rational expectations.

Once we get to period one, there is no more uncertainty, but it may be possible to trade some goods in the state that actually occurs. If there is such a market, we call it a *spot market*.

This contingent economy structure will be used to examine three types of equilibrium, the Arrow-Debreu model of complete forward markets and the Arrow model of securities and spot markets in this chapter, and Radner's more general model of asset and spot markets in Chapter 28.

[5] The term "forward market" is used as a catch-all term for such transactions. Of course, there is the problem of how to insure delivery of the contracted items. *Futures markets* involved standardized forward contracts and (usually) enforcement mechanisms. The first exchange-traded futures contracts were developed by the Chicago Board of Trade in the 1850's and 1860's. For an interesting account of these developments, see Cronon (1991, chapter 3).

27.2 The Arrow-Debreu Model

The Arrow-Debreu model is based on forward markets for every contingent commodity. Such a set of markets, including a market for each contingent good, is called *complete*. There are no spot markets in this equilibrium. In fact, there would be no point to opening the spot markets. At time one, the forward contracts are executed, creating a Pareto optimal allocation *ex post*. Now suppose we opened the spot markets in that state. What trade would occur? Any new equilibrium would have to be Pareto optimal. Also, consumers would trade from their new allocation, and so would need something at least as good. The only Pareto optima like that yield the same utility for everyone as the *ex post* allocation. Unless there are flats in the indifference surfaces, this allocation must be the *ex post* allocation. If there are flats, no one's utility can be increased and the *ex post* allocation remains an equilibrium allocation. There's really no point to adding spot markets to the Arrow-Debreu equilibrium.

27.2.1 Arrow-Debreu Forward Markets

The Arrow-Debreu model works as follows. At time zero, contracts concerning contingent goods are traded and firms commit to production plans. The market at time zero the ex *ante market* involves *forward contracts*—contracts for future delivery of goods. The forward markets are complete in the sense that you can write a contact for delivery of any good in any state.

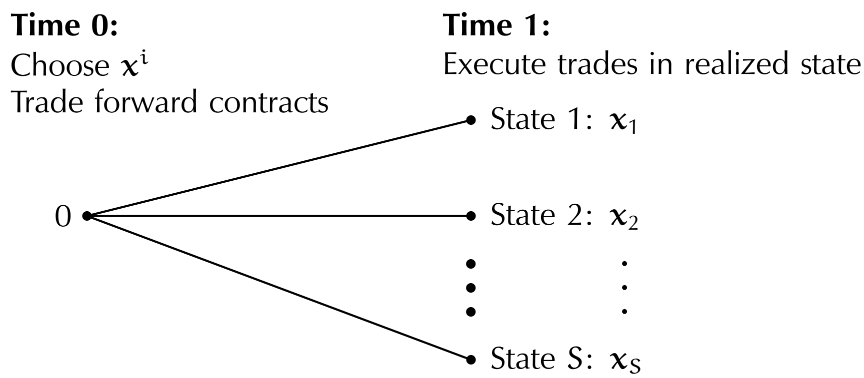

Figure 27.2.1: All choices are made ex *ante*, at time zero, when the future state is unknown. In period one, the actual state is revealed and people now execute the plans made at time zero. Contracts for the unrealized states are not executed.

27.2.2 Execution of Forward Contracts

The seller of a contract on good (ℓ, s) agrees to deliver a specified quantity of good ℓ if state s occurs. The buyer, who will receive good ℓ if state s occurs, pays for it upfront. At time one, one of the states occurs and the contracts involving that state are executed. No contracts on any other states are executed. In this model all monetary transactions take place at time zero, and the prices are formed then. There is no need to have expectations about future prices in the Arrow-Debreu model. All decisions are made in the present. Contracts are executed in the future.

When time one arrives and the state of the world becomes known, production occurs according to plan and the contracted goods are delivered. This delivery includes delivery of goods from the realized endowment and actual production. As mentioned above, there is no trading at time one in this model. Goods are delivered as promised. All trading takes place in the forward markets. There are no spot markets that open once the state is revealed in period one.

27.2.3 Arrow-Debreu Budget Sets

To simplify notation, we will use the period one goods to stand both for the contract delivering those goods in period one and for the goods themselves. This means we interpret the net supply vector $\mathbf{y}^f$ as a bundle of contracts for delivery of good ℓ in the various states s by firm f when $y^f_{\ell,s}$ is positive, and as delivery to firm f when $y^f_{\ell,s}$ is negative. Firm f's profit on those contracts is $\mathbf{p}\cdot\mathbf{y}^f$. Individual i gets share θ^i_f of those profits. Consumer i also gets income from the sale of their endowment via contingent contracts. This income is used to purchase contingent contracts that allow the consumer to consume $\mathbf{x}_s$ if state s occurs. Putting this all together, we obtain the Arrow-Debreu budget set.

Arrow-Debreu Budget Set. The *Arrow-Debreu budget set* $B^i_{AD}(\hat{\mathbf{p}})$ is defined as

$$B^i_{AD}(\mathbf{p}) = \left\{\mathbf{x}^i \in \mathfrak{X}_i : \mathbf{p}\cdot\mathbf{x}^i \le \mathbf{p}\cdot\boldsymbol{\omega}^i + \sum_f \theta^i_f \pi_f(\mathbf{p})\right\}.$$

where π_f is firm f's profit function.

This type of budget set is similar to Hicks' "futures economy" (Hicks, 1946; Chapter 10) and the Fisher Competitive Equilibrium (Becker and Boyd, 1997; Chapter 6).

The consumer's problem is also rather similar to the intertemporal consumer's problem of section 25.2 in that the income side of the budget constraint is known at time zero. We can make the analogy complete by regarding the consumer's purchases as forward contracts made using the present-value prices $\mathbf{p}_t$.

27.2.4 Arrow-Debreu Equilibrium

We use the Arrow-Debreu budget set to define the Arrow-Debreu equilibrium. (Arrow, 1953; Debreu, 1959).

Arrow-Debreu Equilibrium. Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i, \theta^i_f, Y^f)$ be a contingent goods economy. The allocation $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^f) \in \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_I \times Y_1 \times \cdots \times Y_F \subset \mathbb{R}^{mS(I+F)}$ and prices $\hat{\mathbf{p}} = (\hat{p}_{\ell,s}) \in \mathbb{R}^{mS}$ form an *Arrow-Debreu* or *contingent commodity equilibrium* if the following hold.

1. **Profit maximization:** For all f, $\hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f \geq \hat{\mathbf{p}}\cdot\mathbf{y}^f$ for all $\mathbf{y}^f \in Y_f$.
2. **Utility maximization:** For all i, $\hat{\mathbf{x}}^i$ is a best point for i in the Arrow-Debreu budget set
$$B^i_{AD}(\hat{\mathbf{p}}) = \left\{\mathbf{x}^i \in \mathfrak{X}_i : \hat{\mathbf{p}}\cdot\mathbf{x}^i \leq \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i + \sum_f \theta^i_f \hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f\right\}.$$
3. **Markets clear:** $\sum_i \hat{\mathbf{x}}^i = \sum_i \boldsymbol{\omega}^i + \sum_f \hat{\mathbf{y}}^f$.

The Arrow-Debreu budget set has been written slightly differently since profit maximization implies $\pi_f(\hat{\mathbf{p}}) = \hat{\mathbf{p}}\cdot\hat{\mathbf{y}}^f$.

Radner (1973) called a multi-period version of the Arrow-Debreu equilibrium an equilibrium of prices, plans, and expectations—all commodities get priced, whether that state occurs or not, consumption and production plans are made for every contingency. There must be market clearing, both in states that occur and those that do not. Everyone expects that contracts will be fulfilled, regardless of which state actually occurs.

Our Arrow-Debreu model has all the usual properties of a competitive equilibrium. All the previous theorems on Walrasian equilibrium hold in this context. The equilibrium existence theorems of Chapter 16 give us conditions under which an equilibrium exists. Both of the welfare theorems apply. In particular, if preferences are locally non-satiated, equilibrium allocations must be Pareto optimal by the First Welfare Theorem. Indeed, if the technology can be converted to a constant returns to scale production set (see section 16.7), Theorem 21.2.10 shows that the equilibrium allocations are in the core.

27.2.5 An Equilibrium with Production I

April 13, 2023

New Homework: Problems 25.2.6, 25.2.7, 25.5.2, 27.2.2, and 27.3.2 are due on Thursday, April 20.

We know how to find Walrasian equilibria. The Arrow-Debreu model with contingent goods is exactly that, and we find equilibria the same way. Let's see how it's done.

Example 27.2.2: Contingent Market Equilibrium with Production. The economy has two consumers. They both consume two goods in each of two states. There is one firm. Its technology is constant returns to scale, and is defined by the production set

$$Y = \{\mathbf{y} \in \mathbb{R}^4 : y_{1,s} \le -2y_{2,s}, y_{2,s} \le 0, \text{ for } s = 1, 2\}.$$

This technology works independently in each state, and is identical in each state.

Since production is constant returns to scale, profits must be zero and we do not have to worry about how the non-existent profits are distributed. If you wish, you can take $\theta_1^i = 1/2$. The distribution doesn't matter because there is no profit to distribute.

The consumers have identical equal-weighted Cobb-Douglas utility functions in logarithmic form, $u_i(\mathbf{x}) = \sum_{\ell,s} \ln x_{\ell,s}$. Their endowments are $\boldsymbol{\omega}^1 = \big((0,2),(2,1)\big)$, and $\boldsymbol{\omega}^2 = \big((0,1),(3,1)\big)$. This means that the aggregate endowment is

$$\boldsymbol{\omega} = \left(\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right).$$

Denote consumer i's income by m^i. Thus $m^i = \mathbf{p}\cdot\boldsymbol{\omega}^i$ where $\mathbf{p}$ is the price vector. Since the consumers have equal-weighted Cobb-Douglas utility, market demand is

$$\mathbf{x}(\mathbf{p}) = \frac{m}{4}\left(\begin{pmatrix} 1/p_{1,1} \\ 1/p_{2,1} \end{pmatrix}, \begin{pmatrix} 1/p_{1,2} \\ 1/p_{2,2} \end{pmatrix} \right).$$

where $m = m^1 + m^2 = \mathbf{p}\cdot\boldsymbol{\omega}$. Of course, equilibrium prices must be strictly positive to prevent infinite demand.

27.2.6 An Equilibrium with Production II

But what about production? The fact that production is constant returns to scale tells us that production can only occur in state s if $2p_{1,s} = p_{2,s}$, and that if production does not occur in state s, $2p_{1,s} \leq p_{2,s}$ (profits are never positive). We cannot have $2p_{1,s} > p_{2,s}$ for either state s in equilibrium as that would lead to infinite profit in state s.

