28. Radner Equilibrium

The Arrovian securities market allows only one type of security, a security that pays off in good one in exactly one state. Radner's (1972) model of asset markets allows for more general securities, securities that pay off various amounts of goods in various states. We will restrict our attention to securities that pay off in units of good one, but the model is easily generalized to encompass securities that pay off in any good.

The structure of the Radner model is the same as the Arrovian securities model. At time zero, only the assets are traded. Uncertainty is then resolved. Further trade takes place at time one, only in the spot market for the state that actually occurs.

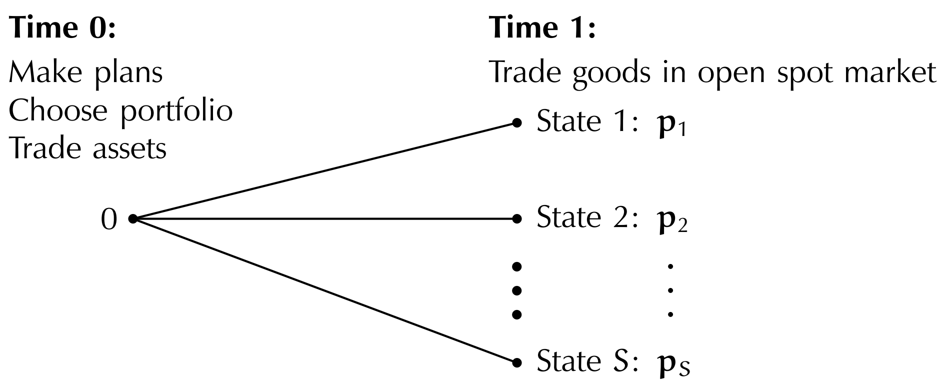

Figure 28.0.1: At time zero, only the trades in securities are made. The trades are based on contingent plans concerning the future. In period one, the state is revealed, a single spot market opens, and people execute the plans made at time zero in the spot market that is open.

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We formalize the Radner equilibrium in section one. It allows for more general securities than in the Arrovian model. The no-arbitrage condition and its consequences are examined in section two, which also gives conditions for Radner equilibrium prices to obey the non-arbitrage condition. We characterize arbitrage-free prices and show that they define market probabilities in section three. The equivalence of Radner equilibria with different sets of assets is examined in section four, which allows us to study the impact of complete and incomplete asset markets. Some simple derivative assets are introduced in section five.

28.1 The Radner Model

As in the Arrovian securities model, we restrict our attention to exchange economies. We use the same contingent commodities economy $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ with m goods in each of S states, where consumers are characterized by their consumption sets, preferences, and endowments. As usual in contingent goods economies, we restrict the consumption sets to be the positive orthant of the contingent commodity space, $\mathfrak{X}_i = \mathbb{R}_+^{mS}$.

An *asset* or *security* written on good one entitles the owner of the asset to receive an amount r_s of good one at $t = 1$ if state s occurs. The seller of the asset is obligated to provide an amount r_s of good one at $t = 1$ if state s occurs. An asset is characterized by its *return* or *payoff vector* $\mathbf{r} = (r_1, \dots, r_S)^\mathsf{T} \in \mathbb{R}^S$.

Each Arrovian security is a contingent contract on good one in a single state. Radner securities involve contingent contracts on good one in one or more states. They can be thought of as bundles of different amounts of Arrovian securities. Examples of Radner assets include $\mathbf{r}^1 = (1, 0, 0)^\mathsf{T}$ (one state), $\mathbf{r}^2 = (3, 0, 2)^\mathsf{T}$ (two states) and $\mathbf{r}^3 = (3, 4, 1)^\mathsf{T}$ (three states). A variety of other payoff patterns can be taken by assets. They include safe assets such as $\mathbf{e} = (1, \dots, 1)^\mathsf{T}$ (paying one unit of good one with certainty) to things like $\mathbf{r} = (1, 2, 0, 5, 1, 100)^\mathsf{T}$.

The bundle of contingent contracts can also include both sales and purchases. If you own a bundle where r_s is positive, you will receive good one in state s, while if r_s is negative, you will provide good one in state s. For example, if you bought an asset of with return vector $\mathbf{r} = (1, -2, -1, 5)^\mathsf{T}$, you would receive good one from the seller if state one or four occurs, and pay good one to the seller if states two or three occur.

28.1.1 The Return Matrix

Let K be the number of assets and z_k^i denote the amount of asset k owned by consumer i. We describe the asset structure by means of the *return matrix* or *payoff matrix* $\mathbf{R}$. It is the $S \times K$ matrix whose columns consist of the return vectors $\mathbf{r}^k$.

$$\mathbf{R} = \left(\mathbf{r}^1 \mid \mathbf{r}^2 \mid \cdots \mid \mathbf{r}^S\right) = \begin{pmatrix} r_1^1 & r_1^2 & \cdots & r_1^K \\ r_2^1 & r_2^2 & \cdots & r_2^K \\ \vdots & \vdots & & \vdots \\ r_S^1 & r_S^2 & \cdots & r_S^K \end{pmatrix}$$

The S rows represent the states and the K columns the assets. Thus r_s^k is the element in row s and column k.

The Arrovian model is a special case of the Radner model. In the Arrovian model, there are S assets with return vectors $\mathbf{r}^s = \mathbf{e}^s$ for $s = 1, \ldots, S$. This yields return matrix $\mathbf{I}_S$, the $S \times S$ identity matrix.

28.1.2 Asset Portfolios

A *portfolio* is a vector $\boldsymbol{z}^i = (z_1^i, \dots, z_K^i)^\mathsf{T} \in \mathbb{R}^K$ that indicates the amount of each Radner security that consumer i buys (positive z_k^i) or sells (negative z_k^i). The vector of payoffs from a portfolio $\boldsymbol{z}^i$ is found by multiplying the return vector for asset k by the amount of asset k owned, and adding up. The portfolio return vector is

$$\sum_{k=1}^{K} \mathbf{r}^k z_k^i = \mathbf{R}\boldsymbol{z}^i.$$

As indicated, the portfolio return vector can be expressed as a matrix product, $\mathbf{R}\boldsymbol{z}^i$.

28.1.3 Spot Markets in the Radner Model

Suppose spot market prices are $\mathbf{p}_s$ and consumer i has portfolio $\boldsymbol{z}^i$. The consumer has two sources of income available in spot market s: The value of the state s endowment and the value of the portfolio return in state s. The resulting budget constraint in spot market s is

$$\mathbf{p}_s \cdot \boldsymbol{x}_s^i \leq \mathbf{p}_s \cdot \boldsymbol{\omega}_s^i + p_{1,s} \left(\sum_k z_k^i r_s^k \right).$$

As in the Arrovian securities model, the value of asset k in state s is the price of good one multiplied by the amount of good one that portfolio pays.

Using the return matrix $\mathbf{R}$ we can write the Radner spot market budget constraints as

$$\mathbf{p}_s \cdot \boldsymbol{x}_s^i \leq \mathbf{p}_s \cdot \boldsymbol{\omega}_s^i + p_{1,s} \left(\mathbf{R} \boldsymbol{z}^i \right)_s \text{ for } s = 1, \ldots, S$$

where $\left(\mathbf{R} \boldsymbol{z}^i \right)_s$ denotes the s^{th} entry of the $S \times 1$ matrix $\mathbf{R}\boldsymbol{z}^i$.

In the Arrovian securities case, the return matrix is the $S \times S$ identity matrix $\mathbf{I}_S$, and the spot market budget constraints reduce to the usual Arrovian spot market constraints:

$$\mathbf{p}_s \cdot \boldsymbol{x}_s^i \leq \mathbf{p}_s \cdot \boldsymbol{\omega}_s^i + p_{1,s} z_s^i \text{ for } s = 1, \ldots, S.$$

The other constraint defining the budget set is the asset budget constraint. It has the same form as with Arrovian securities, $\mathbf{q} \cdot \boldsymbol{z}^i \leq 0$.

28.1.4 The Radner Budget Set

We can now define the Radner budget set (Radner, 1972).

Radner budget set. The *Radner budget set* $B^i_R(\mathbf{p},\mathbf{q};\mathbf{R})$ is defined by

$$B^i_{\mathbf{R}}(\mathbf{p},\mathbf{q};\mathbf{R}) = \Big\{\boldsymbol{x}^i \in \mathbb{R}^{mS}_+ : \text{ there exists } \boldsymbol{z}^i \in \mathbb{R}^K \text{ obeying}$$
$$\sum_k q_k z^i_k \le 0, \text{ such that for all states } s$$
$$\mathbf{p}_s \cdot \boldsymbol{x}^i_s \le \mathbf{p}_s \cdot \boldsymbol{\omega}^i_s + p_{1,s}\big(\mathbf{R}\boldsymbol{z}^i\big)_s\Big\}.$$

We say $\boldsymbol{x}^i \in B^i_R(\mathbf{p},\mathbf{q};\mathbf{R})$ *via a portfolio* $\boldsymbol{z}^i$ if

1. $\boldsymbol{z}^i \in \mathbb{R}^K$
2. $\mathbf{q}\cdot\boldsymbol{z}^i \le 0$
3. $\mathbf{p}_s \cdot \boldsymbol{x}^i_s \le \mathbf{p}_s \cdot \boldsymbol{\omega}^i_s + p_{1,s}\big(\mathbf{R}\boldsymbol{z}^i\big)_s$ for all s.

Of course, the Radner budget set depends on spot market prices ($\mathbf{p}$) and securities prices ($\mathbf{q}$). Normally, the return matrix ($\mathbf{R}$) will be held fixed. But there will be times when the dependence on the return matrix will also be important. Our notation reflects that.

28.1.5 Homogeneity of the Radner Budget Set

Our first result concerning Radner budget sets is that they enjoy the same homogeneity properties as the Arrovian budget sets.

Proposition 28.1.1. *If $\boldsymbol{\lambda} \in \mathbb{R}^S$ with $\boldsymbol{\lambda} \gg \mathbf{0}$ and $\mu > 0$, the Radner budget set obeys $B^i_R(\boldsymbol{\lambda} \odot \mathbf{p}, \mu\mathbf{q}; \mathbf{R}) = B^i_R(\mathbf{p}, \mathbf{q}; \mathbf{R})$.*

Proof. The Radner budget set for $(\mathbf{p}, \mathbf{q})$ is defined by the budget constraints

$$\begin{aligned} \mathbf{q}\cdot\boldsymbol{z}^i &\le 0, \\ \mathbf{p}_s\cdot\boldsymbol{x}^i_s &\le \mathbf{p}_s\cdot\boldsymbol{\omega}^i_s + p_{1,s}\big(\mathbf{R}\boldsymbol{z}^i\big)_s, \text{ for } s = 1,\dots,S \end{aligned}$$

while the Radner budget set for $(\boldsymbol{\lambda} \odot \mathbf{p}, \mu\mathbf{q})$ is defined by the budget constraints

$$\begin{aligned} \mu\mathbf{q}\cdot\boldsymbol{z}^i &\le 0, \\ \lambda_s\mathbf{p}_s\cdot\boldsymbol{x}^i_s &\le \lambda_s\mathbf{p}_s\cdot\boldsymbol{\omega}^i_s + \lambda_s p_{1,s}\big(\mathbf{R}\boldsymbol{z}^i\big)_s, \text{ for } s = 1,\dots,S \end{aligned}$$

Since $\mu > 0$ and each $\lambda_s > 0$, the budget constraints are equivalent for both budget sets. It immediately follows that the budget sets are identical. □

28.1.6 Arrovian Securities in the Radner Model

In fact, we had already seen a true Radner budget set in Example 27.6.2, which was a Radner equilibrium based on a subset of the Arrovian securities.

Example 28.1.2: Radner Budget with Arrovian Securities. In Example 27.6.2, the payoff matrix was

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with two assets and three states. Since the spot prices were $\mathbf{p} = (1, 1, 1)$, the spot market budget constraints were

$$\begin{aligned} x_1^i \le \omega_1^i + z_1^i &= p_1\omega_1^i + p_1(\mathbf{R}z^i)_1 \\ x_2^i \le \omega_2^i + z_2^i &= p_2\omega_2^i + p_2(\mathbf{R}z^i)_2 \\ x_3^i \le \omega_3^i \qquad &= p_3\omega_3^i + p_3(\mathbf{R}z^i)_3 \end{aligned}$$

and the asset market constraint was $q_1 z_1^i + q_2 z_2^i \le 0$. These are exactly the constraints of a Radner budget set. ◄

A full Arrovian securities market yields a Radner budget set where the payoff matrix is the $S \times S$ identity matrix.

28.1.7 Definition of Radner Equilibrium

The definition of the Radner equilibrium is basically the same as the Arrovian equilibrium, except that the Arrovian budget set is replaced by the Radner budget set. We have vectors of spot prices, securities prices, consumption plans, and asset portfolios. In equilibrium, consumers maximize utility and the asset market and all spot markets must clear.

Radner Equilibrium. Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ be an contingent goods exchange economy with m goods in each of S states. Suppose there are K assets defined by a $S \times K$ return matrix $\mathbf{R}$.

A collection $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ of spot prices $\hat{\mathbf{p}} = (\hat{\mathbf{p}}_s) \in \mathbb{R}^{mS}_+$, asset prices $\hat{\mathbf{q}} \in \mathbb{R}^K$, consumption plans $\hat{\mathbf{x}}^i \in \mathbb{R}^{mS}_+$, and asset portfolios $\hat{\mathbf{z}}^i \in \mathbb{R}^S$ is a *Radner equilibrium* with return matrix $\mathbf{R}$ if

1. Each $\hat{\mathbf{x}}^i$ is maximal in i's Radner budget set $B^i_R(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ via the portfolio $\hat{\mathbf{z}}^i$. That is, $\hat{\mathbf{x}}^i \in B^i_R(\hat{\mathbf{p}}, \hat{\mathbf{q}}; \mathbf{R})$ via the portfolio $\hat{\mathbf{z}}^i$, and $\hat{\mathbf{x}}^i \succsim_i \mathbf{x}$ for every $\mathbf{x} \in B^i_R(\hat{\mathbf{p}}, \hat{\mathbf{q}})$.
2. The asset market clears: $\sum_i \hat{z}^i_k \le 0$ for every k.
3. All spot markets clear: $\sum_i \hat{\mathbf{x}}^i_s \le \sum_i \boldsymbol{\omega}^i_s$ for every state s.