If production occurs in state s, it follows that $x_{1,s} = 2x_{2,s}$ because $2p_{1,s} = p_{2,s}$. We will now use market clearing to compute the inputs. By market clearing, $\mathbf{x} = \mathbf{y} + \boldsymbol{\omega}$. This means $y_{1,1} = x_{1,1} = 2x_{2,1} = 2(3 + 2y_{2,1})$ and $5 + y_{1,2} = x_{1,2} = 2x_{2,2} = 2(2 + 2y_{2,2})$. Profit is maximized when $y_{1,s} = -2y_{2,s}$, so we have

$$-2y_{2,1} = 2(3 + y_{2,1}) \quad \text{and} \quad 5 - 2y_{2,2} = 2(2 + 2y_{2,2}),$$

implying $y_{2,1} = -3/2$ and $y_{2,2} = 1/6$. The latter is impossible as good $(2, 2)$ is an input, with $y_{2,2} \leq 0$. There cannot be any production in state two! We must have $y_{\ell,2} = 0$. Consumption must equal the endowment in state 2, forcing $p_{2,1} = (2/5)p_{2,2} < 2p_{2,2}$.

27.2.7 An Equilibrium with Production III

Production does occur in state one, as we must have a non-zero amount of good one in both states. This implies $y_{2,1} = -1$ and $y_{1,1} = 2$. Equilibrium production and consumption are then

$$\hat{y} = \left(\begin{pmatrix} 3 \\ -3/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \quad \text{and} \quad \hat{x} = \left(\begin{pmatrix} 3 \\ 3/2 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right).$$

We normalize prices so that $p_{1,1} = 1$. This yields equilibrium price vector

$$\hat{p} = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3/5 \\ 3/2 \end{pmatrix} \right).$$

Incomes are $m^1 = 67/10$, $m^2 = 53/10$, and $m = 12$. This yields consumption vectors

$$\hat{x}^1 = \left(\begin{pmatrix} 67/40 \\ 67/80 \end{pmatrix}, \begin{pmatrix} 335/12 \\ 67/60 \end{pmatrix} \right) \quad \text{and} \quad \hat{x}^2 = \left(\begin{pmatrix} 53/40 \\ 53/80 \end{pmatrix}, \begin{pmatrix} 265/12 \\ 53/60 \end{pmatrix} \right).$$

◀

So what does the example tell us? The obvious thing is production can be state-dependent, even though the technology used is the same in both states. Here the firm produces in state one, and shuts down in state two.

The differences in production are driven by the differences in aggregate endowments between the two states. Since preferences are identical, there are two possible sources of differences in production between the two states. They can depend on different technologies (ruled out in the example) or different endowments, which happens here. It's not hard to see that different preferences can also lead to state-dependent production. That is not possible here as the preferences are identical.[6]

In the interest of simplicity, we will focus on exchange economies in the rest of the chapter.

[6] See Exercise 27.2.1.

27.3 Insurance in the Arrow-Debreu Model

Although the Arrow-Debreu equilibrium is merely a type of Walrasian equilibrium, the use of contingent goods allows us to consider some new issues that couldn't easily be raised in a basic Walrasian framework. One such issue is insurance.

The presence of contingent commodity markets allow consumers to insure against risk by trading in contingent commodities. With such markets, it's possible to buy an umbrella that will only be delivered when you need it, or even a contingent house to be delivered if yours is destroyed by fire or flood. Contingent commodity markets can include markets for insurance.

In Example 23.3.4, we examined a single insurance market. We found that actuarially fair pricing of insurance would lead to full insurance. The buyer would purchase exactly enough insurance to eliminate their financial risk. Here we have a somewhat different problem in that we are determining both supply and demand for insurance as well as the equilibrium prices.

When consumers have access to contingent commodity markets, we can again ask how much people insure? There are two issues to address here. One is whether full insurance is possible? The second is does full insurance occur in equilibrium?

27.3.1 What is Full Insurance?

To answer these questions, we have to clarify what we mean by full insurance. In Example 23.3.4, everything was stated in monetary terms, and full insurance meant that the monetary wealth was independent of the state. One might try to mimic that in a contingent commodities model by requiring that the value of consumption is the same in every state. This turns out to be a bad choice because in Arrow-Debreu models, the value of consumption in each state is connected to the probability of each state. This happens in Example 27.3.1 below, where the price of consumption in state two relative to state one is $p = 2$, reflecting the fact that state two is twice as likely to occur.[7] This was not an issue in Example 23.3.4, where the distinction between wealth and consumption is blurred.

We will define full insurance **in consumption terms**. A consumer is *fully insured* at x if their consumption is independent of the state, meaning $x_s = x_r$ for all states r and s. In other words, a consumer is fully insured if their consumption vector is certain.[8]

[7] See also Problem 27.2.2.

[8] We could also define full insurance in utility terms. Exercise 27.3.4 examines a case where there are equilibria where utility is the same in every state even though the consumption vector differs across states.

27.3.2 Is Full Insurance Possible?

Now let's consider the possibility of full insurance. If everyone is fully insured, not only is individual consumption certain, but aggregate consumption must be certain. Market clearing then requires that the aggregate endowment also be certain. Full insurance can only occur if there is no aggregate uncertainty, if the total endowment is independent of the state.

So the question sharpens to this: Can everyone can fully insure when there is no aggregate risk? We will see that full insurance requires that consumers agree on the probabilities of each state (the Common Probability Theorem). If they do agree on the probabilities, and if there is no aggregate risk, there is generally a full insurance equilibrium (the Full Insurance Theorem).

Most of our examples use a simple insurance model with a single good in each state. Insurance focuses on mitigating risk. This generally involves trying to make consumption across states more equal, at least if the cost is not too high. Using single-good models allows us to focus on the insurance issues, without getting distracted by questions concerning allocation of goods within the states

27.3.3 No Aggregate Uncertainty—Full Insurance

The first two examples involve consumers that are fully insured in equilibrium.

Example 27.3.1: No Aggregate Uncertainty–Full Insurance. Take a contingent goods exchange economy with one good ($m = 1$) and two states ($S = 2$). The contingent commodity space is $\mathbb{R}^{mS}_+ = \mathbb{R}^2_+$. Endowments are uncertain, with $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (0, 2)$ with aggregate endowment $\boldsymbol{\omega} = (2, 2)$.[9]

Preferences are described by identical expected utility functions: $u_i(\mathbf{x}^i) = \frac{1}{3}\ln x^i_1 + \frac{2}{3}\ln x^i_2$. Since the consumers have von Neumann-Morgenstern utility functions, we have built in the assumption that state two is twice as likely to occur as state one. The probabilities are $\pi_1 = 1/3$ and $\pi_2 = 2/3$. As there is only one good in each state we may write x_s for $x_{1,s}$ to simplify notation since there is no $x_{2,s}$ or $x_{3,s}$.

Similarly, we write $\mathbf{p} = (p_1, p_2)$ instead of $(p_{1,1}, p_{1,2})$. Since preferences are Cobb-Douglas, the price must be positive in both states. This allows us to normalize prices by setting $p_1 = 1$ and $p_2 = p$. In other words, p is the relative price of state-two consumption. Consumer incomes are now $m^1 = 2$ and $m^2 = 2p$. The Cobb-Douglas demand functions are

$$\mathbf{x}^1(p) = \frac{2}{3}\begin{pmatrix}1\\2/p\end{pmatrix} \quad \text{and} \quad \mathbf{x}^2(p) = \frac{2p}{3}\begin{pmatrix}1\\2/p\end{pmatrix}.$$

Market demand is

$$\mathbf{x}(p) = \frac{2(1+p)}{3}\begin{pmatrix}1\\2/p\end{pmatrix}.$$

We set demand equal to supply, $\boldsymbol{\omega} = (2, 2)^{\mathsf{T}}$, and solve for p to find the equilibrium relative price. Using the market for good one, we obtain $2(1 + p)/3 = 2$, so $p = 2$. By Walras' Law, this also clears the market for good two.

The equilibrium allocations are $\mathbf{x}^1 = (2/3, 2/3)$ and $\mathbf{x}^2 = (4/3, 4/3)$. The equilibrium prices are any positive scalar multiple of $\mathbf{p} = (1, 2)$.

Each consumer's equilibrium consumption is independent of the state. This is the hallmark of full insurance.

If the consumers lived in a world of autarky, where no trade was possible, they would consume varying amounts in each state because their endowments differ across the states. In equilibrium there is no uncertainty about consumption. This is made possible by the fact that there is no uncertainty about the aggregate endowment. ◄

[9] Here we are writing the commodity vector as a row (by state) of 1-dimensional columns. The aggregate endowment $\boldsymbol{\omega} = (2, 2)$ is the same in each state, but the individual endowments are very much state dependent.

27.3.4 No Aggregate Uncertainty—Cobb-Douglas Utility SKIPPED

This type of example easily generalizes to markets with many goods per state when consumers have identical Cobb-Douglas preferences.

Example 27.3.2: No Aggregate Uncertainty–Cobb-Douglas Utility. When consumers have identical Cobb-Douglas preferences and there is no aggregate uncertainty, every consumer fully insures. Let $U(\mathbf{x}) = \sum_s \pi_s \big(\sum_\ell \alpha_\ell \ln x_{\ell,s} \big)$ be the common Cobb-Douglas utility with each $\pi_s, \alpha_\ell > 0$ with $\sum_\ell \alpha_\ell = 1$ and $\sum_s \pi_s = 1$. Let $\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}^i$ be the aggregate endowment. Since there is no aggregate uncertainty, we can write $\omega_{\ell,s} = \omega_\ell$.

Consumer i's demand for good ℓ in state s is $x^i_{\ell,s} = \pi_s \alpha_\ell (\mathbf{p}\cdot\boldsymbol{\omega}^i)/p_{\ell,s}$ and market demand for (ℓ, s) is $x_{\ell,s} = \pi_s \alpha_\ell (\mathbf{p}\cdot\boldsymbol{\omega})/p_{\ell,s}$. In equilibrium, $x_{\ell,s} = \omega_\ell$, implying that $p_{\ell,s} = \pi_s \alpha_\ell (\mathbf{p}\cdot\boldsymbol{\omega})/\omega_\ell$. We normalize equilibrium prices by setting $\hat{\mathbf{p}}\cdot\boldsymbol{\omega} = 1$. This yields equilibrium prices $\hat{p}_{\ell,s} = \pi_s \alpha_\ell/\omega_\ell$.

An interesting feature of the equilibrium prices is that there is a price vector $\bar{\mathbf{p}} = (\alpha_\ell/\omega_\ell)$ with $\hat{\mathbf{p}}_s = \pi_s \bar{\mathbf{p}}$. The prices in a given state are the common probability of that state times $\bar{\mathbf{p}}$. We will see this phenomenon again in the Full Insurance Theorem.

It follows that consumer i consumes

$$x^i_{\ell,s} = \frac{\pi_s \alpha_\ell}{\hat{p}_{\ell,s}} (\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i) = \pi_s \alpha_\ell \frac{\omega_\ell}{\pi_s \alpha_\ell} (\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i) = (\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i)\omega_\ell.$$

In sum, $\hat{\mathbf{x}}^i = (\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i)\boldsymbol{\omega}$.

Because the aggregate endowment $\boldsymbol{\omega}$ is certain, so is each $\hat{\mathbf{x}}^i$. Although consumers have uncertain endowments, they are able to eliminate all risk in equilibrium. Consumers fully insure. ◀

27.3.5 Full Insurance: Agreement on Probabilities

One interesting fact about full insurance equilibria is that all consumers must agree on the probabilities of the states. It doesn't matter whether the probabilities are subjective or objective. All that matters is whether the consumers agree on the probabilities.

The key to this is that under full insurance, the price ratios of the same good in different states are also the probability ratios of those state, at least if the marginal rates of substitution make sense.

Common Probability Theorem. *Let $\mathcal{E} = (\mathfrak{X}_i, \pi_s^i, u_i, \boldsymbol{\omega}^i)$ be a contingent goods exchange economy with expected utility where each utility function $u_i \in \mathcal{C}^1$ is concave and obeys $Du_i > \mathbf{0}$ on $\mathfrak{X}_i$. Further, every endowment obeys $\boldsymbol{\omega}^i > \mathbf{0}$, with $\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}^i \gg \mathbf{0}$.*

If the aggregate endowment $\boldsymbol{\omega}$ is certain, and if $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i)$ is an Arrow-Debreu equilibrium with $\hat{\mathbf{p}} \gg \mathbf{0}$ where every consumer is fully insured and if there is a good ℓ with $\hat{x}^i_{\ell,s} > 0$ for every consumer i and state s, then every consumer agrees on the subjective probabilities. That is, there are probabilities π_s with $\pi_s = \pi_s^i$ for every consumer i. The common probability π_s is defined by

$$\pi_s = \frac{\hat{p}_{\ell,s}}{\sum_r \hat{p}_{\ell,r}}. \tag{27.3.2}$$

Moreover, if equilibrium prices obey $\hat{\mathbf{p}} \gg \mathbf{0}$, then $\hat{\mathbf{p}} = (\pi_s \bar{\mathbf{p}})$ for some $\bar{\mathbf{p}} \in \mathbb{R}^m_{++}$.

27.3.6 Proof of Common Probability Theorem

Proof. The smoothness of u_i means that the first-order conditions hold. Since consumption of good ℓ is positive in every state, the Lagrange multipliers for the constraints $x^i_{\ell,s}$ are zero for every state s by complementary slackness. The condition $Du_i > 0$ means that all preferences are monotonic. It follows that the Lagrange multiplier λ for the budget constraint is positive and hence

$$\pi^i_s \partial u_i/\partial x_{\ell,s}(\hat{x}^i) = \lambda \hat{p}_{\ell,s} > 0$$

for every state s. This means that $\text{MRS}^i_{(\ell,s)(\ell,r)}(\hat{x}^i)$ is finite for every consumer i.

Now fix states r and s. We have

$$\frac{\hat{p}_{\ell,r}}{\hat{p}_{\ell,s}} = \text{MRS}^i_{(\ell,r)(\ell,s)}(\hat{x}^i) = \frac{\pi^i_r}{\pi^i_s}\left[\frac{\partial u_i}{\partial x_\ell}(\hat{x}^i_r) \Big/ \frac{\partial u_i}{\partial x_\ell}(\hat{x}^i_s)\right]. \tag{27.3.3}$$

Full insurance means that $\hat{x}^i_r = \hat{x}^i_s$ for all states r and s, so the utility terms are identical. But then $\hat{p}_{\ell,r}/\hat{p}_{\ell,s} = \pi^i_r/\pi^i_s$. We can rearrange this to read $\hat{p}_{\ell,r}(\pi^i_s/\hat{p}_{\ell,s}) = \pi^i_r$. Summing over all r, and using $\sum_r \pi^i_r = 1$, we find

$$1 = \sum_r \pi^i_r = \frac{\pi^i_s}{\hat{p}_{\ell,s}} \sum_r \hat{p}_{\ell,r}.$$

Then

$$\pi^i_s = \frac{\hat{p}_{\ell,s}}{\sum_r \hat{p}_{\ell,r}}.$$

Since the right hand side is independent of i, we set $\pi_s = \pi^i_s$ to obtain equation 27.3.2.

Now suppose $\hat{\mathbf{p}} \gg \mathbf{0}$. The fact that $Du_i > \mathbf{0}$ means that preferences are locally non-satiated. Then Lemma 15.2.4 tells us that markets clear with equality. It follows that $\sum_i \hat{\mathbf{x}}^i = \boldsymbol{\omega} \gg \mathbf{0}$. As a result, equation 27.3.2 holds for every good ℓ. Setting $\bar{\mathbf{p}}_\ell = \sum_r \hat{p}_{\ell,r}$ then implies $\hat{\mathbf{p}}_s = \pi_s \bar{\mathbf{p}}$. □

27.3.7 Prices and Probabilities

When $m = 1$, equation 27.3.2 allows us to normalize equilibrium prices so that the prices are the common subjective probabilities of each state. Applying this to the full insurance Example 27.3.1, we obtain $\hat{\mathbf{p}} = (1/3, 2/3) = (\pi_1, \pi_2)$, which are precisely the subjective probabilities of both consumers.

When $m > 1$, equilibrium prices are proportional to $\hat{\mathbf{p}} = (\pi_s \overline{\mathbf{p}})$. As a result, unless $\overline{\mathbf{p}}_\ell$ is the same for every good ℓ, prices cannot be normalized to be the probabilities of each state. Nonetheless, comparison of prices across states allows us to recover the probabilities.

The interpretation of the price ratios is not so clear-cut in cases where consumers do not fully insure. As above, the price ratios are a marginal rate of substitution.[10] The problem is that the marginal rate of substitution depends on the different consumption levels in the two states as well as the probabilities. With full insurance, the marginal utilities are the same, leaving only the probabilities. The same sort of issue arises with intertemporal problems where the marginal rate of substitution involves consumption levels in addition to the discount factor. We only see the pure discount factor when consumption is equal in both periods. Nonetheless, just as it makes sense to think of marginal rates of substitution as affecting discounting, there is value in thinking of the normalized prices as market probabilities. In particular, when we introduce assets into the model, the price of the asset will be its expected value according to the market probabilities.

[10] If utility is not differentiable there may be a range of marginal rates of substitution, as defined by the supergradient.

27.3.8 Averaging the Economy

There's a natural way to generate a stateless economy from an contingent goods exchange economy with expected utility. We average it!

We start with a contingent goods exchange economy with expected utility, $\mathcal{E} = \left(\mathbb{R}^{mS}_+, \pi_i, u_i, \boldsymbol{\omega}^i\right)$. Define the *average* or *mean economy* $\overline{\mathcal{E}} = \left(\mathbb{R}^m_+, u_i, \overline{\boldsymbol{\omega}}^i\right)$ as the stateless economy with consumption sets $\mathbb{R}^m_+$, utility function u_i (without the expectation) and endowments $\overline{\boldsymbol{\omega}}^i = \mathrm{E}\boldsymbol{\omega}^i = \sum_s \pi_s \boldsymbol{\omega}^i_s$.

We map allocations in $\mathcal{E}$ into allocations in $\overline{\mathcal{E}}$ by taking expectations: $(\mathbf{x}^i)_i \mapsto (\mathrm{E}\mathbf{x}^i)_i$. This map has a right inverse, mapping allocations in $\mathbb{R}^{mI}_+$ into $\mathbb{R}^{mSI}_+$ by mapping $\mathbf{x}^i \in \mathbb{R}^m_+$ to $(\mathbf{x}^i, \dots, \mathbf{x}^i) \in \mathbb{R}^{ms}_+$. Given a price vector $\mathbf{p} \in \mathbb{R}^m_+$ for $\overline{\mathcal{E}}$, we can form a price vector in $\mathbb{R}^{mS}_+$ taking $(\pi_s \mathbf{p})_{s=1}^S$ as in the Common Probability Theorem. We will show that this maps equilibria in $\overline{\mathcal{E}}$ into equilibria in $\mathcal{E}$.

27.3.9 Average and Aggregate Endowments

Full insurance is only possible when the aggregate endowment is certain. In that case, we can say a little more about how the aggregate endowment in the contingent markets economy relates to the endowment in the mean economy.

Lemma 27.3.3. *Let ω be the aggregate endowment in a contingent markets economy $\mathcal{E}$ and $\overline{\omega} = \mathrm{E}\omega$ the aggregate endowment in the related average economy $\overline{\mathcal{E}}$. Then*

$$\omega = (\overline{\omega}, \dots, \overline{\omega}).$$

Proof.

$$\begin{aligned}
\overline{\omega} &= \sum_i \mathrm{E}\omega^i = \sum_i \left(\sum_s \pi_s \omega_s^i \right) \\
&= \sum_s \pi_s \left(\sum_i \omega_s^i \right) \\
&= \left(\sum_s \pi_s \right) \left(\sum_i \omega_r^i \right) \\
&= \sum_i \omega_r^i \\
&= \omega_r = \omega_s
\end{aligned}$$

because the aggregate endowment is certain. This shows that $\overline{\omega}$ is the same as the aggregate endowment in the original economy $\mathcal{E}$ in every state r. In other words,

$$\omega = (\overline{\omega}, \overline{\omega}, \dots, \overline{\omega}).$$

□

27.3.10 Full Insurance Theorem

We know that full insurance requires agreement on probabilities and no aggregate risk. The following theorem shows us that such economies have an equilibrium where all consumers are fully insured—their equilibrium consumption bundle is certain. It doesn't depend on which state occurs. The Common Probability Theorem and the Full Insurance Theorem combine to characterize full insurance equilibria in exchange economies where consumers have identical expected utility functions.

Full Insurance Theorem. *Let $\mathcal{E} = (\mathfrak{X}_i, \pi^i_s, u_i, \boldsymbol{\omega}^i)$ be a contingent goods exchange economy with expected utility where each utility function $u_i \in \mathcal{C}^1$ is concave and obeys $Du_i > 0$ on $\mathfrak{X}_i$. Further, every endowment obeys $\boldsymbol{\omega}^i > 0$, with $\sum_i \boldsymbol{\omega}^i \gg 0$.*

If the aggregate endowment $\boldsymbol{\omega}$ is certain, then there is an Arrow-Debreu equilibrium $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i)$ where $\hat{\mathbf{p}} = (\pi_1 \overline{\mathbf{p}}, \dots, \pi_S \overline{\mathbf{p}})$ and every consumer is fully insured. That is, $\hat{\mathbf{x}}^i_r = \hat{\mathbf{x}}^i_s$ for every r and s.

27.3.11 Proof of Full Insurance Theorem I

Proof. The mean economy $\overline{\mathcal{E}}$ satisfies the hypotheses of Corollary 16.5.5, so it has an equilibrium $(\overline{\mathbf{p}}, \overline{\mathbf{x}}^i)$. We will use this equilibrium to construct an equilibrium in the original economy.

To that end, define $\hat{\mathbf{p}} = (\pi_s \overline{\mathbf{p}})_s$ and $\hat{\mathbf{x}}^i = (\overline{\mathbf{x}}^i, \dots, \overline{\mathbf{x}}^i) \in \mathbb{R}^{mS}_+$. We will show that $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i)$ is an Arrow-Debreu equilibrium for the contingent markets economy $\mathcal{E}$. For this, we must show that (1) each consumer i maximizes utility at $\hat{\mathbf{x}}^i$ over their Arrow-Debreu budget set and (2) markets clear.