The Arrovian securities equilibrium is a then a Radner equilibrium where the payoff matrix is the $S \times S$ identity matrix.

Like the Arrovian securities equilibrium, the Radner equilibrium is time-consistent when preferences are state separable. However, we cannot guarantee that asset prices are always positive because we allow both positive and negative payoffs in various states. Nonetheless, if an asset obeys $\mathbf{r}^k > \mathbf{0}$, $(1, s)$-monotonicity of preferences will guarantee its price is positive because the consumers value it.[1]

[1] Recall that $(1, s)$-monotonicity means that more of good $(1, s)$ makes the consumer better off.

28.1.8 Homogeneity of Radner Equilibrium

As with the Arrovian securities equilibrium, the homogeneity properties of the Radner budget sets translate directly to homogeneity of equilibrium.

Theorem 28.1.3. *If* $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ *is a Radner equilibrium with return matrix* $\mathbf{R}$*, then* $(\boldsymbol{\lambda} \odot \hat{\mathbf{p}}, \mu\hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ *is also an Radner equilibrium with return matrix* $\mathbf{R}$ *for any* $\boldsymbol{\lambda} \gg \mathbf{0}$ *and* $\mu > 0$.

Proof. By Proposition 28.1.1, $B^i_R(\hat{\mathbf{p}}, \hat{\mathbf{q}}; \mathbf{R}) = B^i_R(\boldsymbol{\lambda} \odot \hat{\mathbf{p}}, \mu\hat{\mathbf{q}}; \mathbf{R})$, the fact that $\hat{\mathbf{x}}^i$ and $\hat{\mathbf{z}}^i$ maximize utility over $B^i_R(\hat{\mathbf{p}}, \hat{\mathbf{q}}; \mathbf{R})$ means they also maximize utility over $B^i_R(\boldsymbol{\lambda} \odot \hat{\mathbf{p}}, \mu\hat{\mathbf{q}}; \mathbf{R})$. Since market clearing also holds, $(\boldsymbol{\lambda} \odot \hat{\mathbf{p}}, \mu\hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ is a Radner equilibrium. □

The flexibility of the Radner model allows for some new phenomena that cannot occur in Arrovian securities models. One is the possibility of fewer assets than states. Such markets are called *incomplete*. We examined such a model in Example 27.6.2 and saw that when markets are incomplete, the equilibrium may not be Pareto optimal. Of course, an equilibrium that is not Pareto optimal must involve an allocation that differs from any Arrow-Debreu equilibrium allocation.

28.1.9 Radner Equilibrium with Risk-free Bond I

The next example shows another Radner equilibrium that is not based on Arrovian securities. Rather, both assets pay off in both states, but one is risky and the other is not. In this case, both assets have the same expected payoff. We will be particularly interested in how the extra risk affects the price of the risky asset.

Example 28.1.4: Radner Equilibrium with Risk-free Bond. Consider a Radner Equilibrium with one good, two states, and two consumers. Asset one is a risk-free bond that pays one unit of good one (the only good) in each state. Asset two pays 1/2 unit of the good in state one and 3/2 units in state two. The return matrix is

$$\mathbf{R} = \begin{pmatrix} 1 & 1/2 \\ 1 & 3/2 \end{pmatrix}.$$

Both consumers have utility $u(\mathbf{x}) = \ln x_1 + \ln x_2$. The endowments are $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (0, 2)$. There is no aggregate uncertainty.

Asset one is risk-free. Its mean payoff is one unit of good one, and the variance is zero. Asset two is risky. Its mean payoff is also one, but the variance is

$$\mathrm{var}(\mathbf{r}^2) = \frac{1}{2}(r_1^2 - 1)^2 + \frac{1}{2}(r_2^2 - 1)^2 = \frac{1}{4}$$

where $1/2$ is the probability of state one occurring. Here both consumers agree that the probability is $1/2$.

28.1.10 Radner Equilibrium with Risk-free Bond II

As is usual in these one-good models, we may use Theorem 28.1.3 to set both of the spot prices equal to 1 so $\mathbf{p} = (1, 1)$.

Given portfolio $\boldsymbol{z}^i$,

$$\mathbf{R}\boldsymbol{z}^i = \begin{pmatrix} z_1^i + \frac{1}{2}z_2^i \\ z_1^i + \frac{3}{2}z_2^i \end{pmatrix}.$$

Since $\mathbf{p} = (1, 1)$, it follows that the indirect utility from portfolio $\boldsymbol{z}^i$ is

$$v_i(\boldsymbol{z}^i) = \ln\left(\omega_1^i + z_1^i + \frac{1}{2}z_2^i\right) + \ln\left(\omega_2^i + z_1^i + \frac{3}{2}z_2^i\right).$$

The asset market budget constraint is $q_1 z_1^i + q_2 z_2^i = 0$. If either asset had $q_s \le 0$, infinite utility would be possible as both assets have positive payoffs. Both assets must have a positive price to prevent this. We will use the risk-free asset as numéraire, setting $\mathbf{q} = (1, q)$. Then $z_2^i = -(1/q)z_1^i$. Indirect utility can now be rewritten as a function of a single variable, z_1^i.

$$v_i(z_1^i) = \ln\left(\omega_1^i + \left(1 - \frac{1}{2q}\right)z_1^i\right) + \ln\left(\omega_2^i + \left(1 - \frac{3}{2q}\right)z_1^i\right).$$

Differentiate to obtain the first-order conditions:

$$\frac{1 - 1/2q}{\omega_1^i + (1 - 1/2q)z_1^i} + \frac{1 - 3/2q}{\omega_2^i + (1 - 3/2q)z_1^i} = 0.$$

Then

$$\begin{aligned} z_1^i(q) &= -\frac{1}{2}\left(\frac{\omega_1^i}{1 - 1/2q} + \frac{\omega_2^i}{1 - 3/2q}\right) \\ &= -q\left(\frac{\omega_1^i}{2q - 1} + \frac{\omega_2^i}{2q - 3}\right) \end{aligned}$$

28.1.11 Radner Equilibrium with Risk-free Bond III

Asset market clearing requires $z_1^1 + z_1^2 = 0$. This means that

$$\begin{aligned} 0 &= \frac{\omega_1}{2q-1} + \frac{\omega_2}{2q-3} \\ &= \frac{2}{2q-1} + \frac{2}{2q-3} \\ &= \frac{1}{2q-1} + \frac{1}{2q-3} \end{aligned}$$

where we used the fact that $\omega_1 = \omega_2 = 2$. The last line implies $q = 1$.

Both assets have the same price, so $\hat{\mathbf{q}} = (1, 1)$. Plugging into the asset demand functions, we obtain $\hat{\mathbf{z}}^1 = (-2, +2)$ and $\hat{\mathbf{z}}^2 = (+2, -2)$. It follows that $\hat{\mathbf{x}}^1 = (1, 1)$ and $\hat{\mathbf{x}}^2 = (1, 1)$. The allocation is the same as in Exercise 27.3.1, and is so Pareto optimal.

Both consumers are fully insured in equilibrium. Consumer one's portfolio pays $\mathbf{R}\hat{\mathbf{z}}^1 = (-1, +1)$ and consumer two's portfolio pays $\mathbf{R}\hat{\mathbf{z}}^2 = (+1, -1)$. This is the same solution that we would get with Arrovian securities where consumer one sell one unit of security one and buy one unit of security two. Consumer two would do the opposite, which is what happens in the Arrovian securities version examined in Exercise 27.4.3.

The remarkable thing is that both assets have the same price. It doesn't matter that asset two is a risky asset while asset one is risk-free, both with the same expected return. The usual intuition is that the risky asset should have a lower price since the expected return is the same and the risk higher. Surely our risk-averse consumers would pay less for such an asset. But they don't! The consumers want to equalize their payoffs across states, and both assets are equally valuable for this task. ◀

28.2 Arbitrage in Radner Equilibrium April 20, 2023

Another possibility that Radner equilibrium allows is for there to be more assets than states. While fewer assets than states can cause equilibria to be Pareto suboptimal, too many assets lead to a different complication. The extra assets open the possibility of arbitrage—buying and selling a combination of assets that guarantees a profit. Of course, such things cannot happen in a competitive equilibrium, and conditions that ensure they cannot plays a role in solving such models.

In fact, the possibility of arbitrage does not arise from the number of assets, but rather from the fact that the assets are linearly dependent. When there are more assets than states, they are necessarily linearly dependent and must admit the possibility of arbitrage.

28.2.1 Radner Equilibrium with Too Many Assets I

Let's take a look at such a model.

Example 28.2.1: Radner Equilibrium with Too Many Assets. We start with the same setting as Example 27.3.1, with $\mathcal{E}$ a contingent goods exchange economy containing two states and three assets. There is one good in each period, so consumption sets are $\mathbb{R}^2_+$. There are two consumers with utility $u_i(\mathbf{x}) = \ln x_1 + \ln x_2$ and endowments $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (0, 2)$. So far, it is a familiar model.

To spice it up a bit, we introduce a payoff matrix with more assets than states. Let the return matrix be

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

with a risk-free asset and both Arrovian securities.

28.2.2 Radner Equilibrium with Too Many Assets II

With one good in each state, we can use that good as the numéraire for the spot markets. This yields price vector $\mathbf{p} = (1, 1)$. Let $\boldsymbol{z}^i$ be consumer i's portfolio. The resulting income vectors can be written

$$\mathbf{m}^1 = \begin{pmatrix} z_1^1 + z_2^1 + 2 \\ z_1^1 + z_3^1 \end{pmatrix}, \qquad \mathbf{m}^2 = \begin{pmatrix} z_1^2 + z_2^2 \\ 2 + z_1^2 + z_3^2 \end{pmatrix}.$$

Notice that different choices of $\boldsymbol{z}^1$ and $\boldsymbol{z}^2$ can generate the same income vectors. All that is required is that $z_1^i + z_2^i$ and $z_1^i + z_3^i$ remain the same. With three variables and two equations, there are infinitely many ways to do this. This also means there are infinitely many portfolios that yield the same indirect utility. Utility maximization requires $x_s^i = m_s^i$, yielding indirect utility

$$\begin{aligned} v_1(\boldsymbol{z}^1) &= \ln x_1^1 + \ln x_2^1 \\ &= \ln(z_1^1 + z_2^1 + 2) + \ln(z_1^1 + z_3^1) \\ v_2(\mathbf{x}^2) &= \ln x_1^2 + \ln x_2^2 \\ &= \ln(z_1^2 + z_2^2) + \ln(2 + z_1^2 + z_3^2). \end{aligned}$$

28.2.3 Radner Equilibrium with Too Many Assets III

Each of the assets pays off in a good that is always valuable to both consumers. In equilibrium, this must have a positive price, otherwise utility maximization would not be possible. Since asset prices are positive, we can pick any asset as numéraire. We use asset one as numéraire and set $\mathbf{q} = (1, q_2, q_3)$. The asset budget constraint becomes $z_1^i + q_2 z_2^i + q_3 z_3^i = 0$, allowing us to replace z_1^i with $-q_2 z_2^i - q_3 z_3^i$ in indirect utility.

$$\begin{aligned}
v_1(\mathbf{z}^1) &= \ln x_1^1 + \ln x_2^1 \\
&= \ln\big((1-q_2)z_2^1 - q_3 z_3^1 + 2\big) + \ln\big(-q_2 z_2^1 + (1-q_3)z_3^1\big) \\
v_2(\mathbf{x}^2) &= \ln x_1^2 + \ln x_2^2 \\
&= \ln\big((1-q_2)z_2^1 - q_3 z_3^2\big) + \ln\big(2 + -q_2 z_2^1 + (1-q_3)z_3^2\big).
\end{aligned}$$

Differentiating v_i with respect to z_2^i and z_3^i yields the first-order conditions

$$\frac{1-q_2}{x_1^i} = \frac{q_2}{x_2^i}, \qquad \frac{q_3}{x_1^i} = \frac{1-q_3}{x_2^i}.$$

Recall that these are **necessary** conditions for an optimum. We also know that $x_s^i > 0$ for all consumers i and states s. This allows us to conclude that

$$\frac{1-q_2}{q_2} = \frac{x_1^i}{x_2^i} = \frac{q_3}{1-q_3},$$

which implies $q_2 + q_3 = 1$. Prices must obey this relation whenever utility maximization is possible. **If it fails, utility cannot be maximized.**

28.2.4 Radner Equilibrium with Too Many Assets IV

A little thought shows that this makes perfect sense. Assets two and three are Arrovian securities. If we hold one unit of each (total cost $q_2 + q_3$) we get the same return as from one unit of asset one (costing \$1). So let's compare the portfolios $z = (1, 0, 0)^T$ and $z' = (0, 1, 1)^T$. Both portfolios pay one unit of the consumption good regardless of which state occurs.

Now suppose z and z' had different prices, say $q = 1$ for z and $q' = 2$ for z'. This would make arbitrage possible. We could sell one unit of z', obtaining \$2 of income, and use that income to purchase two units of z. This is certainly affordable (in fact, it is costless) and increases our payoff by one unit of the consumption good in each state since $\mathbf{R}z = (1, 1)^T$. Doubling the transaction doubles the gain, tripling it triples the gain, etc. Because these arbitrages pay off in valuable goods, utility maximization is simply not possible if the portfolios z and z' have different costs.

When the costs are the same ($1 = q_2 + q_3$), it doesn't matter if we shift our asset holdings between z and z'. The two portfolios cost the same and have the same payoff in every state. The consumer has the same budget constraint in every state and makes the same consumption choices. We can make these shifts by adding a scalar multiple of $(z - z')$ to our portfolio.

We must take this into account when finding the equilibrium. There will be many equivalent equilibria, all with the same payoffs for each consumer, differing only in the addition of a multiple of $(z - z')$ to portfolios.