We start with the budget sets, and the budget sets start with income. Consumer i's income in the Arrow-Debreu equilibrium is $\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i$, while it is $\overline{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}}^i = \overline{\mathbf{p}}\cdot E\boldsymbol{\omega}^i$ in the mean economy. We calculate

$$
\begin{aligned}
\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i &= (\pi_1\overline{\mathbf{p}}, \dots, \pi_S\overline{\mathbf{p}})\cdot(\boldsymbol{\omega}^i_1, \dots, \boldsymbol{\omega}^i_S) \\
&= \sum_s \pi_s\, \overline{\mathbf{p}}\cdot\boldsymbol{\omega}^i_s \\
&= \overline{\mathbf{p}}\cdot\left(\sum_s \pi_s \boldsymbol{\omega}^i_s\right) \\
&= \overline{\mathbf{p}}\cdot E\boldsymbol{\omega}^i \\
&= \overline{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}}^i,
\end{aligned}
$$

showing that the two incomes are the same, $\hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i = \overline{\mathbf{p}}\cdot\overline{\boldsymbol{\omega}}^i$.

27.3.12 Proof of Full Insurance Theorem II

Now suppose $\mathbf{x}'$ is in consumer i's Arrow-Debreu budget set $B^i_{AD}(\hat{\mathbf{p}})$. Then

$$\begin{aligned}
\bar{\mathbf{p}}\cdot\bar{\boldsymbol{\omega}}^i &= \hat{\mathbf{p}}\cdot\boldsymbol{\omega}^i \\
&\geq \hat{\mathbf{p}}\cdot\mathbf{x}' \\
&= \sum_s \pi_s\,\bar{\mathbf{p}}\cdot\mathbf{x}'_s \\
&= \bar{\mathbf{p}}\cdot\left(\sum_s \pi_s\mathbf{x}'_s\right) \\
&= \bar{\mathbf{p}}\cdot E\mathbf{x}'.
\end{aligned}$$

This shows that i's expected consumption $E\mathbf{x}'$ is in the budget set in the average economy $\bar{\mathcal{E}}$.

27.3.13 Proof of Full Insurance Theorem III

We know that $\bar{\mathbf{x}}^i$ solves the consumer's problem in $\bar{\mathcal{E}}$ and that

$$u_i(\bar{\mathbf{x}}^i) = \sum_s \pi_s u_i(\bar{\mathbf{x}}^i) = Eu(\hat{\mathbf{x}}^i) = U_i(\hat{\mathbf{x}}^i).$$

By utility maximization in $\bar{\mathcal{E}}$,

$$U_i(\hat{\mathbf{x}}^i) = u_i(\bar{\mathbf{x}}^i) \geq u_i(E\mathbf{x}').$$

By concavity of u_i and Jensen's Inequality.

$$\begin{aligned} U_i(\hat{\mathbf{x}}^i) &\geq u_i(E\mathbf{x}') \\ &\geq Eu_i(\mathbf{x}') \\ &= \sum_s \pi_s u_i(\mathbf{x}'_s) \\ &= U_i(\mathbf{x}'), \end{aligned}$$

showing $\hat{\mathbf{x}}^i$ maximizes utility in the Arrow-Debreu budget set.

27.3.14 Proof of Full Insurance Theorem IV

We need only show that markets clear in the Arrow-Debreu economy $\mathcal{E}$ to complete the proof that $(\hat{\mathbf{p}}, \bar{\mathbf{x}}^i)$ is an Arrow-Debreu equilibrium in $\mathcal{E}$.

Recall that $(\overline{\mathbf{p}}, \bar{\mathbf{x}}^i)$ is an equilibrium in $\overline{\mathcal{E}}$. Market clearing in the mean economy $\overline{\mathcal{E}}$ means $\sum_i \bar{\mathbf{x}}^i \le \sum_i \overline{\boldsymbol{\omega}}^i = \overline{\boldsymbol{\omega}}$. Now

$$
\begin{aligned}
\sum_i \hat{\mathbf{x}}^i &= \sum_i \left(\bar{\mathbf{x}}^i, \dots, \bar{\mathbf{x}}^i\right) \\
&\le \sum_i (\overline{\boldsymbol{\omega}}, \dots, \overline{\boldsymbol{\omega}}) \\
&= \boldsymbol{\omega}
\end{aligned}
$$

by Lemma 27.3.3. This establishes market clearing in $\mathcal{E}$. Then $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i)$ is a equilibrium in $\mathcal{E}$. Finally, since each consumer's consumption is independent of the state, every consumer is fully insured. $\square$

The Full Insurance Theorem does not say that all Arrow-Debreu equilibria in economies without aggregate uncertainty are fully insured. There may be equilibria that are not fully insured in our sense. Exercise 27.3.4 examines a case satisfying the assumptions of the theorem where some Arrow-Debreu equilibria are not full insurance equilibria.

27.3.15 Partial Insurance with Different Priors SKIPPED

We have considered the case where all consumers have the same subjective prior probability distribution. But what if the consumers have different priors? In that case, the Common Probability Theorem tells us that consumers will not fully insure. The following example shows how different priors may lead to partial insurance even when the underlying utility function is the same.

Example 27.3.4: Partial Insurance with Different Priors. We start with Example 27.3.1, but alter one of the utility functions. Again there is one good and two states. Endowments are $\boldsymbol{\omega}^1 = (2,0)$ and $\boldsymbol{\omega}^2 = (0,2)$. We retain the utility function $u_1(\mathbf{x}^1) = \frac{1}{3}\ln x_1^1 + \frac{2}{3}\ln x_2^1$ for consumer one, but consumer two's utility is changed to $u_2(\mathbf{x}^2) = \frac{1}{2}\ln x_1^2 + \frac{1}{2}\ln x_2^2$.

One interpretation is that both consumers have expected utility functions with the same underlying utility $\ln x$, but the consumers have different opinions about the probabilities of the two states. Consumer one believes that the states have probabilities $(1/3, 2/3)$ while consumer two believes the probabilities are equal, $(1/2, 1/2)$. Since there is disagreement about the probabilities, at least one consumer's beliefs are incorrect. As we will see, the market probabilities that the equilibrium gives us will differ from both consumer's prior probabilities.

As in Example 27.3.1, we normalize prices to $\mathbf{p} = (1, p)$, yielding incomes $m^1 = 2$ and $m^2 = 2p$. Consumer demands are now

$$\mathbf{x}^1(p) = \frac{2}{3}\begin{pmatrix} 1 \\ 2/p \end{pmatrix} \quad \text{and} \quad \mathbf{x}^2(p) = p\begin{pmatrix} 1 \\ 1/p \end{pmatrix}.$$

Market demand is then

$$\mathbf{x}(p) = \begin{pmatrix} p + \frac{2}{3} \\ 1 + \frac{4}{3}p \end{pmatrix}.$$

Market clearing for good one requires $2 = p + 2/3$, so $p = 4/3$.

At the equilibrium price of $\mathbf{p} = (1, 4/3)$, demands are then $\mathbf{x}^1 = (2/3, 1)$ and $\mathbf{x}^2 = (4/3, 1)$. Neither consumer is fully insured! They both reach a point where the marginal benefit from further insurance is outweighed by the marginal cost.

If we normalize prices so that $p_1 + p_2 = 1$, we can interpret the prices as market probabilities. This yields $\mathbf{p} = (3/7, 4/7)$. The market probabilities are a compromise between the subjective probabilities of the two consumers.[11] ◀

[11] Keep in mind that these market probabilities are affected by the fact that consumption varies over the states, and affects the two consumers in an opposite way.

27.3.16 Aggregate Uncertainty—Partial Insurance

Another question that can be addressed is insurance under aggregate uncertainty, when the total endowment differs by state.

Example 27.3.5: Aggregate Uncertainty–Partial Insurance. We again modify Example 27.3.1. Again there is one good and two states. This time we boost the endowment of consumer one to $\boldsymbol{\omega}^1 = (3,0)$ instead of $(2,0)$ but leave consumer two's endowment unchanged at $\boldsymbol{\omega}^2 = (0,2)$. This introduces aggregate uncertainty. The new aggregate endowment of $\boldsymbol{\omega} = (3,2)$ is higher in state one than in state two.

We retain the utility functions $u_i(\mathbf{x}^1) = \frac{1}{3}\ln x_1^i + \frac{2}{3}\ln x_2^i$ for the two consumers $i = 1,2$. As in Example 27.3.1, we normalize prices to $\mathbf{p} = (1,p)$, now yielding incomes $m^1 = 3$ and $m^2 = 2p$. Consumer demands are now

$$\mathbf{x}^1(p) = \left(1, \frac{2}{p}\right) \quad \text{and} \quad \mathbf{x}^2(p) = \frac{2p}{3}\left(1, \frac{2}{p}\right).$$

Market demand is

$$\mathbf{x}(p) = \frac{(3+2p)}{3}\left(1, \frac{2}{p}\right).$$

Market clearing in state one requires that $(3+2p)/3 = 3$, implying $p = 3$. The normalized equilibrium price vector is $\mathbf{p} = (1,3)$.

In equilibrium, consumer one has demand $\mathbf{x}^1 = (1, 2/3)$ and consumer two has demand $\mathbf{x}^2 = (2, 4/3)$. Consumption varies by state for both consumers. Compared to Example 27.3.1, both consumers have gained from the increase in consumer one's endowment. ◄

In a sense, the consumers in Example 27.3.5 have insured as fully as mutually possible because each consumer's allocation is proportional to the aggregate endowment. As we saw in Example 15.3.5, identical Cobb-Douglas utility leads to an equilibrium where everyone's allocation is some fraction of the aggregate endowment. Individual endowments affect the share of aggregate income that each consumer receives, but do not affect the relative consumption of goods. When the aggregate endowment varies by state, the consumers can not fully insure.

27.4 Spot Markets: Arrovian Securities Model

We are not forced to use the Arrow-Debreu equilibrium when there are contingent goods. There are other kinds of equilibria in contingent goods economies. One alternate equilibrium for contingent economies was proposed by Arrow (1953).

Arrow's equilibrium adds a another type of market: spot markets. As in the Arrow-Debreu model, trading in the forward market takes place at time zero. However, the forward market involves only a single good. These forward contracts traded at time zero are called *Arrovian securities*. Once uncertainty is resolved, the world is in a specific state. The forward contracts pertaining to that state are executed and those paying off in other states expire. A spot market opens in the state that actually occurs. All goods may be traded there.

To get the intuition, we'll think of the Arrovian securities as paying off in gold, which also functions as numéraire. At time one, some state occurs. Call it state s. In state s we have our income (in gold) from the sale of our state s endowment. To that, we either add or subtract the gold that we have either received or paid out to settle our securities contracts. This gives us the income available for purchase of goods on the spot market. By trading in the various securities, we are able *ex ante* to move money between one possible spot market and another.

Although securities trading allows consumers to modify their income in the various states of the world, consumers may not trade forward contracts for goods other than gold. In such a world, you cannot contract for an umbrella to be delivered in the states where it rains. What you can do is buy securities that pay you the price of a umbrella in the states where it rains.

Unlike the Arrow-Debreu model, where we trade at the forward prices, in the Arrovian world we have to forecast the future price of umbrellas in the states where it rains. The Arrovian securities equilibrium is a perfect foresight equilibrium, where can perfectly forecast such prices.

27.4.1 The Structure of the Arrovian Market

To model this idea, we start with consumers that can only trade in forward markets for good 1 at time zero. At time one, uncertainty is resolved, revealing that the world is in state s. Consumers receive their endowments for state s. They also fulfill any contracts concerning good 1 for state s. Contacts concerning other states are now null and void. Consumers take or make delivery of any units of good one resulting from their holdings of Arrovian security s. Spot markets open where consumers can trade using their endowments as modified by their receipts or payments of good 1. Prior to the securities trading at time zero, consumers have perfect foresight expectations concerning prices. They correctly foresee what prices will be in any possible spot market, even though most of these markets will never open.

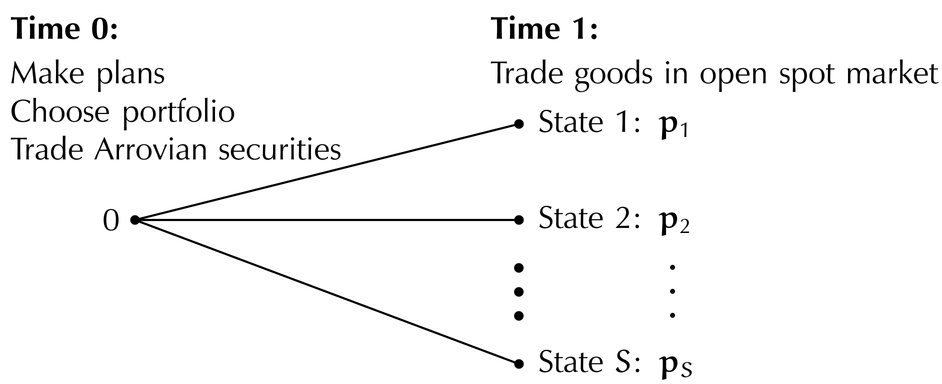

Figure 27.4.1: At time zero, only the trades in securities are made. The trades are based on contingent plans concerning the future. In period one, the state is revealed, a single spot market opens, and people execute only the plans made at time zero in the spot market that is open. Plans for other spot markets are abandoned.

27.4.2 More About Arrovian Securities

We will focus on the case of pure exchange, using a contingent exchange economy $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ comprised of m goods in each of S states. Consumers are characterized by their consumption sets, preferences, and endowments. As usual, the consumption sets are the positive orthant of the contingent commodity space, $\mathfrak{X}_i = \mathbb{R}_+^{mS}$.

Let $\mathbf{p}_s \in \mathbb{R}^m$ denote the price vector for spot market s and $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_S) \in \mathbb{R}_+^{mS}$ be the corresponding vector of price expectations for all the spot markets. Contingent trade takes place only for forward contracts on good one, the Arrovian securities. Since good one is traded in both the contingent and spot markets it has an extra price, its forward (time zero) price. Let $q_s \in \mathbb{R}$ be the time zero price of a unit of good one that will be delivered if state s occurs. The Arrovian securities prices are the vector $\mathbf{q} = (q_1, \ldots, q_S) \in \mathbb{R}^S$. Since there is an Arrovian security for every state, we say the Arrovian assets are *complete*.

A trading plan for i is expressed by a vector $\boldsymbol{z}^i = (z_1^i, \ldots, z_S^i) \in \mathbb{R}^S$ of demands for the contingent good (the Arrovian securities) and vectors of planned demands $\mathbf{x}_s^i \in \mathbb{R}_+^m$ in each of the potential spot markets. We can write the trading plan as $(\boldsymbol{z}^i, \mathbf{x}_s^i)$. The vector $\boldsymbol{z}^i$ is referred to as consumer i's *securities portfolio*.

While the demands in the spot markets are for non-negative quantities of goods, demands for securities can be either positive or negative. Buying a unit of Arrovian security s ($z_s = +1$) entitles you to one unit of good one if state s occurs, otherwise it pays nothing. Selling a unit of Arrovian security s ($z_s = -1$) commits you to providing (to the buyer) one unit of good one if state s occurs.

27.4.3 Creation of Arrovian Securities

At time zero, consumers trade *Arrovian securities*, contingent claims for good one in the various states. But where do they obtain the income needed to purchase these securities? After all, there is no endowment at time zero that can be sold to obtain income. And where do the securities themselves come from?

We start with the second question. The securities are forward contracts. They are created by the participants. This requires a appropriate legal framework to prevent people from selling securities they cannot pay off, or refusing to deliver contracted goods when due.

With no time zero endowment, the only source of income at time zero is the sale of Arrovian securities. The income used to purchase one Arrovian security is generated from the sale of other Arrovian securities. It is impossible to buy securities if you do not sell securities. This trade-off drives the creation of securities.

Because payoffs in the Arrovian securities model are in units of good one, it is tempting to think of good one as money. However, we cannot model fiat currencies that way. If the commodity used to trade in is itself valueless to consumers, the Arrovian securities model may break down, as in Example 27.6.1. There are models of nominal assets that could include fiat money, but they involve further complications, especially if the money is modeled more realistically. However, good one could be a commodity money that does have value to consumers (e.g., gold or silver).

27.4.4 Arrovian Budget Sets

To facilitate the definition of an Arrovian securities equilibrium, we start with the budget set. The budget set depends on both the asset prices $\mathbf{q}$ and expected spot market prices $\mathbf{p}$. We construct it as described above.

Arrovian Budget Set. Consumer i's *Arrovian budget set* is defined by

$$B_A^i(\mathbf{p},\mathbf{q}) = \Big\{\mathbf{x} \in \mathbb{R}_+^{mS} : \text{ there is } \mathbf{z} \in \mathbb{R}^S \text{ with } \sum_s q_s z_s \le 0$$
$$\text{and } \mathbf{p}_s \cdot \mathbf{x}_s \le \mathbf{p}_s \cdot \boldsymbol{\omega}_s^i + p_{1,s} z_s^i \text{ for all } s\Big\}.$$

We say $\mathbf{x}^i \in B_A^i(\mathbf{p},\mathbf{q})$ *via a portfolio* $\mathbf{z}^i$ if

1) $\mathbf{z}^i \in \mathbb{R}^S$
2) $\mathbf{q} \cdot \mathbf{z}^i \le 0$
3) $\mathbf{p}_s \cdot \mathbf{x}_s \le \mathbf{p}_s \cdot \boldsymbol{\omega}_s^i + p_{1,s} z_s^i$ for all s.

As advertised, income in each state s combines endowment income ($\mathbf{p}_s \cdot \boldsymbol{\omega}_s^i$) with the value of any contingent holdings of good one ($p_{1,s} z_s^i$). Notice that we multiply the amount of good one received (z_s^i) by its state s price ($p_{1,s}$). This income is used to pay for consumption in state s. Of course, the consumer's problem is to maximize utility over the Arrovian budget set.

27.4.5 Properties of Arrovian Budget Sets

The Arrovian budget set has some interesting homogeneity properties. Not only do we have the usual degree zero homogeneity in prices $(\mathbf{p}, \mathbf{q})$, $B^i_A(\lambda\mathbf{p}, \lambda\mathbf{q}) = B^i_A(\mathbf{p}, \mathbf{q})$, but there is some extra homogeneity.

We find that $B^i_A(\mathbf{p}, \lambda\mathbf{q}) = B^i_A(\mathbf{p}, \mathbf{q})$ and if $\mathbf{p}'_s = \mathbf{p}_s$ for $s \neq r$ and $\mathbf{p}'_r = \lambda\mathbf{p}_r$, then $B^i_A(\mathbf{p}', \mathbf{q}) = B^i_A(\mathbf{p}, \mathbf{q})$. The separate homogeneity in each spot market and in the securities market means we can separately normalize prices in each of the spot markets and in the Arrovian securities market.

State Scalar Product. Given a scalar λ_s for each state and a contingent commodity vector $\mathbf{x} = (\mathbf{x}_s)$, the *state scalar product* of $\boldsymbol{\lambda}$ and $\mathbf{x}$ is $\boldsymbol{\lambda} \odot \mathbf{x} = (\lambda_s \mathbf{x}_s)$.[12]

We can now express separate homogeneity more concisely.

Proposition 27.4.2. *If $\boldsymbol{\lambda} \in \mathbb{R}^S$ with $\boldsymbol{\lambda} \gg \mathbf{0}$ and $\mu > 0$, the Arrovian budget set obeys $B^i_A(\boldsymbol{\lambda} \odot \mathbf{p}, \mu\mathbf{q}) = B^i_A(\mathbf{p}, \mathbf{q})$.*

Proof. The budget constraints are equivalent, so the budget sets are the same. □

Proposition 27.4.2 includes all the cases discussed in the preceding paragraph. For example, setting $\boldsymbol{\lambda} = (\lambda, \dots, \lambda)$ and $\mu = \lambda$ is ordinary degree zero homogeneity. The case $\boldsymbol{\lambda} = (1, \dots, 1)$ captures degree zero homogeneity in asset prices alone. Setting $\mu = 1$ and $\boldsymbol{\lambda} = (1, \dots, 1, \lambda, 1, \dots, 1)$ where λ is in the r^{th} position, involve homogeneity only in state r.

[12] The state scalar product does not have a standard name. It may seem an odd construction, but if you think in terms of random variables, it is quite natural. The mapping $s \mapsto \mathbf{x}_s$ is a vector-valued random variable while $s \mapsto \lambda_s$ is a scalar random variable. We can form the scalar product, which is the vector random variable $s \mapsto (\boldsymbol{\lambda} \odot \mathbf{x})_s = \lambda_s \mathbf{x}_s$.

27.4.6 Arrovian Securities Equilibrium

We can now use the Arrovian budget to define the Arrovian Securities Equilibrium in the usual way. Consumers maximize utility and markets clear.

Arrovian Securities Equilibrium. Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ be an contingent goods exchange economy with m goods in each of S states. A collection $\big(\hat{\mathbf{p}}, \hat{\mathbf{q}}, (\hat{\mathbf{x}}^i), (\hat{\boldsymbol{z}}^i)\big)$ of spot prices $\hat{\mathbf{p}} = (\hat{\mathbf{p}}_s) \in \mathbb{R}_+^{mS}$, securities prices $\hat{\mathbf{q}} \in \mathbb{R}^S$, consumption plans $\hat{\mathbf{x}}^i \in \mathfrak{X}_i$, and portfolios $\hat{\boldsymbol{z}}^i \in \mathbb{R}^S$ is an *Arrovian securities equilibrium* if

1. Each $\hat{\mathbf{x}}^i$ is maximal in i's Arrovian budget set $B_A^i(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ via the portfolio $\hat{\boldsymbol{z}}^i$. That is, $\hat{\mathbf{x}}^i \in B_A^i(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ via the portfolio $\hat{\boldsymbol{z}}^i$ and $\hat{\mathbf{x}}^i \succsim_i \mathbf{x}$ for every $\mathbf{x} \in B_A^i(\hat{\mathbf{p}}, \hat{\mathbf{q}})$.
2. The asset market clears: $\sum_i \hat{z}_s^i \leq 0$ for every Arrovian security s.
3. All spot markets clear: $\sum_i \hat{\mathbf{x}}_s^i \leq \sum_i \boldsymbol{\omega}_s^i$ for every state s.

27.4.7 Time Consistency

There are several basic properties that equilibrium must have. The first is time consistency.