28.2.5 Radner Equilibrium with Too Many Assets V

One easy way to find all the equilibria starts by setting $z_1^i = 0$ for both consumers. We then solve for the equilibrium. Since $q_2 + q_3 = 1$, we may set $q_2 = q$ when $q_3 = 1 - q$. Asset market clearing becomes $qz_2^i + (1 - q)z_3^i = 0$, so $z_3^i = -qz_2^i/(1 - q)$.

We substitute in the first-order conditions for each consumer $i = 1, 2$, obtaining

$$\begin{aligned} q(2 + z_2^1) &= (1 - q)z_3^1 = -qz_2^1 \\ qz_2^2 &= (1 - q)(2 + z_3^2) = 2(1 - q) - qz_2^2 \end{aligned}$$

We solve these to find $z_2^1 = -1$ and $z_2^2 = 2(1-q)/q$. Then market clearing, $z_2^1 + z_2^2 = 0$, yields $q = 1/2$.

The equilibrium asset prices are $\hat{\mathbf{q}} = (1, 1/2, 1/2)$. Equilibrium asset demands are $\hat{\boldsymbol{z}}^1 = (0, -1, +1)$ and $\hat{\boldsymbol{z}}^2 = (0, +1, -1)$. The corresponding consumption vectors are $\hat{\mathbf{x}}^1 = (1, 1)$ and $\hat{\mathbf{x}}^2 = (1, 1)$.

We will still have the same equilibrium price and consumption vectors (but not portfolios) with any other portfolios that leave income the same. We saw that happens if $z_1^i + z_2^i$ and $z_1^i + z_3^i$ are held constant. So if we set $z_1^i = \alpha$ instead of zero, we must replace z_2^i by $z_2^i - \alpha$ and z_3^i by $z_3^i - \alpha$.

It follows that the other equilibria with $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{x}}$ have portfolios with $\boldsymbol{z}^1 = (\alpha, -1 - \alpha, 1 - \alpha)$ and $\boldsymbol{z}^2 = (-\alpha, 1 + \alpha, -1 + \alpha)$. For both consumers, the portfolio costs and returns are unchanged. This means consumption is unchanged because the budget set is unchanged. ◄

28.2.6 The No-Arbitrage Condition

Example 28.2.1 showed us that arbitrage can play an important role in determining asset prices. In our two period model, we can arbitrage if we can find a portfolio where asset purchases in time zero are no more than sales and that guarantees a positive payoff at time one. In more complex models where there is consumption at time zero, there is another way to arbitrage, increasing consumption at time zero without losing utility in the future.

Example 28.2.1 also showed us that if a Radner asset model has more assets than states, we will have many different equilibrium asset allocations that lead to the same consumption outcomes. Moreover, the asset prices are pinned down by the fact that certain portfolios give the same returns. This is all because we must avoid the possibility of arbitrage.

We now formalize the notion of arbitrage by using the payoff matrix.

Arbitrage. Given an $S \times K$ return matrix $\mathbf{R}$, we say asset prices $\mathbf{q} \in \mathbb{R}^K$ *admit arbitrage* if there is a portfolio $\boldsymbol{z} \in \mathbb{R}^K$ with $\mathbf{R}\boldsymbol{z} > \mathbf{0}$ and $\mathbf{q} \cdot \boldsymbol{z} \leq 0$. We refer to such a portfolio $\boldsymbol{z}$ as an *arbitrage*.[2]

Our definition of an arbitrage is that it is a portfolio that can be acquired at zero cost (or less) that guarantees a profit in at least one state, without losses in any state.

By definition, the zero vector cannot be an arbitrage. Asset prices $\mathbf{q}$ satisfy the *no-arbitrage condition* if they do not admit arbitrage. In this case we sometimes say that the asset price vector $\mathbf{q}$ is *arbitrage-free*. Of course, whether $\mathbf{q}$ is arbitrage-free is not a properly of the price vector in isolation, but of the asset price vector in conjunction with a return matrix $\mathbf{R}$.

[2] If the model included consumption at time zero, we would adjust the definition of arbitrage to allow an increase in consumption either in time zero or one without cost. In such models, asset prices *admit arbitrage* if there is a portfolio $\boldsymbol{z} \in \mathbb{R}^K$ with $\mathbf{R}\boldsymbol{z} \geq \mathbf{0}$, $\mathbf{q} \cdot \boldsymbol{z} \leq 0$ and at least one of the inequalities is strict. If the first is strict we can generate free income in period one. If the second is strict we can generate free income in period zero.

28.2.7 Arbitrage with More Assets than States

We already encountered the idea of arbitrage in Example 28.2.1. We examine this in more detail.

Example 28.2.2: Arbitrage with More Assets than States. In Example 28.2.1, the return matrix was

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

If $q_1 > q_2 + q_3$, form the portfolio

$$\boldsymbol{z} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \frac{q_1}{q_2 + q_3} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Then $\boldsymbol{z}$ is an arbitrage because $\mathbf{q} \cdot \boldsymbol{z} = 0$ and

$$\mathbf{R}\boldsymbol{z} = \frac{q_1 - q_2 - q_3}{q_2 + q_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

On the other hand, if $q_1 < q_2 + q_3$, the portfolio $-\boldsymbol{z}$ is an arbitrage. The asset prices are only arbitrage-free when $q_1 = q_2 + q_3$, exactly the optimality condition we found in Example 28.2.1. ◀

28.2.8 Example 28.2.3: The No-arbitrage Condition

The no-arbitrage condition usually places restrictions on asset prices. This is similar to the way that both monotonicity and the production technology can place restrictions on equilibrium prices in Walrasian equilibrium.

Example 28.2.3: The No-arbitrage Condition. Let the return matrix be

$$\mathbf{R} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix},$$

which guarantees arbitrage is possible (e.g., $\mathbf{R}\boldsymbol{e} = (2,3,4)^{\mathsf{T}} > \mathbf{0}$).

I claim the price vector $\mathbf{q} = (1,2)$ satisfies the no-arbitrage condition. Suppose we have an arbitrage. Then $\mathbf{R}\boldsymbol{z} > \mathbf{0}$ and $\mathbf{q}\cdot\boldsymbol{z} \le 0$.

Now

$$\mathbf{R}\boldsymbol{z} = \begin{pmatrix} z_1 + z_2 \\ z_1 + 2z_2 \\ z_1 + 3z_2 \end{pmatrix} > 0 \quad \text{and} \quad \mathbf{q}\cdot\boldsymbol{z} = z_1 + 2z_2 \le 0.$$

Taking the second component of $\mathbf{R}\boldsymbol{z}$, we find $z_1 + 2z_2 \ge 0$. But we also have $z_1 + 2z_2 = \mathbf{q}\cdot\boldsymbol{z} \le 0$. Thus $z_1 + 2z_2 = 0$.

Now if $z_2 > 0$, the first component of $\mathbf{R}\boldsymbol{z}$, $z_1 + z_2$, is negative, which is impossible. But if $z_2 < 0$, the third component of $\mathbf{R}\boldsymbol{z}$, $z_1 + 3z_2$, is negative, which is also impossible. It follows that $z_2 = 0$.

But then $z_1 \ge 0$ from the second component of $\mathbf{R}\boldsymbol{z} \ge 0$, and $z_1 \le 0$ from $\mathbf{q}\cdot\boldsymbol{z} \le 0$. It must be that $z_1 = 0$, in which case $\mathbf{R}\boldsymbol{z} = \mathbf{0}$. That means that $\mathbf{q} = (1,2)$ is arbitrage-free. ◄

28.2.9 The Possibility of Arbitrage

Let's take a closer look at the effect of the return matrix. Some return matrices simply do not allow portfolios that have even the possibility of arbitrage. This happens when the return matrix is zero. Then $\mathbf{R}z = 0$ for all z. Since no future payoff is positive, the no-arbitrage condition holds for every vector of asset prices.

In fact, if $\mathbf{R}z \not> 0$ for all z, no arbitrage is ever possible. This type of condition can easily be satisfied. The range of $\mathbf{R}$, $\operatorname{ran}\mathbf{R} = \{\mathbf{R}z : z \in \mathbb{R}^K\}$ is a vector subspace of $\mathbb{R}^S$. When $S = 2$, if $\operatorname{ran}\mathbf{R}$ is a negatively sloped line through the origin, no arbitrage is possible. For example, this happens when $K = 1$ and $\mathbf{R} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

As the previous example suggests, arbitrage may not be possible even when $\operatorname{ran}\mathbf{R}$ has dimension as high as $S - 1$, when it is a hyperplane in $\mathbb{R}^S$. All that is required is that $\operatorname{ran}\mathbf{R}$ not contain any positive elements. When the return matrix never allows arbitrage, the no-arbitrage condition has no real effect. It cannot place restrictions on prices.

When arbitrage is possible, when there is a portfolio z with $\mathbf{R}z > 0$, there will always be at least one asset price vector that admits arbitrage—the zero vector.

28.2.10 The Arbitrage Alternative

Are there non-trivial asset price vectors that admit arbitrage? The answer is yes, which is the key to the Arbitrage Alternative Theorem.

Arbitrage Alternative Theorem. *Let $\mathbf{R}$ be an $S \times K$ return matrix. Exactly one of the following holds:*

1. *There is a $\boldsymbol{z}^*$ with $\mathbf{R}\boldsymbol{z}^* > \mathbf{0}$ and there are non-zero asset price vectors $\mathbf{q}$ that permit arbitrage*
2. *Every vector of asset prices is arbitrage-free.*

Proof. **Part (1):** Clearly $\boldsymbol{z}^* \neq \mathbf{0}$. Let $C = \{-\boldsymbol{z}^*\}$ and use Separation Theorem A to find a non-zero vector $\mathbf{q}'$ and real number α with $\mathbf{q}'\cdot\boldsymbol{z}^* > \alpha$ and $\mathbf{q}'\cdot(-\boldsymbol{z}^*) < \alpha$. Then $\mathbf{q}'\cdot\boldsymbol{z}^* > -\alpha$. It follows that $\mathbf{q}'\cdot\boldsymbol{z}^* > |\alpha| \geq 0$. Then $\mathbf{q} = -\mathbf{q}'$ permits arbitrage.

Part (2): In this case there no $\boldsymbol{z} \in \mathbb{R}^K$ with $\mathbf{R}\boldsymbol{z} > \mathbf{0}$. It follows that no $\mathbf{q} \in \mathbb{R}^K$ permits arbitrage since there is no $\boldsymbol{z}$ with $\mathbf{R}\boldsymbol{z} > \mathbf{0}$. $\square$

28.2.11 Arbitrage Portfolios and Arbitrage-Free Prices

We refer to any portfolio $\mathbf{z}$ with $\mathbf{Rz} > \mathbf{0}$ as an *arbitrage portfolio for* $\mathbf{R}$, meaning that it will be an arbitrage for some non-zero vector of asset prices.

When the return matrix allows arbitrage, it puts restrictions on the set of possible equilibrium prices. We saw similar restrictions in Walrasian models with constant returns to scale production, where equilibrium prices had to be in the polar cone of the production set.

In fact, if $\mathbf{Rz} > \mathbf{0}$, any arbitrage-free asset prices $\mathbf{q}$ must lie in the open half-space $\{\mathbf{q} : \mathbf{q}\cdot\mathbf{z} > 0\}$. It follows that any arbitrage-free prices must lie in the convex cone

$$\bigcap_{\{\mathbf{z}:\mathbf{Rz}>\mathbf{0}\}} \{\mathbf{q} : \mathbf{q}\cdot\mathbf{z} > 0\}.$$

28.2.12 No Arbitrage in Radner Equilibrium

Arbitrage will typically be ruled out in equilibrium, as we saw in Example 28.2.1. The point is that if arbitrage is possible at a utility maximum, consumers are able to indefinitely increase their income in some state. This increases utility, which means they couldn't have been maximizing utility.

There are two main ways this argument can fail. It fails if good one is free. The securities pay off in units of good one, not directly in income. If good one is free, the extra good one obtained through arbitrage does not matter. It does not affect the budget constraint. The other way the argument can fail is if the increase in income fails to increase utility. This can happen if there is a bliss point in the state where the extra income occurs.

Recall that preferences are *strongly monotonic in good* ℓ or *strongly* ℓ*-monotonic* if $\mathbf{x}' \succ \mathbf{x}$ whenever $x'_k = x_k$ for $k \neq \ell$ and $x'_\ell > x_\ell$.

Our concern is with good one in every state s, good $(1, s)$. If for every state s, there is a consumer with strongly $(1, s)$-monotonic preferences, both problems can be solved. If $p_{1,s} = 0$, demand for good $(1, s)$ will be infinite, an impossibility in equilibrium. This means that the equilibrium price of good $(1, s)$ must be positive. That same consumer will also value extra income in state s, so the possibility of arbitrage would again lead to infinite demand.

No-Arbitrage Theorem. *Let* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ *be a contingent goods exchange economy with* m *goods in each of* S *states. Suppose there are* K *assets defined by a* $S \times K$ *return matrix* $\mathbf{R}$.

Let $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ *be a Radner equilibrium. If for every state* s *there is a consumer with strongly* $(1, s)$*-monotonic preferences, then* $\hat{\mathbf{q}}$ *is arbitrage-free.*

28.2.13 Proof of the No-arbitrage Theorem

No-Arbitrage Theorem. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ be a contingent goods exchange economy with m goods in each of S states. Suppose there are K assets defined by a $S \times K$ return matrix $\mathbf{R}$.*

Let $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ be a Radner equilibrium. If for every state s there is a consumer with strongly $(1,s)$-monotonic preferences, then $\hat{\mathbf{q}}$ is arbitrage-free.

Proof. We first show that $\hat{p}_{1,s} > 0$ for every state s. Suppose there is a state s with $\hat{p}_{1,s} = 0$. By hypothesis, there is also some consumer with strongly $(1,s)$-monotonic preferences. That consumer will have infinite demand for good $(1,s)$, contradicting the fact that $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ is a Radner equilibrium.