Time Consistency. We have assumed that consumers actually execute their plans when the spot market opens. But do they? Or do consumers want to revise their choices *ex post*? This is the issue of time consistency raised by Strotz (1955-56).

To address this we need to know how consumers will behave once uncertainty has been resolved. We need to know something about *ex post* preferences. If the preferences are separable relative to the state partition, they will have chosen the best consumption bundle in each state given the income available in that state. They will not have any reason to revisit their choice. If preferences are not separable relative to the state partition, they might wish to choose otherwise once they get there.

So when utility is separable across states, it is clear that consumers make time consistent choices in the Arrovian securities equilibrium. They don't want to revise their choice once the state is revealed.

27.4.8 Homogeneity of Equilibrium

Second, asset prices will be non-negative provided we have sufficient monotonicity of preferences. The point is that the Arrovian securities pay off in a commodity that consumers value. See Problem 27.4.1.

Third, the various homogeneities of the Arrovian budget sets translate directly to homogeneity of equilibrium.

Theorem 27.4.3. *If* $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ *is an Arrovian securities equilibrium, then* $(\boldsymbol{\lambda} \odot \hat{\mathbf{p}}, \mu\hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ *is also an Arrovian securities equilibrium for any* $\boldsymbol{\lambda} \gg \mathbf{0}$ *and* $\mu > 0$.

Proof. By Proposition 27.4.2, $B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = B^i_A(\boldsymbol{\lambda}\odot\hat{\mathbf{p}}, \mu\hat{\mathbf{q}})$. The fact that $\hat{\mathbf{x}}^i$ and $\hat{\mathbf{z}}^i$ maximize utility over $B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ means they also maximize utility over $B^i_A(\boldsymbol{\lambda} \odot \hat{\mathbf{p}}, \mu\hat{\mathbf{q}})$. Since market clearing also holds, $(\boldsymbol{\lambda} \odot \hat{\mathbf{p}}, \mu\hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ is an Arrovian securities equilibrium. □

27.4.9 Separability and Arrovian Equilibrium

When preferences are separable across states, as happens with expected utility, there is a step-by-step strategy for finding the equilibrium. The holdings of security s determine income in state s. This means that the spot markets are independent of one another once we know z_s. The homogeneity means we can even normalize prices separately in each spot market. We then solve for equilibrium in each state, conditional on a portfolio of securities. This gives us an indirect utility function defined over portfolios.

We maximize this indirect utility under the securities budget constraint to get asset portfolio demands, then solve for equilibrium securities prices. The equilibrium portfolios are then plugged into each of the spot markets to determine the equilibria there. The following example illustrates the procedure in a simple one-good, two-state economy.

This is a type of two-stage budgeting, with asset holdings in state s taking the role of income in state s.

27.4.10 A Simple Arrovian Equilibrium I

April 18, 2023

We return to the Arrow-Debreu equilibrium of Example 27.3.1. We'll recast it as an Arrovian securities market. We'll first solve it using asset holdings, and then solve it a second time in Example 27.4.5 using a more conventional two-stage budgeting.

Example 27.4.4: A Simple Arrovian Equilibrium. Recall that in Example 27.3.1, the endowments are $\boldsymbol{\omega}^1 = (2,0)$ and $\boldsymbol{\omega}^2 = (0,2)$. Preferences are described by the utility functions $u_i(\mathbf{x}^i) = \frac{1}{3}\ln x_1^i + \frac{2}{3}\ln x_2^i$ where x_s^i denotes consumption of good one (the only good) by individual i in state s.

Following our strategy laid out above, we first find all equilibria in the two spot markets, conditional on securities holdings. This allows us to compute utility as a function of the securities portfolio. Then we choose an optimal portfolio given the securities prices.

The first step is simple in this one-good model. There is only one price in each state $s = 1,2$. Because utility is increasing in the only good, its price must be positive. By Theorem 27.4.3, it can be normalized to 1. Thus we may take $\mathbf{p} = (1,1)$. Each consumer maximizes utility by using the income from their endowment plus any receipts from their asset holding, minus any payments due from their asset holding. Since there is only one good in each state, $x_s^i = \omega_s^i + z_s^i$ in the spot equilibrium.

We can now write a type of indirect utility, where utility depends on the asset portfolio. It is

$$v_i(\mathbf{z}^i) = \frac{1}{3}\ln(\omega_1^i + z_1^i) + \frac{2}{3}\ln(\omega_2^i + z_2^i).$$

This completes stage one.

27.4.11 A Simple Arrovian Equilibrium II

Stage two starts by finding the demand for securities. We maximize indirect utility subject to the budget constraint that $q_1 z_1^i + q_2 z_2^i = 0$. Since both Arrovian securities pay off in positive amounts of a valuable good (and no negative amounts), their prices must be strictly positive in equilibrium (no infinite utility in equilibrium). This allows us to use the Arrovian security for state one as numéraire and we write $\mathbf{q} = (1, q)$.

Moreover, since the securities pay off in a valuable good, the asset market budget constraint must hold with equality. The constraint is $z_1^i + q z_2^i = 0$. Indirect utility can now be rewritten as a function of z_1^i as follows:[13]

$$v_i(z_1^i) = \frac{1}{3}\ln(\omega_1^i + z_1^i) + \frac{2}{3}\ln(\omega_2^i - z_1^i/q).$$

After a little simplification, the first-order conditions are

$$\frac{1}{\omega_1^i + z_1^i} = \frac{2}{q\omega_2^i - z_1^i}.$$

Solving for the asset demands yields

$$\begin{aligned} z_1^i(q) &= \frac{q\omega_2^i - 2\omega_1^i}{3} \quad \text{and} \\ z_2^i(q) &= -\frac{z_1^i(q)}{q} = \frac{-\omega_2^i + 2\omega_1^i/q}{3}. \end{aligned}$$

[13] If we had more states, we would use a Lagrangian to maximize utility.

27.4.12 A Simple Arrovian Equilibrium III

Market demand for security one is then

$$z_1(q) = \frac{q\omega_2 - 2\omega_1}{3} = \frac{2q-4}{3}.$$

Market clearing requires this be zero, so $q = 2$. Normalized asset prices are $\mathbf{q} = (1, 2)$. If we had normalized so that the asset prices sum to one, we would have obtained the market probabilities $(1/3, 2/3)$ which coincides with the consumers' subjective probabilities in this full insurance case.

Asset demands are now

$$\boldsymbol{z}^1 = (-4/3, +2/3) \quad \text{and} \quad \boldsymbol{z}^2 = (+4/3, -2/3).$$

Consumer one sells asset one and buys asset two. Consumer two does the opposite. This makes sense as consumer one must buy asset two in order to have any consumption in state two. But that requires selling asset one to raise the funds to buy asset two. Consumer two is in the opposite situation.

This results in consumption

$$\mathbf{x}^1 = (2/3, 2/3) \quad \text{and} \quad \mathbf{x}^2 = (4/3, 4/3),$$

exactly as in the Arrow-Debreu equilibrium of Example 27.3.1. Both consumers are fully insured and have used the Arrovian securities to transfer income from the state where they have everything to the state where they have nothing. Consumer two benefits from owning the more valuable endowment. His endowment is more valuable because both consumers agree that the state where he gets a positive endowment is twice as likely as the state where he doesn't. For consumer one, the chance the endowment is valuable is one-half that of consumer two. ◀

27.4.13 Arrovian Two-Stage Budgeting SKIPPED

Another way to approach Arrovian equilibrium is to treat it as a two-stage budgeting problem using ordinary indirect utility. We illustrate it when preferences are given by expected utility.[14] Suppose consumer i has utility $\sum_{s=1}^{S} \pi_s^i u^i(\mathbf{x}_s)$ where $\pi_s^i \geq 0$ and $\sum_{s=1}^{S} \pi_s^i = 1$. Spot prices are $\mathbf{p}$ and asset prices are $\mathbf{q}$. Define consumer i's overall income m^i by $m^i = \sum_{s=1}^{S} \mathbf{p}_s \cdot \boldsymbol{\omega}_s^i = \mathbf{p} \cdot \boldsymbol{\omega}^i$. We write $\mathbf{m}^i = (m_1^i, \ldots, m_S^i)$ where m_s^i is the income consumer i receives in state s. If the consumer does not trade in the asset market, m_s^i will represent endowment income $m_s^i = \mathbf{p}_s \cdot \boldsymbol{\omega}_s^i$. Consumer i's indirect utility in spot market s is defined in the conventional way. When spot market s has prices $\mathbf{p}_s$ and consumer i has income m_s^i, indirect utility v_s^i is defined by

$$\begin{aligned} v^i(\mathbf{p}_s, m_s^i) = \max\ & u^i(\mathbf{x}_s) \\ \text{s.t. } & \mathbf{p}_s \cdot \mathbf{x}_s \leq m_s^i \\ & \mathbf{x}_s \geq 0. \end{aligned}$$

Consumer i has the same indirect utility in every spot market. Once we have indirect utility in the spot markets, we can calculate consumer i's overall indirect utility. For goods prices $\mathbf{p}$, asset prices $\mathbf{q}$, and overall income $m^i = \mathbf{p} \cdot \boldsymbol{\omega}^i$, we solve

$$\begin{aligned} v^i(\mathbf{p}, m^i) = \max\ & \sum_{s=1}^{S} \pi^i v_s^i(\mathbf{p}_s, m_s^i) \\ \text{s.t. } & \mathbf{q} \cdot \mathbf{m} \leq m^i \\ & \mathbf{m} \geq 0 \end{aligned}$$

to obtain consumer i's indirect utility.

The equilibrium is then found by using market clearing in the spot markets to find equilibrium spot prices conditional on the income distribution within each state. Then we find equilibrium prices for the asset market by using asset market clearing. This determines the income distribution(s) in each state, and so determines the equilibrium allocation(s). Finally, we use the fact that $m_s^i = \mathbf{p}_s \cdot \boldsymbol{\omega}_s^i + p_{1,s} z_s^i$ to find the securities portfolios $\mathbf{z}^i$.

[14] A similar approach works whenever utility is additive separable across states.

27.4.14 A Simple Arrovian Equilibrium, Take 2 SKIPPED

This approach to finding the Arrovian securities equilibrium makes clear that the function of the Arrovian securities is to move income between the various states, and that the asset prices are the prices of income in the various states. Since we have taken security one as our asset numéraire, the asset prices are relative to the price of the Arrovian security on state one. Equivalently, the income prices are relative to the price of income in state one.

We illustrate by redoing Example 27.4.4.

Example 27.4.5: A Simple Arrovian Equilibrium, Take 2. We consider the same two person, two period, one good economy as in Example 27.4.4. Indirect utility of consumer i in state s is found by maximizing $\ln x_s^i$ subject to the constraint that $p_s x_s^i \le m_s^i$ where $m_s^i = p_s\omega_s^i + p_s z_s^i$ is consumer i's income in state s. Since there is only one good, the consumer spends all of their income on it, $x_s^i = m_s^i/p_s$. Thus consumer i's indirect utility in state s is $v^i(p_s, m_s^i) = \ln(m_s^i/p_s)$.

We know that spot prices can be normalized to $\hat{\mathbf{p}} = (1,1)$. Then $x_s^i = m_s^i$ and $v_s^i(1, m_s^i) = \ln m_s^i$. Given choices $\mathbf{m} = (m_1^i, m_2^i)$, overall utility is $\frac{1}{3}\ln m_1^i + \frac{2}{3}m_2^i$. This must be maximized subject to an appropriate constraint on incomes.

To find the income constraint, notice that $m_1^i + qm_2^i = (\omega_1^i + q\omega_2^i) + (z_1^i + qz_2^i)$. The latter term is zero due to the asset budget constraint. We define $m^i = m_1^i + qm_2^i = \omega_1^i + q\omega_2^i$. Thus $m^1 = 2$ and $m^2 = 2q$.

Consumer i's overall maximization problem can be written

$$\begin{aligned} v^i(\hat{\mathbf{p}}, m^i) = \max\ & \frac{1}{3}v^i(\hat{\mathbf{p}}_1, m_1^i) + \frac{2}{3}v^i(\hat{\mathbf{p}}_2, m_2^i) \\ \text{s.t. } & \mathbf{q}\cdot\mathbf{m}^i \le m^i \\ & \mathbf{m}^i \ge 0. \end{aligned}$$

Since $\hat{\mathbf{p}} = (1,1)$, we must maximize the Cobb-Douglas utility $\frac{1}{3}\ln m_1^i + \frac{2}{3}\ln m_2^i$ under the constraint $m_1^i + qm_2^i \le m^i$. The solution is $\mathbf{m}^i = (m^i/3, 2m^i/3q)$.

We now use market clearing to determine q. Since $\mathbf{m}^i = \mathbf{x}^i$, $\mathbf{m}^1 + \mathbf{m}^2 = \mathbf{x}^1 + \mathbf{x}^2 = \boldsymbol{\omega}^1 + \boldsymbol{\omega}^2$. By Walras' Law, we need only clear one market. We choose market one and set $m_1^1 + m_1^2 = 2$. Then $(m^1 + m^2)/3 = 2$. Substituting the values for m^i, we obtain $(2 + 2q)/3 = 2$, implying $q = 2$, just as in Example 27.4.4. The demands are the same as in Example 27.4.4, and yield the same asset demands. ◀

27.5 The Arrovian Equivalence Theorem

The equivalence of the Arrow-Debreu and Arrovian equilibria in Example 27.4.4 is not unusual. The general case is covered in the Arrovian Equivalence Theorem. It not only tells us that the equilibrium allocations are the same, but also that we can use the securities prices as above to obtain the Arrow-Debreu equilibrium from the Arrovian securities equilibrium, or use the Arrow-Debreu prices to generate securities prices and demands. This should not be surprising as the Arrovian securities allow us to move income between the various states as needed to maximize utility, just as the markets for contingent commodities allow in the Arrow-Debreu equilibrium.

Arrovian Equivalence Theorem. *Let* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ *be a contingent exchange economy with* m *goods in each of S states.*

1. *If* $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i)$ *form an Arrow-Debreu equilibrium with* $\hat{\mathbf{p}} \gg \mathbf{0}$ *and* $\hat{\mathbf{x}}^i \in \mathbb{R}_+^{mS}$*, then the securities prices* $\hat{q}_s = \hat{p}_{1,s}$ *and portfolios* $\hat{z}_s^i = \hat{q}_s^{-1}\hat{\mathbf{p}}_s \cdot (\hat{\mathbf{x}}_s^i - \boldsymbol{\omega}_s^i)$ *make* $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ *an Arrovian securities equilibrium.*
2. *If* $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ *is an Arrovian securities equilibrium with* $\hat{\mathbf{p}} \gg \mathbf{0}$*,* $\hat{\mathbf{q}} \gg \mathbf{0}$*, and* $\hat{\mathbf{x}}^i \in \mathbb{R}_+^{mS}$*, then defining* $\mu_s = \hat{q}_s/\hat{p}_{1,s} > 0$ *makes* $(\boldsymbol{\mu} \odot \hat{\mathbf{p}}, \hat{\mathbf{x}}^i)$ *an Arrow-Debreu equilibrium.*

27.5.1 Proof of Arrovian Equivalence Theorem I

Proof. Given a price vector $\mathbf{p} \in \mathbb{R}^{mS}_{+}$, the Arrow-Debreu budget set is $B^i_{AD}(\mathbf{p}) = \{\mathbf{x}^i \in \mathbb{R}^{mS}_{+} : \sum_s \mathbf{p}_s \cdot (\mathbf{x}^i_s - \boldsymbol{\omega}^i_s) \le 0\}$.

The key is to show $B^i_{AD} = B^i_A$ in each part. The consumers will then make the same consumption choices, regardless of whether it is an Arrow-Debreu economy or an Arrovian securities economy. Goods markets will then clear in both cases.

(1A) We start with an Arrow-Debreu equilibrium $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i)$. Define the securities prices by $\hat{q}_s = \hat{p}_{1,s}$ for each s. The first step is to show $B^i_{AD}(\hat{\mathbf{p}}) = B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}})$. Let $\mathbf{x}^i \in B^i_{AD}(\hat{\mathbf{p}})$ and define the asset holdings by

$$z^i_s = \frac{1}{\hat{q}_s}\hat{\mathbf{p}}_s \cdot (\mathbf{x}^i_s - \boldsymbol{\omega}^i_s) = \frac{1}{\hat{p}_{1,s}}\hat{\mathbf{p}}_s \cdot (\mathbf{x}^i_s - \boldsymbol{\omega}^i_s).$$

Then the asset market budget constraint is satisfied since

$$\sum_s \hat{q}_s z^i_s = \sum_s \hat{\mathbf{p}}_s \cdot (\mathbf{x}^i_s - \boldsymbol{\omega}^i_s) \le 0$$

by the Arrow-Debreu budget constraint. It follows that $\mathbf{x}^i \in B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ so $B^i_{AD}(\hat{\mathbf{p}}) \subset B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}})$.

27.5.2 Proof of Arrovian Equivalence Theorem II

(1B) Conversely, let $\mathbf{x}^i \in B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}})$. Then there are portfolios $\boldsymbol{z}^i$ with $\sum_s \hat{q}_s \boldsymbol{z}^i_s \le 0$ and $\hat{\mathbf{p}}_s \cdot (\mathbf{x}^i_s - \boldsymbol{\omega}^i_s) \le \hat{p}_{1,s} \boldsymbol{z}^i_s$. Summing over s yields

$$\sum_s \hat{\mathbf{p}}_s \cdot (\mathbf{x}^i_s - \boldsymbol{\omega}^i_s) \le \sum_s \hat{p}_{1,s} \boldsymbol{z}^i_s = \sum_s \hat{q}_s \boldsymbol{z}^i_s \le 0$$

since $\hat{p}_{1,s} = \hat{q}_s$. Then $\mathbf{x}^i \in B^i_{AD}(\hat{\mathbf{p}})$ and so $B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}}) \subset B^i_{AD}(\hat{\mathbf{p}})$.

Combining the two inclusions shows $B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = B^i_{AD}(\hat{\mathbf{p}})$. Because the budget set is unchanged, consumers choose the same consumption bundles in each economy.

Further, if we define $\hat{z}^i_s = \frac{1}{\hat{q}_s} \hat{\mathbf{p}}_s \cdot (\hat{\mathbf{x}}^i_s - \boldsymbol{\omega}^i_s)$, market clearing implies

$$\sum_i \hat{z}^i_s = \frac{1}{\hat{q}_s} \sum_i \hat{\mathbf{p}}_s \cdot (\hat{\mathbf{x}}^i_s - \boldsymbol{\omega}^i_s) \le 0.$$

This shows the asset market clears and $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\boldsymbol{z}}^i)$ is an Arrovian securities equilibrium.

27.5.3 Proof of Arrovian Equivalence Theorem III

(2) Define $\mu_s = \hat{q}_s/\hat{p}_{1,s} > 0$. This choice of μ_s allows us to unify the separate spot markets under the same budget constraint. We already know from part (1) and separate degree-zero homogeneity of the Arrovian budget set that

$$B^i_{AD}(\boldsymbol{\mu} \odot \hat{\mathbf{p}}) \subset B^i_A(\boldsymbol{\mu} \odot \hat{\mathbf{p}}, \hat{\mathbf{q}}) = B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}}).$$

The Arrow-Debreu budget constraint implies the Arrovian budget constraints. We must show the converse, that the Arrovian budget constraints imply the Arrow-Debreu budget constraint, that $B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}}) \subset B^i_{AD}(\boldsymbol{\mu} \odot \hat{\mathbf{p}})$.

Suppose $\mathbf{x}^i \in B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}})$. Then

$$\begin{aligned}
(\boldsymbol{\mu} \odot \hat{\mathbf{p}}) \cdot (\mathbf{x}^i - \boldsymbol{\omega}^i) &= \sum_s \mu_s \hat{\mathbf{p}}_s \cdot (\mathbf{x}^i_s - \boldsymbol{\omega}^i_s) \\
&\le \sum_s \mu_s \hat{p}_{1,s} z^i_s \\
&= \sum_s \hat{q}_s z^i_s \le 0
\end{aligned}$$

showing that $\mathbf{x}^i \in B^i_{AD}(\boldsymbol{\mu} \odot \hat{\mathbf{p}})$. It follows that $B^i_A(\hat{\mathbf{p}}, \hat{\mathbf{q}}) \subset B^i_{AD}(\boldsymbol{\mu} \odot \hat{\mathbf{p}})$.

Once again, the budget sets are the same and consumers again choose the same consumption bundles in both economies. Market clearing for the Arrow-Debreu economy follows immediately from market clearing in the securities economy, so we have an Arrow-Debreu equilibrium. □

27.5.4 Existence of Arrovian Securities Equilibrium

An immediate corollary is that Arrovian securities equilibrium allocations exist under fairly mild conditions.

Corollary 27.5.1. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ be a contingent goods exchange economy with m goods in each of S states.*

If each consumer has strictly monotonic, convex, continuous preferences, and if $\boldsymbol{\omega} \gg \mathbf{0}$, then an Arrovian securities equilibrium exists where both goods and securities have strictly positive prices.

Proof. We add $Y = -\mathbb{R}_+^{mS}$ to make this a production economy. It is irreducible by Proposition 16.4.2. By the Equilibrium Existence Theorem: Production Economies it has an Arrow-Debreu equilibrium $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i)$ with $\hat{\mathbf{p}} > \mathbf{0}$. Due to strong monotonicity, there will be excess demand if any price is zero, so $\hat{\mathbf{p}} \gg \mathbf{0}$. Since $q_s = \hat{p}_{1,s}$, $\mathbf{q} \gg \mathbf{0}$. The Arrovian Equivalence Theorem yields an equivalent Arrovian securities equilibrium. $\square$

27.5.5 Pareto Optimality of Arrovian Equilibrium

Another corollary of the Arrovian Equivalence Theorem is that Arrovian securities equilibrium allocations are Pareto optimal when prices are strictly positive.

Corollary 27.5.2. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ be a contingent goods exchange economy with m goods in each of S states.*

If $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}, \hat{\mathbf{z}})$ is an Arrovian securities equilibrium with strictly positive prices, then $(\hat{\mathbf{x}})$ is a Pareto optimal allocation.