We now prove that $\hat{\mathbf{q}}$ is arbitrage-free. By way of contradiction, **suppose $\hat{\mathbf{q}}$ is not arbitrage-free**. Then there is an arbitrage portfolio $\mathbf{z}$ with $\hat{\mathbf{q}}\cdot\mathbf{z} \le 0$ and $\mathbf{R}\mathbf{z} > \mathbf{0}$. We must have $(\mathbf{R}\mathbf{z})_s > 0$ for some state s. Let i be a consumer that is strongly $(1,s)$-monotonic. The portfolio $\hat{\mathbf{z}}^i + \mathbf{z}$ will yield at least as much income in every state and more income in state s since $\hat{p}_{1,s}\left(\mathbf{R}\mathbf{z}\right)_s > 0$.

Define $\mathbf{x}'$ by setting $x'_{\ell,r} = \hat{x}^i_{\ell,r}$ for every $\ell = 1, \dots, m$ and $r \neq s$; $x'_{\ell,s} = \hat{x}^i_{\ell,s}$ for $\ell \neq 1$; and $x'_{1,s} = \hat{x}^i_{1,s} + (\mathbf{R}\mathbf{z})_s$. Consumer i can afford $\mathbf{x}'$ by choosing portfolio $\hat{\mathbf{z}}^i + \mathbf{z}$. Moreover, consumer i's utility increases, contradicting the fact that $\hat{\mathbf{x}}^i$ maximizes utility over the Radner budget set. This shows that $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ is not a Radner equilibrium.

This **contradiction shows** that $\hat{\mathbf{q}}$ must be arbitrage-free. □

If no consumers are strongly $(1,s)$-monotonic, Radner equilibria might allow arbitrage. Recall Example 27.6.1, the Arrovian equilibrium where good one was valueless. We found an equilibrium with $\mathbf{q} = (0,0)$. Since the return matrix was $\mathbf{I}_2$, there were plenty of possible arbitrages, including $\mathbf{z} = (1,1)^\mathsf{T}$.

28.2.14 The Arbitrage Pricing Theorem

The restrictions on asset prices become quite severe when an asset's return vector is a linear combination of other asset return vectors. In that case the Arbitrage Pricing Theorem allows us to price the first asset in terms of the others.

The Arbitrage Pricing Theorem applies whenever prices are arbitrage-free. It allows us to price one or more assets in terms of other assets. When we can attain the same returns with two different portfolios, the Arbitrage Pricing Theorem will imply that the portfolios must cost the same (Portfolio Law of One Price). Portfolios that give the same returns can then substituted willy-nilly without affecting the equilibrium.

We start with the Arbitrage Pricing Theorem. Since we are working in a vector space, it is enough to consider portfolios yielding a zero payoff in every state.

Arbitrage Pricing Theorem. *Suppose there is a z^* with $\mathbf{R}z^* > 0$. If $\mathbf{q}$ is arbitrage-free and $\mathbf{R}z = \mathbf{0}$, then $\mathbf{q}\cdot z = 0$.*

Proof. Let $\mathbf{q}$ be arbitrage-free and z obey $\mathbf{R}z = \mathbf{0}$. Now suppose $\mathbf{q}\cdot z < 0$. For small $\varepsilon > 0$, $\mathbf{q}\cdot(z + \varepsilon z^*) < 0$, but $\mathbf{R}(z + \varepsilon z^*) = \varepsilon \mathbf{R}z^* > 0$. This is impossible as it violates the no-arbitrage condition. On the other hand, if $\mathbf{q}\cdot z > 0$, then $\mathbf{q}\cdot(-z) < 0$ which also violates the no-arbitrage conditions. It follows that $\mathbf{q}\cdot z = 0$. $\square$

28.2.15 Arbitrage Pricing and Linear Combinations

When one asset is a linear combination of other assets, the Arbitrage Pricing Theorem implies that its price is the same linear combination of the prices of the other assets, a phenomenon we previously saw in Example 28.2.1.

Corollary 28.2.4. *Suppose there is a $\boldsymbol{z}^*$ with $\mathbf{R}\boldsymbol{z}^* > 0$ and that $\mathbf{q}$ is arbitrage-free. If $\mathbf{r}^1 = \sum_{k=2}^K \alpha_k \mathbf{r}^k$, then $q_1 = \sum_{k=2}^K \alpha_k q_k$.*

Proof. We can rewrite the condition that $\mathbf{r}^1 = \sum_{k=2}^K \alpha_k \mathbf{r}^k$ as

$$\mathbf{R}\begin{pmatrix} -1 \\ \alpha_2 \\ \vdots \\ \alpha_K \end{pmatrix} = 0.$$

Since $\mathbf{q}$ is arbitrage free and there is $\boldsymbol{z}^*$ with $\mathbf{R}\boldsymbol{z}^* > 0$, the Arbitrage Pricing Theorem yields

$$0 = \mathbf{q}\cdot\begin{pmatrix} -1 \\ \alpha_2 \\ \vdots \\ \alpha_K \end{pmatrix} = -q_1 + \left(\sum_{k=2}^{K} \alpha_k q_k\right).$$

Thus $q_1 = \sum_{k=2}^K \alpha_k q_k$. $\square$

28.2.16 Portfolio Law of One Price

Another corollary to the Arbitrage Pricing Theorem is the Portfolio Law of One Price. It states that any two portfolios that give the same portfolio return vector, $\mathbf{R}z = \mathbf{R}z'$, must have the same price, $\mathbf{q}\cdot z = \mathbf{q}\cdot z'$. We used a special case of the Portfolio Law of One Price in Example 28.2.1.

Portfolio Law of One Price. *Suppose there is a z^* with $\mathbf{R}z^* > 0$. If $\mathbf{q}$ is arbitrage-free and $\mathbf{R}z = \mathbf{R}z'$, then $\mathbf{q}\cdot z = \mathbf{q}\cdot z'$.*

Proof. Apply the Arbitrage Pricing Theorem to $z - z'$ to obtain the result. □

28.2.17 Same Payoffs, Same Radner Equilibrium Allocations

One immediate consequence is that if equilibrium asset portfolios are replaced by new portfolios that have the same payoff vectors and satisfy market clearing, then there is also a Radner equilibrium with the same prices and allocations which uses the new portfolios. We already used this idea in Example 28.2.1 to find multiple Radner equilibria with different portfolios but the same consumption allocation.

Corollary 28.2.5. *Let* $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ *be a Radner equilibrium and suppose* $\mathbf{z}^i$ *are portfolios whose payoff vectors obey* $\mathbf{R}\mathbf{z}^i = \mathbf{R}\hat{\mathbf{z}}^i$ *for every consumer* i *and that* $\sum_i \mathbf{z}^i \leq \mathbf{0}$. *Then* $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \mathbf{z}^i)$ *is also a Radner equilibrium.*

Proof. We start with the Radner equilibrium $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$. Here the spot markets clear and $\hat{\mathbf{x}}^i$ maximizes utility for consumer i. By hypothesis, the asset markets also clear at $\mathbf{z}^i$. All that we have to do to show $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \mathbf{z}^i)$ is a Radner equilibrium is to show that each consumer i can realize the choice $\hat{\mathbf{x}}^i$ by using the new portfolio $\mathbf{z}^i$.

The Portfolio Law of One Price tells us that $\hat{\mathbf{q}} \cdot \mathbf{z}^i = \hat{\mathbf{q}} \cdot \hat{\mathbf{z}}^i \leq 0$, so $\mathbf{z}^i$ is a feasible portfolio at asset prices $\hat{\mathbf{q}}$. All that is left is to show that $\mathbf{z}^i$ allows consumer i to purchase $\hat{\mathbf{x}}^i$ in the spot markets. But they can because

$$
\begin{aligned}
\hat{\mathbf{p}}_s \cdot \hat{\mathbf{x}}_s^i &\leq \hat{\mathbf{p}}_s \cdot \boldsymbol{\omega}_s^i + \hat{p}_{1,s} \left(\mathbf{R}\hat{\mathbf{z}}^i\right)_s \\
&= \hat{\mathbf{p}}_s \cdot \boldsymbol{\omega}_s^i + \hat{p}_{1,s} \left(\mathbf{R}\mathbf{z}^i\right)_s
\end{aligned}
$$

where we obtain the second line by using $\mathbf{R}\hat{\mathbf{z}}^i = \mathbf{R}\mathbf{z}^i$. $\square$

28.3 Arbitrage-Free Prices: Characterization

With the main theorems concerning arbitrage under control, it is time to examine arbitrage-free asset prices in more detail. We will characterize the arbitrage-free asset prices.

The Arbitrage Alternative Theorem tells us that that when there is no portfolio $\boldsymbol{z}^*$ with $\mathbf{R}\boldsymbol{z}^* > 0$, when no arbitrage is ever possible, every vector of asset prices must be arbitrage-free. In this case there is no characterization problem to solve. Every asset price vector is arbitrage-free, period.

In the other case, there is a portfolio $\boldsymbol{z}^*$ with a positive payoff, with $\mathbf{R}\boldsymbol{z}^* > 0$. The Arbitrage Alternative Theorem tells us that arbitrage is possible for some non-zero asset price vectors. Our task is to characterize those price vectors.

One important case is when $\operatorname{rank} \mathbf{R} = S$, implying that $\operatorname{ran} \mathbf{R} = \mathbb{R}^S$. Since the entire positive orthant is in the range of $\mathbf{R}$, there is no problem finding a portfolio $\boldsymbol{z}^*$ with $\mathbf{R}\boldsymbol{z}^* > 0$.

28.3.1 Arbitrage-free Prices: Necessary Conditions

The necessary conditions hold where there is a portfolio z^* with strictly positive payoffs, $\mathbf{R}z^* \gg 0$. Then arbitrage-free prices are positive linear combinations of the asset payoffs. More precisely, we can use the same positive linear combination of asset payoffs to determine the price of each asset. All arbitrage-free prices can be written as $\mathbf{q} = \mathbf{R}^\mathsf{T}\boldsymbol{\mu}$ for some $\boldsymbol{\mu} > 0$.[3]

Theorem 28.3.1. *Suppose there is a portfolio z^* with return vector $\mathbf{R}z^* \gg 0$. If the asset price vector $\mathbf{q}$ is non-zero and arbitrage-free, then there is $\boldsymbol{\mu} \in \mathbb{R}^S$, $\boldsymbol{\mu} > 0$ so that $\mathbf{q} = \mathbf{R}^\mathsf{T}\boldsymbol{\mu}$. Equivalently,*

$$q_k = \sum_{s=1}^{S} \mu_s r_s^k$$

for each asset k.

[3] Although not identical, the following result is similar to Farkas's Lemma.

28.3.2 Arb-free Prices: Proof of Necessary Conditions I

Theorem 28.3.1. *Suppose there is a portfolio* $\boldsymbol{z}^*$ *with return vector* $\mathbf{R}\boldsymbol{z}^* \gg \mathbf{0}$. *If the asset price vector is non-zero and arbitrage-free, then there is* $\boldsymbol{\mu} \in \mathbb{R}^S$, $\boldsymbol{\mu} > \mathbf{0}$ *so that* $\mathbf{q} = \mathbf{R}^\mathsf{T}\boldsymbol{\mu}$. *Equivalently,*

$$q_k = \sum_{s=1}^{S} \mu_s r_s^k$$

for each asset k.

Proof. Let

$$V = \mathbf{R}(\ker \mathbf{q}) = \{\boldsymbol{v} \in \mathbb{R}^S : \boldsymbol{v} = \mathbf{R}\boldsymbol{z} \text{ for some } \boldsymbol{z} \in \mathbb{R}^K \text{ with } \mathbf{q}\cdot\boldsymbol{z} = 0\}.$$

The set V is a vector subspace of $\mathbb{R}^S$, hence a closed convex set. Then define $W = \{\boldsymbol{w} \in \mathbb{R}^S : \boldsymbol{w} \gg \mathbf{R}\boldsymbol{z}^*\}$. The set W is open and convex.

The intersection $V \cap W$ is empty, for if $\boldsymbol{w} \in V \cap W$, we have $\mathbf{0} \ll \boldsymbol{w} = \mathbf{R}\boldsymbol{z}$ for some $\boldsymbol{z}$ with $\mathbf{q}\cdot\boldsymbol{z} = 0$, meaning that $\boldsymbol{z}$ is an arbitrage. Since $\mathbf{q}$ is arbitrage-free, this is impossible.

We can now employ Separation Theorem C to find $\boldsymbol{\xi} \neq \mathbf{0}$ and a real number α with $\boldsymbol{\xi}\cdot\boldsymbol{v} \le \alpha$ for every $\boldsymbol{v} \in V$ and $\boldsymbol{\xi}\cdot\boldsymbol{w} > \alpha$ for every $\boldsymbol{w} \in W$. In fact, since W is anti-comprehensive, $\boldsymbol{\xi} > \mathbf{0}$ by Corollary 7.2.6.

Since V is a vector space, $\mathbf{0} \in V$, implying that $\alpha \ge \boldsymbol{\xi}\cdot\mathbf{0} = 0$. I claim $\boldsymbol{\xi}\cdot\boldsymbol{v} = 0$ for every $\boldsymbol{v} \in V$. Suppose there is $\boldsymbol{v} \in V$ with $\boldsymbol{\xi}\cdot\boldsymbol{v} > 0$. For every positive integer n, $n\boldsymbol{v} \in V$, so $\boldsymbol{\xi}\cdot(n\boldsymbol{v}) \le \alpha$. But this is impossible since the left hand side converges to $+\infty$. If there is $\boldsymbol{v} \in V$ with $\boldsymbol{\xi}\cdot\boldsymbol{v} < 0$, we use the above argument on $(-\boldsymbol{v})$. Only $\boldsymbol{\xi}\cdot\boldsymbol{v} = 0$ is possible.

Finally, $2\mathbf{R}\boldsymbol{z}^* \gg \mathbf{R}\boldsymbol{z}^*$, implying that $2\mathbf{R}\boldsymbol{z}^* \in W$. By separation, $2\boldsymbol{\xi}\cdot(\mathbf{R}\boldsymbol{z}^*) > \alpha \ge 0$, so $(\mathbf{R}^\mathsf{T}\boldsymbol{\xi})\cdot\boldsymbol{z}^* = \boldsymbol{\xi}\cdot(\mathbf{R}\boldsymbol{z}^*) > 0$. It follows that $\mathbf{R}^\mathsf{T}\boldsymbol{\xi} \neq \mathbf{0}$.

28.3.3 Arb-free Prices: Proof of Necessary Conditions II

Proof continues. Let

$$\boldsymbol{z} = \mathbf{R}^{\mathsf{T}}\boldsymbol{\xi} - \left(\frac{\mathbf{q}\cdot(\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi})}{\|\mathbf{q}\|^2}\right)\mathbf{q}.$$

Notice that $\mathbf{q}\cdot\boldsymbol{z} = 0$.

By way of contradiction, **suppose $\mathbf{q}$ is not proportional to $\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi}$**. By the Cauchy-Schwartz inequality,[4]

$$\begin{aligned}(\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi})\cdot\boldsymbol{z} &= (\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi})\cdot\left(\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi} - \left(\frac{\mathbf{q}\cdot(\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi})}{\|\mathbf{q}\|^2}\right)\mathbf{q}\right)\\ &= \|\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi}\|^2 - \frac{|\mathbf{q}\cdot(\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi})|^2}{\|\mathbf{q}\|^2}\\ &> 0.\end{aligned}$$

Since $\mathbf{q}\cdot\boldsymbol{z} = 0$, $\mathbf{R}\boldsymbol{z} \in V$. But then $\boldsymbol{\xi}\cdot(\mathbf{R}\boldsymbol{z}) > 0$, which contradicts the properties of $\boldsymbol{\xi}$. This **contradiction shows that $\mathbf{q}$ and $\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi}$ are proportional**. Since $\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi} \neq \mathbf{0}$, we may write $\mathbf{q} = \lambda\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi}$ for some $\lambda \neq 0$.[5]

Finally, $\lambda\boldsymbol{\xi}\cdot(\mathbf{R}\boldsymbol{z}^*) = \mathbf{q}\cdot\boldsymbol{z}^* > 0$ because $\mathbf{q}$ is arbitrage-free. It follows that $\lambda > 0$. Then set $\boldsymbol{\mu} = \lambda\boldsymbol{\xi} > \mathbf{0}$ because $\boldsymbol{\xi} > \mathbf{0}$, completing the proof. □

[4] The Cauchy-Schwartz inequality states that $|\boldsymbol{x}\cdot\boldsymbol{y}|^2 \leq \|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2$ with equality only if $\boldsymbol{x}$ is proportional to $\boldsymbol{y}$ or vice-versa.

[5] An alternative to using the Cauchy-Schwartz inequality is to note that $\boldsymbol{\xi}\cdot(\mathbf{R}\boldsymbol{z}) = 0$ for all $\boldsymbol{z} \in \ker\mathbf{q}$. Now $\boldsymbol{\xi}\cdot(\mathbf{R}\boldsymbol{z}) = (\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi})\cdot\boldsymbol{z}$, so $\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi}$ is perpendicular to $\ker\mathbf{q}$. But then, since $\ker\mathbf{q}$ is a hyperplane, either $\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi} = \mathbf{0}$ or $\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi}$ is proportional to $\mathbf{q}$. Since the first is impossible, we can write $\mathbf{q} = \lambda\mathbf{R}^{\mathsf{T}}\boldsymbol{\xi}$ and set $\boldsymbol{\mu} = \lambda\boldsymbol{\xi}$.

28.3.4 Theorem 28.3.1 is Not Enough

There's a problem with Theorem 28.3.1. It requires that there is a portfolio z^* with $\mathbf{R}z^* \gg 0$ to characterize arbitrage-free prices. However we know that arbitrage is possible whenever there is a portfolio with $\mathbf{R}z > 0$. What if arbitrage is possible, but there is no z^* with $\mathbf{R}z^* \gg 0$? Can we still characterize arbitrage-free prices?

Indeed we can. We strengthen Theorem 28.3.1 as follows.

Theorem 28.3.2. *Suppose there is a portfolio z^* with return vector $\mathbf{R}z^* > 0$. If the asset price vector $\mathbf{q}$ is non-zero and arbitrage-free, then there is $\mu \in \mathbb{R}^S$, $\mu > 0$ so that $\mathbf{q} = \mathbf{R}^\mathsf{T}\mu$. Equivalently,*

$$q_k = \sum_{s=1}^{S} \mu_s r_s^k$$

for each asset k.

28.3.5 Proof of Theorem 28.3.2

Theorem 28.3.2. *Suppose there is a portfolio $\boldsymbol{z}^*$ with return vector $\mathbf{R}\boldsymbol{z}^* > \mathbf{0}$. If the asset price vector $\mathbf{q}$ is non-zero and arbitrage-free, then there is $\boldsymbol{\mu} \in \mathbb{R}^S$, $\boldsymbol{\mu} > \mathbf{0}$ so that $\mathbf{q} = \mathbf{R}^T\boldsymbol{\mu}$. Equivalently,*

$$q_k = \sum_{s=1}^{S} \mu_s r_s^k$$

for each asset k.

Proof. Define the set T by

$$T = \{s : (\mathbf{R}\boldsymbol{z})_s = 0 \text{ for all } \boldsymbol{z} \text{ with } \mathbf{R}\boldsymbol{z} > \mathbf{0}\}$$

It consists of the states where arbitrage portfolios always have zero payoff. Moreover, since $\mathbf{R}\boldsymbol{z}^* > \mathbf{0}$, $T \neq \{1, \dots, S\}$.

For $s \notin T$, let $\boldsymbol{z}_s$ be a arbitrage portfolio with $(\mathbf{R}\boldsymbol{z}_s)_s > 0$. Such exist by the definition of T. Let $\boldsymbol{z}^{**} = \sum_{s \notin T} \boldsymbol{z}_s$. It is an arbitrage portfolio because $\mathbf{R}\boldsymbol{z}^{**} = \sum_{s \notin T} \mathbf{R}\boldsymbol{z}_s > \mathbf{0}$. It also obeys $(\mathbf{R}\boldsymbol{z}^{**})_s > 0$ for every $s \notin T$.

Now form $\hat{\mathbf{R}}$ by removing every row $s \in T$ from $\mathbf{R}$. Let $S' = S - \#T$, so that $\hat{\mathbf{R}}$ is an $S' \times K$ matrix and let $\hat{\boldsymbol{z}}$ be the vector obtained by deleting all rows in T from $\boldsymbol{z}^{**}$. Apply Theorem 28.3.1 to $\hat{\mathbf{R}}$ and $\hat{\boldsymbol{z}}$ to find a S'-vector $\boldsymbol{\mu}'$ with $\mathbf{q} = \hat{\mathbf{R}}^T\boldsymbol{\mu}'$. Set $\mu_s = 0$ for $s \in T$ and $\mu_s = \mu'_s$ for $s \notin T$.

Finally, if $\mathbf{R}\boldsymbol{z} > \mathbf{0}$,

$$\begin{aligned}
\mathbf{q}\cdot\boldsymbol{z} = (\hat{\mathbf{R}}^T\boldsymbol{\mu})\cdot\boldsymbol{z} &= \sum_{k=1}^{K}\left(\sum_{s=1}^{S}\mu_s r_s^k\right)z^k \\
&= \sum_{s=1}^{S}\mu_s\left(\sum_{k=1}^{K} r_s^k z^k\right) \\
&= \sum_{s\notin T}\mu_s\left(\sum_{k=1}^{K} r_s^k z^k\right) > 0,
\end{aligned}$$

showing that arbitrage is impossible. $\square$

28.3.6 Restrictions on Asset Prices

Combining the Arbitrage Alternative Theorem and Theorem 28.3.2, we see that the no-arbitrage condition will place restrictions on asset prices if and only if there is a portfolio z^* with $\mathbf{R}z^* > 0$. In other words, whenever there is a arbitrage portfolio, a portfolio with positive return vector, the no arbitrage condition places restrictions on possible asset prices. If there is no such portfolio, all asset prices are arbitrage-free.

28.3.7 Positive Returns mean Positive Asset Prices

Theorem 28.3.2 excluded the case when $\mathbf{q} = \mathbf{0}$. In that case asset prices can still be written as $\mathbf{q} = \mathbf{R}^T \boldsymbol{\mu}$ by setting $\boldsymbol{\mu} = \mathbf{0}$, regardless of whether arbitrage might be possible.

One consequence of Theorem 28.3.2 is that any asset with a positive return vector will have a positive price provided asset prices are not all zero.

Proposition 28.3.3. *Suppose there is a portfolio $\mathbf{z}^*$ with return vector $\mathbf{R}\mathbf{z}^* > \mathbf{0}$ and that the asset price vector $\mathbf{q}$ is non-zero and arbitrage-free. If $r_s^k \geq 0$ for all $s = 1, \ldots, S$, then $q_k \geq 0$. Moreover, if $r_s^k > 0$ for all $s = 1, \ldots, S$, then $q_k > 0$.*

Proof. Here

$$q_k = \sum_{s=1}^{S} \mu_s r_s^k$$

with $\mu_s \geq 0$ for all s and $\mu_s > 0$ for some s, when the result follows. □

28.3.8 Arbitrage-free Prices: Sufficient Conditions

Theorem 28.3.2 gave us necessary conditions for arbitrage-free prices. Showing that such prices could be written as a positive linear combination of the return vectors.

Theorem 28.3.4 states sufficient conditions for non-trivial price vectors to prohibit arbitrage. It shows any **strictly** positive linear combination of the return vectors is an arbitrage-free asset price vector.

Theorem 28.3.4. *If* $\mathbf{q} = \mathbf{R}^\mathsf{T}\boldsymbol{\mu}$ *for some* $\boldsymbol{\mu} \gg \mathbf{0}$*, then* $\mathbf{q}$ *is arbitrage-free.*

Proof. Using the expression for $\mathbf{q}$, we find $\mathbf{q}\cdot\boldsymbol{z} = \mathbf{q}^\mathsf{T}\boldsymbol{z} = \boldsymbol{\mu}^\mathsf{T}\mathbf{R}\boldsymbol{z} = \boldsymbol{\mu}\cdot(\mathbf{R}\boldsymbol{z})$.

Now suppose we have an arbitrage $\boldsymbol{z}$. Then $\mathbf{R}\boldsymbol{z} > \mathbf{0}$ and $0 \geq \mathbf{q}\cdot\boldsymbol{z}$. But when $\mathbf{R}\boldsymbol{z} > 0$, $\mathbf{q}\cdot\boldsymbol{z} = \boldsymbol{\mu}\cdot(\mathbf{R}\boldsymbol{z}) > 0$ because $\boldsymbol{\mu} \gg \mathbf{0}$. This contradicts the fact that $\mathbf{q}\cdot\boldsymbol{z} \leq 0$. We can conclude that no arbitrage is possible at asset prices $\mathbf{q} = \mathbf{R}^\mathsf{T}\boldsymbol{\mu}$ whenever $\boldsymbol{\mu} \gg \mathbf{0}$. □

The prices in Example 28.2.3 were obtained this way, by using $\boldsymbol{\mu} = (1/3, 1/3, 1/3)^\mathsf{T}$.

28.3.9 Necessary vs. Sufficient Conditions

Comparing Theorems 28.3.2 and 28.3.4, we find a gap between the necessary and sufficient conditions to write asset prices as a sum of payoffs r_s^k times weights μ_s, with

$$q_k = \sum_{s=1}^{S} \mu_s r_s^k$$

for $k = 1, \ldots, K$, or more concisely, $\mathbf{q} = \mathbf{R}^{\mathsf{T}}\boldsymbol{\mu}$.

By Theorem 28.3.4, if the weight vector $\boldsymbol{\mu}$ is **strictly positive**, then $\mathbf{q}$ is arbitrage-free. In contrast, Theorem 28.3.2 tells us arbitrage-free asset prices $\mathbf{q}$ can always be written using a weight vector that is merely **positive**. Indeed, Theorem 28.3.2 sometimes constructs the weights so that they are **not** strictly positive.

So what can we say about this gap? We start with the following example where weights that are merely positive can generate prices that allow arbitrage.

Example 28.3.5: Arbitrage with Non-negative Linear Combinations. Let the return matrix be

$$\mathbf{R} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The return matrix is symmetric, so $\mathbf{R}^{\mathsf{T}} = \mathbf{R}$. Now define $\boldsymbol{\mu} = (0, 1)^{\mathsf{T}}$. The resulting prices

$$\mathbf{q} = \mathbf{R}^{\mathsf{T}}\boldsymbol{\mu} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

admit arbitrage. In fact, $\boldsymbol{z} = (1, -1/2)^{\mathsf{T}}$ is an arbitrage for these prices since $\mathbf{R}\boldsymbol{z} = (1/2, 0)^{\mathsf{T}} > \mathbf{0}$ and $\mathbf{q}\cdot\boldsymbol{z} = 0$.

We get a similar result if the weight vector is $(1, 0)^{\mathsf{T}}$. Then asset prices are $(1, 1)^{\mathsf{T}}$. Then $\boldsymbol{z} = (-1, 1)^{\mathsf{T}}$ is an arbitrage as $\mathbf{R}\boldsymbol{z} = (0, 1)^{\mathsf{T}} > \mathbf{0}$ and $\mathbf{q}\cdot\boldsymbol{z} = 0$. ◄

28.3.10 Positive Linear Combinations may be Insufficient

So what causes weights that are merely positive to be insufficient?

Example 28.3.5 is not an isolated case. The following proposition shows that such examples can be constructed whenever there is a portfolio z with positive return vector, $\mathbf{R}z > 0$ that is not strictly positive ($\mathbf{R}z \not\gg 0$). This means that there will be some $\mu > 0$, where $\mu \not\gg 0$ that do not generate arbitrage-free prices.