Proof. By the Arrovian Equivalence Theorem, there is an Arrow-Debreu equilibrium with the same allocation of goods. By the First Welfare Theorem, that allocation is Pareto optimal. □

27.5.6 Market Probabilities

One way to interpret the weights μ_s in the Arrovian Equivalence Theorem is to convert them to probabilities. Set

$$\pi_s = \frac{\mu_s}{\sum_s \mu_s}.$$

The π's are non-negative and sum to one. They can be interpreted as probabilities. In fact, they are *market probabilities*, derived from the equilibrium market prices.

By Theorem 27.4.3 there is an Arrovian securities equilibrium with these weights and the same allocations. When these probabilities are used as weights, the Arrow-Debreu price of any security is just its expected value.

In other words, we use the Arrow-Debreu price system $\pi \odot \hat{\mathbf{p}}$. Then the corresponding Arrovian price system has $q_s = \pi_s \hat{p}_{1,s}$. Since one unit of security s pays off with value $\hat{p}_{1,s}$ in state s and value 0 otherwise, its expected value is

$$\sum_{r=1}^{S} \pi_s \delta_{rs} \hat{p}_{1,r} = \pi_s \hat{p}_{1,s}.$$

27.5.7 Arrovian Market Probabilities

The market probabilities are illustrated in the following example.

Example 27.5.3: Arrovian Market Probabilities. We redo the partial insurance example (Example 27.3.4) as an Arrovian securities model. Recall that here are two consumers, two states, and one good. Endowments are $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (0, 2)$. Consumer one has utility $u_1(\mathbf{x}^1) = \frac{1}{3}\ln x_1^1 + \frac{2}{3}\ln x_2^1$, and consumer two's utility is $u_2(\mathbf{x}^2) = \frac{1}{2}\ln x_1^2 + \frac{1}{2}\ln x_2^2$.

In Example 27.3.4, we obtained an Arrow-Debreu equilibrium with $\hat{\mathbf{p}} = (1, 4/3)$, $\hat{\mathbf{x}}^1 = (2/3, 1)$, and $\hat{\mathbf{x}}^2 = (4/3, 1)$. We use the Arrovian Equivalence Theorem to find that $\hat{\mathbf{q}} = \hat{\mathbf{p}} = (1, 4/3)$, $\hat{\mathbf{z}}^1 = (-4/3, +1)$, and $\hat{\mathbf{z}}^2 = (+4/3, -1)$ gives us an Arrovian securities equilibrium.

We can separately normalize spot prices so that $\mathbf{p} = (1, 1)$ without changing the equilibrium. This means that good one is a numéraire in both spot markets. Now $\hat{p}_1 + \hat{p}_2 = 7/3$, so we divide the asset prices by $7/3$, obtaining $\boldsymbol{\pi} = (3/7, 4/7)$. These are the market probabilities. This normalization allows us to use $\boldsymbol{\pi}$ as the securities prices, yielding $(\mathbf{p}, \boldsymbol{\pi}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ as an Arrovian securities equilibrium with the same allocation of goods.

Moreover, security one has an expected payoff of $3/7$ and security two's expected payoff is $4/7$. Their prices are precisely their expected payoffs.

Here the market probabilities lie between the subjective probabilities of the two states. In fact, we can write them as weighted averages of the probabilities:

$$\frac{3}{7} = \frac{3}{7}\left(\frac{1}{3}\right) + \frac{4}{7}\left(\frac{1}{2}\right) \quad \text{and} \quad \frac{4}{7} = \frac{3}{7}\left(\frac{2}{3}\right) + \frac{4}{7}\left(\frac{1}{2}\right).$$

In each state there is a weight of $3/7$ on consumer one's subjective probability and $4/7$ on consumer two's subjective probability. ◀

27.6 Two Unusual Arrovian Equilibria SKIPPED

Is it important in the Arrovian Equivalence Theorem that prices are strictly positive? Yes! When prices are not strictly positive, Arrovian securities equilibria need not correspond to Arrow-Debreu equilibria. Indeed, they might not even be Pareto optimal.

27.6.1 A Non-Optimal Equilibrium I

Example 27.6.1: A Non-Optimal Arrovian Equilibrium. Suppose an economy has two consumers, two goods $\ell = 1, 2$, and two states, $s = 1, 2$. The consumers have identical utility functions, $u(\mathbf{x}) = \ln x_{2,1} + \ln x_{2,2}$. Notice that good one does not appear in the utility function. Endowments are $\boldsymbol{\omega}^1 = \big((2,2),(1,1)\big)^{\mathsf{T}}$ and $\boldsymbol{\omega}^1 = \big((1,1),(2,2)\big)^{\mathsf{T}}$.

Since good one is intrinsically valueless, it must have equilibrium price zero in each spot market. If the price were positive, there would be excess supply. The price of good two must be positive, otherwise consumers would try to consume an infinite amount. We take good two as numéraire in each spot market. The spot market budget constraints are then $x^i_{2,s} \le \omega^i_{2,s}$, so utility is maximized at $\mathbf{x}^i = \boldsymbol{\omega}^i$. Since utility is independent of asset holdings, portfolio demand can be anything, regardless of asset prices. We only require that the asset market clear. Once that is satisfied, asset prices can be anything. Since the assets are valueless, the obvious choice is $\mathbf{q} = (0, 0)$.

27.6.2 A Non-Optimal Equilibrium II SKIPPED

It follows that spot prices $\mathbf{p} = \big((0,1),(0,1)\big)$, asset prices $\mathbf{q} = (0,0)$, portfolios $\mathbf{z}^i = \mathbf{0}$, and consumption vectors $\mathbf{x}^i = \boldsymbol{\omega}^i$ constitute an Arrovian equilibrium.

The resulting allocation,

$$\left\{\left(\begin{pmatrix}2\\2\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\\2\end{pmatrix}\right)\right\}$$

is not Pareto optimal because the marginal rates of substitution between good two in states one and two differ across consumers. In fact, $\mathrm{MRS}^1_{21,22} = 1/2$ and $\mathrm{MRS}^2_{21,22} = 2$. This means that consumer one should trade good two in state one for good two in state two, and vice-versa for consumer two. We do that in the following allocation,

$$\left\{\left(\begin{pmatrix}2\\3/2\end{pmatrix},\begin{pmatrix}1\\3/2\end{pmatrix}\right),\left(\begin{pmatrix}1\\3/2\end{pmatrix},\begin{pmatrix}2\\3/2\end{pmatrix}\right)\right\}$$

which is both a Pareto improvement and Pareto optimal. To see the latter, note that $\mathrm{MRS}^i_{21,22} = 1$ for both consumers. The allocation of good one doesn't matter because it is not in the utility function.

Since the equilibrium allocation is not Pareto optimal, it cannot be an Arrow-Debreu allocation. This means there cannot be an Arrow-Debreu equilibrium corresponding to this Arrovian securities equilibrium. All this happens because the price of good one is zero. ◀

27.6.3 Missing Arrovian Security I SKIPPED

Before we end our examination of Arrovian securities models, we will push the model to its limits by eliminating the standard requirement that every state have its own Arrovian security.

Example 27.6.2: Missing Arrovian Security. Suppose an economy has two consumers and three states, $s = 1, 2, 3$. The consumers have identical utility functions, $u_i(\mathbf{x}^i) = \ln x_1^i + \ln x_2^i + \ln x_3^i$. Endowments are $\boldsymbol{\omega}^1 = (1, 0, 3)$ and $\boldsymbol{\omega}^2 = (3, 4, 1)$.

Although there are three states, we only have two Arrovian securities, one each for states one and two. There is no Arrovian security for state three. This set of Arrovian securities is incomplete because there are fewer securities than states. Nonetheless, we will find an Arrovian securities equilibrium.[15]

Of course, we can normalize the spot prices so that $p_s = 1$ in every state s. The lack of a security for state three means that the consumers simply consume their endowment in state three. The resulting indirect utility is

$$
\begin{aligned}
v_1(\mathbf{z}^1) &= \ln(1 + z_1^1) + \ln z_2^1 + \ln 3 \\
v_2(\mathbf{z}^2) &= \ln(3 + z_1^2) + \ln(4 + z_2^2) + \ln 1.
\end{aligned}
$$

[15] Technically, this is a Radner equilibrium as in section 28.1 with return matrix $\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

27.6.4 Missing Arrovian Security II SKIPPED

Both asset prices must be positive as there would otherwise be infinite demand for them. We can normalize prices so that $q_1 = 1$ and $q_2 = q$. For each consumer, we maximize indirect utility using the budget constraint $q_1 z_1^i + q_2 z_2^i = z_1^i + q z_2^i = 0$. This time we use a Lagrangian:

$$\mathcal{L} = v_i(\mathbf{z}^i) - \lambda_i(\mathbf{q}\cdot\mathbf{z}^i).$$

The first-order conditions are

$$\frac{1}{1+z_1^1} = \lambda_1, \quad \frac{1}{z_2^1} = \lambda_1 q$$

and

$$\frac{1}{3+z_1^2} = \lambda_2, \quad \frac{1}{4+z_2^2} = \lambda_2 q.$$

Combining the first-order conditions for consumer one shows that $\lambda_1(1+z_1^1+qz_2^1) = 2$ while consumer two yields $\lambda_2(3 + 4q + z_1^2 + qz_2^2) = 2$. Using the budget constraints $z_1^i + qz_2^i = 0$, we find $\lambda_1 = 2$ and $\lambda_2(3 + 4q) = 2$. That means $\lambda_1 = 2$ and $\lambda_2 = 2/(3 + 4q)$.

Now asset demand by consumer one is $z_1^1 = -1/2$ and $z_2^1 = 1/2q$, while asset demand by consumer two is $z_1^2 = -3/2 + 2q$ and $z_2^2 = -2 + 3/2q$. Market clearing for good one requires $-1/2 - 3/2 + 2q = 0$, so $q = 1$.

We can now state the equilibrium.

Asset prices:	$\mathbf{q} = (1, 1)$
Portfolio, consumer 1:	$\mathbf{z}^1 = (-1/2, 1/2)$
Portfolio, consumer 2:	$\mathbf{z}^2 = (1/2, -1/2)$
Goods prices:	$\mathbf{p} = (1, 1, 1)$
Demands, consumer 1:	$\mathbf{x}^1 = (1/2, 1/2, 3)$
Demands, consumer 2:	$\mathbf{x}^2 = (7/2, 7/2, 1)$.

We have used the fact that spot prices can be any positive numbers. Since we took good one as numéraire in each spot market, we used $\mathbf{p} = (1, 1, 1)$.

27.6.5 Missing Arrovian Security III: Pareto Optimality SKIPPED

The missing asset has proven no barrier to finding an Arrovian securities equilibrium. However, it does have consequences. The equilibrium allocation is not Pareto optimal, and so not an Arrow-Debreu equilibrium allocation.

Since allocation is interior, we can see this by calculating marginal rates of substitution. Good three is going to be the problem because the consumers must eat their endowments of it, they can't take advantage of trade to alter their consumption of good three. The marginal rates of substitution are $\text{MRS}^1_{13} = 6$ and $\text{MRS}^2_{13} = 2/7$. As these are not equal, the equilibrium allocation is not Pareto optimal. ◀

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