Proposition 28.3.6. *Let $\mathbf{R}$ be a return matrix and z a portfolio with $\mathbf{R}z > 0$ but $\mathbf{R}z \not\gg 0$. Then there is a vector $\mu > 0$ such that the asset price vector $\mathbf{q} = \mathbf{R}^T\mu$ admits arbitrage using the portfolio z.*

Proof. Under the hypothesis, there must be at least one state r with $(\mathbf{R}z)_r = 0$. Define μ by $\mu_r = 1$ and $\mu_s = 0$ otherwise and set $\mathbf{q} = \mathbf{R}^T\mu$. Then

$$\mathbf{q}\cdot z = (\mathbf{R}^T\mu)\cdot z = \mu\cdot\mathbf{R}z = (\mathbf{R}z)_r = 0.$$

Now z is an arbitrage as $\mathbf{R}z > 0$ and $\mathbf{q}\cdot z = 0$. $\square$

28.3.11 Return Matrices with Rank S

One special case occurs when the return matrix has rank S. Then an asset price vector is arbitrage-free if and only if it is generated by strictly positive weights. Since we already known this is sufficient, it is enough to show it is necessary, which we do in the following proposition.

Proposition 28.3.7. *Suppose the return matrix has rank equal to the number of states S and that $\mathbf{q} = \mathbf{R}^{\mathsf{T}}\boldsymbol{\mu}$ is arbitrage-free. Then $\boldsymbol{\mu} \gg \mathbf{0}$.*

Proof. Because rank $\mathbf{R} = S$, the range of $\mathbf{R}$ must be all of $\mathbb{R}^S$. Let s be any state and take $\boldsymbol{z}_s$ with $\mathbf{R}\boldsymbol{z}_s = \boldsymbol{e}^s$. Since $\mathbf{R}\boldsymbol{z}_s = \boldsymbol{e}^s > \mathbf{0}$ and $\mathbf{q}$ is arbitrage-free, we must have $\mathbf{q}\cdot\boldsymbol{z}_s > 0$. We rewrite this as

$$0 < (\mathbf{R}^{\mathsf{T}}\boldsymbol{\mu})\cdot\boldsymbol{z}_s = \boldsymbol{\mu}\cdot(\mathbf{R}\boldsymbol{z}_s) = \mu_s.$$

So $0 < \mu_s 0$.

Since s was arbitrary, this holds for every state s, showing that $\boldsymbol{\mu} \gg \mathbf{0}$. □

In other words, there is no gap between the necessary and sufficient conditions when the range of $\mathbf{R}$ is all of $\mathbb{R}^S$.

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28.3.12 Market Probabilities

We can also use Theorem 28.3.2 to interpret equilibrium asset prices in terms of probabilities, as we had previously done with Arrovian securities in Example 27.5.3.

Given an equilibrium asset price $\mathbf{q}$, we say that a probability distribution $\{\pi_s\}$ is a *market probability distribution for* $\mathbf{q}$ if the price of any asset k is its expected return under that distribution, meaning

$$q_k = \sum_s \pi_s r_s^k = E\mathbf{r}^k$$

for every asset $k = 1, \ldots, K$.

This is a bit different from the Arrovian case because the Radner model makes a clear distinction between assets and states. In the Arrovian model, the two are bound together. In the Radner model we make the probabilities using the weights μ_s that pertain to the states, rather than the asset prices q_k. In contrast, the two are identical in the Arrovian model.

28.3.13 Market Probability Theorem

Theorem 28.3.8 shows that arbitrage-free prices generate a market probability distribution whenever there is an arbitrage portfolio.

Theorem 28.3.8. *Suppose there is a portfolio $\mathbf{z}^*$ with $\mathbf{R}\mathbf{z}^* > \mathbf{0}$. If $\mathbf{q}$ is arbitrage-free, then there is a constant $\lambda > 0$ and probabilities π_s so that for $\hat{\mathbf{q}} = \lambda\mathbf{q}$, the price of each asset is its expected return, $\hat{q}_k = \sum_s \pi_s r_s^k$ for each asset k. Equivalently, $\hat{\mathbf{q}} = \mathbf{R}^T\boldsymbol{\pi}$.*

Proof. Since $\mathbf{q}$ are arbitrage-free asset prices, Theorem 28.3.2 allows us to write $\mathbf{q} = \mathbf{R}^T\boldsymbol{\mu}$ for some $\mu > \mathbf{0}$. Because $\boldsymbol{\mu} > \mathbf{0}$, $\sum_r \mu_r > 0$. Define the market probabilities π_s by

$$\pi_s = \frac{\mu_s}{\sum_r \mu_r}.$$

Then $\boldsymbol{\pi} > \mathbf{0}$ and $\sum_r \pi_r = 1$. This allows us to interpret π as a probability distribution on the states $s = 1, \dots, S$.

Set $\lambda = \left(\sum_r \mu_r\right)^{-1}$. Then $\hat{\mathbf{q}} = \lambda\mathbf{q}$ is also an equivalent arbitrage-free asset price vector and obeys $\hat{q}_k = \sum_s \pi_s r_s^k$. The asset price $\hat{q}_k$ is the expected return to asset k. $\square$

We had previously used this interpretation of asset prices for Arrovian securities in Example 27.5.3. Theorem 28.3.8 shows that we can also interpret asset prices this way in the Radner model.

28.3.14 Market Probabilities with Arrovian Securities

When there are Arrovian securities, $r_s^k = \delta_s^k$. This means that the asset prices themselves can be interpreted as the market probability π_s of each state. As such, $\sum_s q_s = \sum_s \pi_s = 1$. For Arrovian securities equilibria, we can restate Theorem 28.3.8 as follows.

Corollary 28.3.9. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ be a contingent goods exchange economy with m goods in each of S states.*

If for every state s at least one consumer is strongly $(1, s)$-monotonic and $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ is an Arrovian securities equilibrium. Then there are weights $\pi_s \geq 0$ with $\sum_s \pi_s = 1$ so that $(\hat{\mathbf{p}}, \mathbf{q}', \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ is also a Arrovian equilibrium, where $\mathbf{q}'$ is defined by $q'_s = \pi_s$. In other words, the price of Arrovian security s is its market probability π_s. Moreover, $\sum_s q'_s = 1$.

Proof. The return matrix for Arrovian securities is $\mathbf{I}_S$. Then $\mathbf{I}_S \boldsymbol{e} > \mathbf{0}$, so we can apply Theorem 28.3.8. □

We saw this when interpreting the results of Example 27.5.3. In that example the return matrix was $\mathbf{I}_2$ and the securities prices $(1, 4/3)$ yielding $\boldsymbol{\mu} = (1, 4/3)$. Normalizing, we obtain equilibrium probabilities of $\pi_1 = 3/7$ and $\pi_2 = 4/7$. The Radner equilibrium has generated market probabilities that differ from the beliefs of either consumer.

28.3.15 Market Probabilities vs. Common Probabilities I

Even when consumers are agreed on the subjective probabilities of each state, these common subjective probabilities need not be the market probabilities. The following example shows how this can occur.

Example 28.3.10: Market Probabilities and Common Probabilities. Let's return to Example 27.3.5. There is one good and two states. The endowment of consumer one is $\boldsymbol{\omega}^1 = (3, 0)$ while consumer two's endowment is $\boldsymbol{\omega}^2 = (0, 2)$. The aggregate endowment is $\boldsymbol{\omega} = (3, 2)$, indicating there is aggregate uncertainty. Utility is identical for the two consumers, with $u_i(\mathbf{x}^1) = \frac{1}{3}\ln x_1^i + \frac{2}{3}\ln x_2^i$ for $i = 1, 2$.

Rather than considering the Arrow-Debreu equilibrium as in Example 27.3.5, we treat this as an Arrovian securities model. The Arrow-Debreu equilibrium was $\mathbf{p} = (1, 3)$ with $\mathbf{x}^1 = (1, 2/3)$ and $\mathbf{x}^2 = (2, 4/3)$.

We use the Arrovian Equivalence Theorem to convert this to an Arrovian securities equilibrium by setting $\hat{\mathbf{q}} = (1, 3)$. We can separately renormalize the spot prices so that $\hat{\mathbf{p}} = (1, 1)$. The resulting allocation is $\hat{\mathbf{x}}^1 = (1, 2/3)$ and $\hat{\mathbf{x}}^2 = (2, 4/3)$ and the consumer's asset portfolios are $\hat{\mathbf{z}}^1 = -\hat{\mathbf{z}}^2 = (-2, +2/3)$.

28.3.16 Market Probabilities vs. Common Probabilities II

Although consumers use the same probabilities $(1/3, 2/3)$ for the two states, these are not the market probabilities. We use $\hat{\mathbf{q}} = (1, 3)$ to find the market probabilities, $\boldsymbol{\pi} = (1/4, 3/4)$.

So why does this happen? In the Common Probability Theorem we derived the first-order conditions in equation 27.3.2. Applying it to good one in the Arrovian securities model, it becomes

$$\frac{\hat{q}_r}{\hat{q}_s} = \mathrm{MRS}^i_{(1,r)(1,s)}(\hat{\mathbf{x}}^i) = \frac{\pi^i_r}{\pi^i_s}\left[\frac{\partial u_i}{\partial x_1}(\hat{x}^i_{1,r}) \bigg/ \frac{\partial u_i}{\partial x_1}(\hat{x}^i_{1,s})\right].$$

The asset price ratio equals the marginal rate of substitution between states. And the marginal rate of substitution between states is the product of the ratio of those states probabilities and the ratio of the marginal utilities in the two states.

When there is full insurance, $\hat{x}^i_{1,r} = \hat{x}^i_{1,s}$. The marginal utilities are the same, leaving us with the probability ratio. Thus $\hat{q}_r/\hat{q}_s = \pi_r/\pi_s$. In that case the market probabilities are the common subjective probabilities.

But we don't have full insurance. Consumption differs by state. The ratio of marginal utilities is not one. This drives a wedge between the asset price ratio and the probability ratio. When $i = 1$, the asset price ratio, which is the market probability ratio, becomes

$$\frac{\hat{q}_1}{\hat{q}_2} = \frac{1/3}{2/3}\left[\frac{u'(1)}{u'(2/3)}\right] = \frac{1}{2}\left[\frac{2}{3}\right] = \frac{1}{3}$$

where $1/2$ is the subjective probability ratio and $2/3$ is the marginal rate of substitution between states. Their product is the market probability ratio $1/3$. Of course, using $i = 2$ also yields the same ratio.

The fact that consumption of good one varies by state means that the market probabilities differ from the common subjective probabilities. ◀

28.4 Complete and Incomplete Asset Markets

We saw in sections 28.2 and 28.3 that the no-arbitrage conditions have a number of implications, depending on the asset structure and the properties of the return matrix. One important distinction concerning return matrices is whether the asset structure is complete or incomplete.

Complete and Incomplete Assets. The return matrix of a Radner model is *complete* if there are as many independent assets as there are states. In other words, $\mathbf{R}$ is *complete* if $\operatorname{rank}\mathbf{R} = S$, or equivalently, $\operatorname{ran}\mathbf{R} = \mathbb{R}^S$. If a return matrix has $\operatorname{rank}\mathbf{R} < S$, we say the asset structure is *incomplete*.

Of course, $\operatorname{rank}\mathbf{R} > S$ is impossible. The Arrovian securities model has a complete asset structure—its return matrix is the $S \times S$ identity matrix, which has rank S. In Example 27.4.5, we were able to freely move income between states in the Arrovian securities model. This is the key to showing that the resulting allocation is also an Arrow-Debreu equilibrium (Arrovian Equivalence Theorem) and must be Pareto optimal. Radner equilibria with complete markets also allow us to freely move income around between states. This will allow to show that when markets are complete, any Radner equilibrium is equivalent to an Arrow-Debreu equilibrium (Theorem 28.4.6), and so is Pareto optimal.

In Example 27.6.2 we examined a model with incomplete assets ($\operatorname{rank}\mathbf{R} = S - 1$). The resulting equilibrium allocation was neither an Arrow-Debreu equilibrium nor a Pareto optimum. In some cases, a Radner equilibrium will turn out to be Pareto optimal (maybe even equivalent to an Arrow-Debreu equilibrium), but these are special cases.

We will see that Radner equilibrium behaves like the Arrovian securities equilibrium when the asset structure is complete, but is often not Pareto optimal when the asset structure is not complete.

Whether or not the asset structure is complete is not the end of the story. In section 28.5 we will see how certain derivative assets, namely options, can sometimes complete an incomplete set of assets.

28.4.1 Equivalent Asset Structures

A given asset matrix $\mathbf{R}$ defines the possibilities for moving income between states. Pick a price system for the spot markets using good one as the numéraire in every spot market. A portfolio $\boldsymbol{z}$ generates a payoff vector $\mathbf{R}\boldsymbol{z}$, which describes the possibilities the consumer has to move income between states. The range of $\mathbf{R}$, $\operatorname{ran}\mathbf{R} = \{\mathbf{R}\boldsymbol{z} : \boldsymbol{z} \in \mathbb{R}^K\}$, defines the possible ways we can move income between states using the given asset structure $\mathbf{R}$.

We will focus on asset structures that yield the same set of possible payoffs. We define two return matrices $\mathbf{Q}$ and $\mathbf{R}$ to be *Radner equivalent* if $\operatorname{ran}\mathbf{Q} = \operatorname{ran}\mathbf{R}$.[6] In other words, two return matrices are Radner equivalent if they have the same set of possible payoffs.

Radner equivalence of $\mathbf{R}$ and $\mathbf{Q}$ implies that they have the same rank. It also requires that $\mathbf{R}$ and $\mathbf{Q}$ have the same number of states. It does not require that the number of assets be the same.

Example 28.4.1: Equivalent or Inequivalent Asset Structures?. Consider the return matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ defined by

$$\mathbf{Q} = \begin{pmatrix} 1 & -3 \\ -2 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} -5 & 3 \\ 2 & 1 \\ 3 & -4 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is easily verified that all three matrices have rank 2. This means their range has dimension 2. The range is a hyperplane in $\mathbb{R}^3$. In fact, the range of $\mathbf{Q}$ and $\mathbf{R}$ is the hyperplane perpendicular to the vector $(1,1,1)^{\mathbf{T}}$. This follows from the fact that $(1,1,1)^{\mathbf{T}}$ is perpendicular to each of the individual return vectors in both $\mathbf{Q}$ and $\mathbf{R}$. As a result, we can write $\operatorname{ran}\mathbf{Q} = \operatorname{ran}\mathbf{R} = \{\boldsymbol{x} : x_1 + x_2 + x_3 = 0\}$, so $\mathbf{Q}$ and $\mathbf{R}$ are Radner equivalent.

However, $\mathbf{S}$ is not Radner equivalent to either $\mathbf{Q}$ or $\mathbf{R}$. The problem is that the second asset in $\mathbf{S}$ is not perpendicular to $(1,1,1)^{\mathbf{T}}$. ◄

It is actually quite easy to find return matrices that are not Radner equivalent to a given return matrix. None of the asset structures in the example is Radner equivalent to either $\mathbf{I}_3$ (with range $\mathbb{R}^3$) or to $(1,2,3)^{\mathbf{T}}$ (with 1-dimensional range).

[6] This defines an equivalence relation on the set of return matrices. In other words, Radner equivalence is reflexive, symmetric, and transitive.

28.4.2 Return Matrices and Budget Sets

We start with a preliminary lemma relating Radner budget sets for different return matrices.

Lemma 28.4.2. *Let* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ *be a contingent goods exchange economy with* m *goods in each of* S *states. Suppose that for every state* s *at least one consumer is strongly* $(1, s)$*-monotonic.*

Let $\mathbf{Q}$ *and* $\mathbf{R}$ *be return matrices with* $\operatorname{ran}\mathbf{R} \subset \operatorname{ran}\mathbf{Q}$. *Suppose there is a portfolio* $\boldsymbol{z}^*$ *with* $\mathbf{R}\boldsymbol{z}^* > 0$ *and* $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\boldsymbol{z}}^i)$ *is a Radner equilibrium for the return matrix* $\mathbf{R}$. *Then there is weight vector* $\boldsymbol{\mu} > 0$ *with* $\hat{\mathbf{q}} = \mathbf{R}^{\mathsf{T}}\boldsymbol{\mu}$ *and* $B_R^i(\hat{\mathbf{p}}, \hat{\mathbf{q}}; \mathbf{R}) \subset B_R^i(\hat{\mathbf{p}}, \bar{\mathbf{q}}; \mathbf{Q})$ *where* $\bar{\mathbf{q}} = \mathbf{Q}^{\mathsf{T}}\boldsymbol{\mu}$.

Proof. By the No-Arbitrage Theorem, the asset prices $\hat{\mathbf{q}}$ obey the no-arbitrage condition. By Theorem 28.3.2 there is a weight vector $\boldsymbol{\mu} > 0$ with $\hat{\mathbf{q}} = \mathbf{R}^{\mathsf{T}}\boldsymbol{\mu}$. Now define $\bar{\mathbf{q}} = \mathbf{Q}^{\mathsf{T}}\boldsymbol{\mu}$.

Suppose $\mathbf{x}$ is in consumer i's Radner budget set for $\mathbf{R}$, $\mathbf{x} \in B_R^i(\hat{\mathbf{p}}, \hat{\mathbf{q}}; \mathbf{R})$. There is then a portfolio $\boldsymbol{z}$ with $\hat{\mathbf{q}}\cdot\boldsymbol{z} \le 0$ and $\hat{\mathbf{p}}_s\cdot\mathbf{x}_s \le \hat{\mathbf{p}}_s\cdot\boldsymbol{\omega}_s + p_{1,s}\big(\mathbf{R}\boldsymbol{z}\big)_s$. Since $\operatorname{ran}\mathbf{R} \subset \operatorname{ran}\mathbf{Q}$, we may find another portfolio $\boldsymbol{z}'$ with $\mathbf{R}\boldsymbol{z} = \mathbf{Q}\boldsymbol{z}'$. Now

$$\bar{\mathbf{q}}\cdot\boldsymbol{z}' = \bar{\mathbf{q}}^{\mathsf{T}}\boldsymbol{z}' = \boldsymbol{\mu}^{\mathsf{T}}\mathbf{Q}\boldsymbol{z}' = \boldsymbol{\mu}^{\mathsf{T}}\mathbf{R}\boldsymbol{z} = \hat{\mathbf{q}}\cdot\boldsymbol{z},$$

so $\mathbf{x}$ is also in the Radner budget set for $\mathbf{Q}$. $\square$

28.4.3 Return Matrices and Asset Pricing

An immediate corollary is:

Corollary 28.4.3. *Let* $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ *be a contingent goods exchange economy with* m *goods in each of* S *states. Suppose that for every state* s *at least one consumer is strongly* $(1, s)$*-monotonic.*

Let $\mathbf{Q}$ *and* $\mathbf{R}$ *be Radner equivalent return matrices. Suppose there is a portfolio* $\boldsymbol{z}^*$ *with* $\mathbf{R}\boldsymbol{z}^* > 0$ *and* $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\boldsymbol{z}}^i)$ *is a Radner equilibrium for the return matrix* $\mathbf{R}$*. Then there is weight vector* $\boldsymbol{\mu} > 0$ *with* $\hat{\mathbf{q}} = \mathbf{R}^T\boldsymbol{\mu}$ *and* $B_R^i(\hat{\mathbf{p}}, \hat{\mathbf{q}}; \mathbf{R}) = B_R^i(\hat{\mathbf{p}}, \bar{\mathbf{q}}; \mathbf{Q})$ *when* $\bar{\mathbf{q}} = \mathbf{Q}\boldsymbol{\mu}$.

Proof. Since $\operatorname{ran}\mathbf{R} = \operatorname{ran}\mathbf{Q}$, there is also a portfolio $\boldsymbol{z}'$ with $\mathbf{Q}\boldsymbol{z}' > 0$. Construct $\bar{\mathbf{q}}$ as in Lemma 28.4.2. Then Lemma 28.4.2 applies in both directions, establishing that the budget sets are equal. $\square$

28.4.4 Radner Equivalence Theorem

These preliminary results are the basis of an equivalence theorem for Radner equilibria. They let us show that if two return matrices are Radner equivalent, then they have the same equilibrium allocations of consumption goods. This theorem is similar to the Arrovian Equivalence Theorem and holds regardless of whether the asset structure is complete.

Radner Equivalence Theorem. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ be a contingent goods exchange economy with m goods in each of S states. Suppose that for every state s at least one consumer is strongly $(1, s)$-monotonic.*

If $\mathbf{R}$ and $\mathbf{Q}$ are Radner equivalent return matrices with $\mathbf{R}\mathbf{z}^* > 0$ for some portfolio $\mathbf{z}^*$, and if $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ is a Radner equilibrium with return matrix $\mathbf{R}$, then there are asset prices $\bar{\mathbf{q}}$ and asset holdings $\bar{\mathbf{z}}$ so that $(\hat{\mathbf{p}}, \bar{\mathbf{q}}, \hat{\mathbf{x}}^i, \bar{\mathbf{z}}^i)$ is a Radner equilibrium with return matrix $\mathbf{Q}$. Moreover, there is a weight vector $\boldsymbol{\mu} > 0$ with $\hat{\mathbf{q}} = \mathbf{R}^T\boldsymbol{\mu}$ and $\bar{\mathbf{q}} = \mathbf{Q}^T\boldsymbol{\mu}$ and the portfolios $\bar{\mathbf{z}}^i$ obey $\mathbf{Q}\bar{\mathbf{z}}^i = \mathbf{R}\hat{\mathbf{z}}^i$.

28.4.5 Proof of Radner Equivalence Theorem

Radner Equivalence Theorem. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ be a contingent goods exchange economy with m goods in each of S states. Suppose that for every state s at least one consumer is strongly $(1,s)$-monotonic.*

If $\mathbf{R}$ and $\mathbf{Q}$ are Radner equivalent return matrices with $\mathbf{R}z^* > 0$ for some portfolio z^*, and if $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{z}^i)$ is a Radner equilibrium with return matrix $\mathbf{R}$, then there are asset prices $\bar{\mathbf{q}}$ and asset holdings $\bar{z}$ so that $(\hat{\mathbf{p}}, \bar{\mathbf{q}}, \hat{\mathbf{x}}^i, \bar{z}^i)$ is a Radner equilibrium with return matrix $\mathbf{Q}$. Moreover, there is a weight vector $\boldsymbol{\mu} > 0$ with $\hat{\mathbf{q}} = \mathbf{R}^{\mathsf{T}}\boldsymbol{\mu}$ and $\bar{\mathbf{q}} = \mathbf{Q}^{\mathsf{T}}\boldsymbol{\mu}$ and the portfolios $\bar{z}^i$ obey $\mathbf{Q}\bar{z}^i = \mathbf{R}\hat{z}^i$.

Proof. By the No-Arbitrage Theorem, the asset prices $\hat{\mathbf{q}}$ obey the no-arbitrage condition. By Theorem 28.3.2 there is a weight vector $\boldsymbol{\mu} > 0$ with $\hat{\mathbf{q}} = \mathbf{R}^{\mathsf{T}}\boldsymbol{\mu}$. Now define $\bar{\mathbf{q}} = \mathbf{Q}^{\mathsf{T}}\boldsymbol{\mu}$.

By Corollary 28.4.3, we know $B_R^i(\hat{\mathbf{p}}, \hat{\mathbf{q}}; \mathbf{R}) = B_R^i(\hat{\mathbf{p}}, \bar{\mathbf{q}}; \mathbf{Q})$. Since the budget sets are identical, the optimal choice for each consumer is identical.

All that remains is to find portfolios $\bar{z}^i$ where asset markets clear and each $\hat{\mathbf{x}}^i$ is affordable using $\bar{z}^i$. Use the fact that $\operatorname{ran}\mathbf{R} = \operatorname{ran}\mathbf{Q}$ to choose $\bar{z}^i$ with $\mathbf{Q}\bar{z}^i = \mathbf{R}\hat{z}^i$ for $i = 1, \dots, I-1$. This ensures that $\hat{\mathbf{x}}^i$ is affordable for $i = 1, \dots, I-1$ using $\bar{z}^i$. Then set $\bar{z}^I = -\sum_{i=1}^{I-1} \bar{z}^i$. This guarantees that asset markets clear in the $\mathbf{Q}$ equilibrium.

We only need to check affordability for consumer I. For this, we need to show that consumer I's is unchanged, that $\mathbf{Q}\bar{z}^I = \mathbf{R}\hat{z}^I$. Now

$$\mathbf{Q}\bar{z}^I = -\sum_{i=1}^{I-1} \mathbf{Q}\bar{z}^i = -\sum_{i=1}^{I-1} \mathbf{R}\hat{z}^i = \mathbf{R}\hat{z}^I$$

because the asset market clears in the original equilibrium. Each consumer's payoffs in each state under $\mathbf{Q}$ using $\bar{z}^i$ are the same payoffs that $\hat{z}^i$ yields under $\mathbf{R}$. It follows that $(\hat{\mathbf{p}}, \bar{\mathbf{q}}, \hat{\mathbf{x}}, \bar{z})$ is a Radner equilibrium. □

One consequence of the Radner Equivalence Theorem is that when return matrices are Radner equivalent, the set of equilibrium consumption allocations is exactly the same.

28.4.6 Radner Equilibrium Exists with Complete Asset Markets

When we have a complete set of assets, it is easy to show a Radner equilibrium exists. One way to do that is to use the existence of Arrovian equilibria and apply the Radner Equivalence Theorem.

Theorem 28.4.4. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ be a contingent goods exchange economy with m goods in each of S states. Suppose $\mathbf{R}$ is return matrix with a complete set of assets.*

If each consumer has strictly monotonic, convex, continuous preferences, and if $\boldsymbol{\omega} \gg 0$, then an Radner securities equilibrium exists where every good has a strictly positive price and the securities prices obey $\hat{\mathbf{q}} = \mathbf{R}^\mathsf{T}\boldsymbol{\mu}$ for some strictly positive $\boldsymbol{\mu}$.

Proof. By Corollary 27.5.1, an Arrovian equilibrium exists. Write it as $(\bar{\mathbf{p}}, \bar{\mathbf{q}}, \bar{\mathbf{x}}^i, \bar{\boldsymbol{z}}^i)$. Both goods and Arrovian securities prices will be strictly positive.

Under these hypotheses, each consumer is $(1, s)$ strongly monotonic. Take the return matrices $\mathbf{I}_S$ and $\mathbf{R}$. Of course, $\mathbf{I}_S\boldsymbol{e} > 0$, so we can apply the proof of the Radner Equivalence Theorem to obtain a Radner equilibrium $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\boldsymbol{z}}^i)$ with $\hat{\mathbf{p}} = \bar{\mathbf{p}}$, $\hat{\mathbf{q}} = \mathbf{R}^\mathsf{T}\bar{\mathbf{q}}$, and $\hat{\mathbf{x}}^i = \bar{\mathbf{x}}^i$. Moreover, the $\hat{\boldsymbol{z}}^i$ obey $\mathbf{R}\hat{\boldsymbol{z}}^i = \bar{\boldsymbol{z}}^i$. Set $\boldsymbol{\mu} = \bar{\mathbf{q}} \gg 0$ to complete the proof. □

Needless to say, such equilibria also inherit Pareto optimality from the Arrovian equilibrium.

28.4.7 Radner Equivalence and Complete Assets

We can also sharpen the results of the Radner Equivalence Theorem when assets are complete.

When $\mathbf{R}$ is complete, one of the hypotheses of the Radner Equivalence Theorem is automatically satisfied. That is the requirement there is a portfolio z^* with $\mathbf{R}z^* > 0$. This is automatically satisfied for complete return matrices as $\operatorname{ran}\mathbf{R} = \mathbb{R}^S$. Indeed, there are then many portfolios z with $\mathbf{R}z \gg 0$.

We can go a bit further in modifying the Radner Equivalence Theorem when the return matrices are not only complete, but invertible. If we examine the proof when $\mathbf{Q}$ and $\mathbf{R}$ are invertible, we find $\boldsymbol{\mu} = (\mathbf{R}^\mathsf{T})^{-1}\hat{\mathbf{q}}$, so $\bar{\mathbf{q}} = \mathbf{Q}(\mathbf{R}^\mathsf{T})^{-1}\hat{\mathbf{q}}$. Further, $\mathbf{Q}\bar{z}^i = \mathbf{R}\hat{z}^i$ has a unique solution, $\bar{z}^i = (\mathbf{Q})^{-1}\mathbf{R}\hat{z}^i$. This can be applied to all $i = 1, \dots, I$, not just $i < I$. We don't have to define z^I indirectly because $\sum_i \bar{z}^i = (\mathbf{Q})^{-1}\mathbf{R}\big(\sum_i \hat{z}^i\big) = 0$.

Summing up, we have proved the following form of the Radner Equivalence Theorem, which requires the return matrices be invertible.

Theorem 28.4.5. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ be a contingent goods exchange economy with m goods in each of S states. Suppose that for every state s at least one consumer is strongly $(1, s)$-monotonic.*

If $\mathbf{R}$ and $\mathbf{Q}$ are invertible return matrices and if $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{z}^i)$ is a Radner equilibrium with return matrix $\mathbf{R}$, then $(\hat{\mathbf{p}}, \bar{\mathbf{q}}, \hat{\mathbf{x}}, \bar{z})$ is a Radner equilibrium with return matrix $\mathbf{Q}$ when $\bar{\mathbf{q}}$ and $\bar{z}^i$ are defined by $\bar{\mathbf{q}} = \mathbf{Q}(\mathbf{R}^\mathsf{T})^{-1}\hat{\mathbf{q}}$ and $\bar{z}^i = (\mathbf{Q})^{-1}\mathbf{R}\hat{z}^i$ for $i = 1, \dots, I$.

Theorem 28.4.5 applies when one of the return matrices is Arrovian, hence invertible, and the other payoff matrix is invertible. We take $\mathbf{R} = \mathbf{I}_S$. Then the theorem gives us a simpler form for $\bar{\mathbf{q}}$ and $\bar{z}^i$. Specifically, $\bar{\mathbf{q}} = \mathbf{Q}\hat{\mathbf{q}}$ and $\bar{z}^i = (\mathbf{Q})^{-1}\hat{z}^i$ where $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{z}^i)$ is the Arrovian securities equilibrium.

28.4.8 Radner Equivalence Theorem II

We don't get such a nice formula when $\mathbf{R}$ is Arrovian and $\mathbf{Q}$ is merely complete, but not necessarily invertible. In that case, we only know that ran $\mathbf{Q} = \mathbb{R}^S = \operatorname{ran} \mathbf{I}_S$.

This situation is summed up in the following theorem.

Theorem 28.4.6. *Let $\mathcal{E} = (\mathfrak{X}_i, \succsim_i, \boldsymbol{\omega}^i)_{i=1}^I$ be a contingent goods exchange economy with m goods in each of S states. Suppose that for every state s at least one consumer is strongly $(1,s)$-monotonic and that the return matrix $\mathbf{R}$ is complete (rank $\mathbf{R} = S$).*

1. *Suppose $\hat{\mathbf{x}}^i \in \mathbb{R}_+^{mS}$ and $\hat{\mathbf{p}} \in \mathbb{R}_+^{mS}$ with $\hat{\mathbf{p}} \gg 0$ form an Arrow-Debreu equilibrium. Define $\mathbf{q} = (\hat{p}_{1,s})^\mathsf{T} \in \mathbb{R}^S$ and $\hat{\mathbf{q}} = \mathbf{R}^\mathsf{T}\mathbf{q}$. Then there are portfolios $\hat{\mathbf{z}}^i \in \mathbb{R}^S$ such that $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ is a Radner equilibrium with return matrix $\mathbf{R}$.*
2. *If $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ with $\hat{\mathbf{p}} \in \mathbb{R}_+^{mS}$, $\hat{\mathbf{p}} \gg 0$, $\hat{\mathbf{q}} \in \mathbb{R}^S$, and $\hat{\mathbf{x}}^i \in \mathbb{R}_+^{mS}$ are a Radner equilibrium with return matrix $\mathbf{R}$. Then there are weights $\mu_s > 0$ such that $(\boldsymbol{\mu} \odot \hat{\mathbf{p}}, \hat{\mathbf{x}})$ forms an Arrow-Debreu equilibrium.*

Proof. For part one, suppose $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^i)$ are an Arrow-Debreu equilibrium with $\hat{\mathbf{p}} \gg 0$. By the Arrovian Equivalence Theorem, if the securities prices $\mathbf{q}$ are defined by $q_s = \hat{p}_{1,s}$, there are portfolios $\mathbf{z}^i \in \mathbb{R}^S$ so that $(\hat{\mathbf{p}}, \mathbf{q}, \hat{\mathbf{x}}^i, \mathbf{z}^i)$ is an Arrovian securities equilibrium. Note that $\mathbf{q} \gg 0$ because $\hat{\mathbf{p}} \gg 0$.

The Arrovian securities equilibrium has return matrix $\mathbf{I}_S$, so we can write $\mathbf{q} = \mathbf{I}_S^\mathsf{T}\mathbf{q}$. Now rank $\mathbf{R} = S$, so ran $\mathbf{R} = \mathbb{R}^S = \operatorname{ran} \mathbf{I}_S$. By the Radner Equivalence Theorem, $\hat{\mathbf{q}} = \mathbf{R}^\mathsf{T}\mathbf{q}$ will be the asset prices in a Radner equilibrium. Further, the Radner Equivalence Theorem yields portfolios $\hat{\mathbf{z}}^i$ so that $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ is a Radner equilibrium.

For part two, we reverse the process. Since $\hat{\mathbf{q}}$ is arbitrage-free, there is $\mathbf{q} > 0$ with $\hat{\mathbf{q}} = \mathbf{R}^\mathsf{T}\mathbf{q}$. By the Radner Equivalence Theorem, there are portfolios $\mathbf{z}^i$ so that $(\hat{\mathbf{p}}, \mathbf{q}, \hat{\mathbf{x}}^i, \mathbf{z}^i)$ is an Arrovian securities equilibrium. The Arrovian Equivalence Theorem tells us there are $\mu_s > 0$ so that $(\boldsymbol{\mu} \odot \hat{\mathbf{p}}, \hat{\mathbf{x}})$ is an Arrow-Debreu equilibrium. □

28.4.9 Alternate Method of Finding Radner Equilibria

These results lead to additional ways of finding Radner equilibria when there is a complete set of assets. First find an Arrow-Debreu or Arrovian securities equilibrium and then convert it to a Radner equilibrium, using Corollary 28.2.4 to derive the asset prices in the original problem from Arrovian securities prices. In fact, if an asset has return vector $\mathbf{r}^k = (r_1^k, \dots, r_S^k)$, it has price

$$\hat{q}_k = \sum_{s=1}^{S} r_s^k \alpha_s \tag{28.4.1}$$

where α_s denotes the price of Arrovian security s. Under complete markets, all Radner equilibria can be found this way.

To understand why this works, consider the return matrix

$$\mathbf{Q} = \left(\mathbf{R} \vdots \mathbf{I}_S\right).$$

Since $\mathbf{R}$ is already complete, $\mathbf{Q}$ is also complete. They both have the same range, making them Radner equivalent. Every Radner equilibrium for $\mathbf{R}$ translates to a Radner for $\mathbf{Q}$ and vice-versa. Now in the $\mathbf{Q}$ economy, Corollary 28.2.4 to the Arbitrage Pricing Theorem yields equation 28.4.1 for the $\mathbf{Q}$ equilibrium asset prices. Because $\mathbf{R}$ is complete, the equilibrium pattern of portfolio payoffs can be realized through the assets in $\mathbf{R}$, implying that the $\mathbf{Q}$ equilibrium asset prices that pertain to $\mathbf{R}$ are also $\mathbf{R}$ equilibrium asset prices.

28.4.10 Radner Equilibrium via Alternate Method I

Here's an example of the alternate method.

Example 28.4.7: Radner Equilibrium via Alternate Method. A contingent goods exchange economy has two consumers and one good that may be consumed in any of three states. Both consumers have the same utility function $u(\mathbf{x}) = \ln x_1 + \ln x_2 + \ln x_3$. Their endowments are $\boldsymbol{\omega}^1 = (2, 1, 0)$ and $\boldsymbol{\omega}^2 = (0, 1, 2)$. There is no aggregate uncertainty. The return matrix is

$$\mathbf{R} = \begin{pmatrix} 1/2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

Since there is a complete set of assets (rank $\mathbf{R} = 3$), we start by finding the Arrow-Debreu equilibria. With equal-weighted Cobb-Douglas utility, the demands are

$$\mathbf{x}^1(\mathbf{p}) = (2p_1 + p_2) \begin{pmatrix} 1/p_1 \\ 1/p_2 \\ 1/p_3 \end{pmatrix} \text{ and } \mathbf{x}^2(\mathbf{p}) = (p_2 + 2p_3) \begin{pmatrix} 1/p_1 \\ 1/p_2 \\ 1/p_3 \end{pmatrix}.$$

We add to get market demand and set equal to supply.

$$2(p_1 + p_2 + p_3) \begin{pmatrix} 1/p_1 \\ 1/p_2 \\ 1/p_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

It follows that $p_1 = p_2 = p_3$, and we may take $\hat{\mathbf{p}} = (1, 1, 1)$. The corresponding allocations are $\hat{\mathbf{x}}^1 = \hat{\mathbf{x}}^2 = (1, 1, 1)^{\mathbf{T}}$.

28.4.11 Radner Equilibrium via Alternate Method II

To turn this into a Radner equilibrium, we next consider the corresponding Arrovian securities equilibrium. By Arrovian Equivalence Theorem, the corresponding asset prices are $\mathbf{q} = (1, 1, 1)^{\mathsf{T}}$.

Then we employ the Radner Equivalence Theorem, starting from the Arrovian securities equilibrium. Here the return matrix is the 3×3 identity matrix I_3, so the corresponding weights are $\boldsymbol{\mu} = \mathbf{q}$. The new Radner asset prices for return matrix $\mathbf{R}$ are $\hat{\mathbf{q}} = \mathbf{R}^{\mathsf{T}}\mathbf{q}$. Thus $\hat{\mathbf{q}} = (3/2, 3/2, 1/2)$.

The only thing left to determine are the equilibrium portfolios. We use the fact that $\mathbf{x}^i = \boldsymbol{\omega}^i + \mathbf{R}\mathbf{z}^i$ to find them. Thus

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} z_1^1/2 \\ z_1^1 + z_2^1 \\ (z_2^1 + z_3^1)/2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} z_1^2/2 \\ z_1^2 + z_2^2 \\ (z_2^2 + z_3^2)/2 \end{pmatrix}.$$

Some easy calculations yield $\hat{\mathbf{z}}^1 = (-2, 2, 0)^{\mathsf{T}}$ and $\hat{\mathbf{z}}^2 = (2, -2, 0)^{\mathsf{T}}$. Notice that we did not need to calculate the Arrovian portfolios to find the Radner equilibrium. It was enough to find the Arrovian prices.

The Radner equilibrium is $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$ or any positive state scalar multiple of $\hat{\mathbf{p}}$ and any positive scalar multiple of $\hat{\mathbf{q}}$. ◄

28.4.12 Alternate Method: More Assets than States I

We can also use the alternate method when there are more assets than states.

Example 28.4.8: More Assets than States. We consider a contingent goods exchange economy with one good in each of three states. There are two consumers with identical utility: $u(\mathbf{x}) = \ln x_1 + \ln x_2 + \ln x_3$. Endowments are $\boldsymbol{\omega}^1 = (3,1,1)^\mathsf{T}$ and $\boldsymbol{\omega}^2 = (1,3,3)^\mathsf{T}$. The return matrix is

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Since there are 4 assets and 3 states, the assets are linearly dependent. In fact, it is easy to see that $\mathbf{r}^1 = \mathbf{r}^2 + \mathbf{r}^3$. Since asset prices must be arbitrage-free, $q_1 = q_2 + q_3$. Moreover, if $\mathbf{R}\boldsymbol{z} = \mathbf{R}\boldsymbol{z}'$, we have

$$\begin{aligned} z_1 + z_2 &= z_1' + z_2' \\ z_1 + z_3 &= z_1' + z_3' \\ z_1 + z_2 + z_4 &= z_1' + z_2' + z_4' \end{aligned}$$

It follows that $z_2' - z_2 = z_3' - z_3 = -(z_1' - z_1)$ and $z_4 = z_4'$ We can write $\Delta\boldsymbol{z} = (\alpha, -\alpha, -\alpha, 0)$ for some $\alpha \in \mathbb{R}$. Keep in mind that $\mathbf{q}\cdot\Delta\boldsymbol{z} = 0$.

We can now simplify the problem and consider only equilibria where $z_1 = 0$. This allows us to temporarily eliminate asset one from consideration, yielding return matrix $\mathbf{Q}$ defined by

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The most efficient way to proceed is to find the Arrow-Debreu equilibrium, which has $\mathbf{p} = (1,1,1)$. Then incomes are $m_1 = 5$ and $m_2 = 7$. This yields consumption allocation $\mathbf{x}^1 = (5/3)(1,1,1)^\mathsf{T}$ and $\mathbf{x}^2 = (7/3)(1,1,1)^\mathsf{T}$.

Now Arrovian asset prices would be $\mu_2 = \mu_3 = \mu_4 = 1$, so by Theorem 28.4.5, the Radner asset prices (except asset one) are

$$\begin{pmatrix} q_2 \\ q_3 \\ q_4 \end{pmatrix} = \mathbf{Q}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Then $q_1 = q_2 + q_3 = 2$, so $\mathbf{q} = (2,1,1,2)^\mathsf{T}$.

28.4.13 Radner Equilibrium: More Assets than States II

Next, we compute the holdings of the Radner assets other than one. We set $z_1^i = 0$ to obtain

$$\begin{aligned} 5/3 &= 3 + z_2^1 \\ 5/3 &= 1 + z_3^1 \\ 5/3 &= 1 + z_2^1 + z_4^1. \end{aligned}$$

and

$$\begin{aligned} 7/3 &= 1 + z_2^2 \\ 7/3 &= 3 + z_3^2 \\ 7/3 &= 3 + z_2^2 + z_4^2. \end{aligned}$$

This implies $\boldsymbol{z}^1 = (0, -4/3, +2/3, 2)$ and $\boldsymbol{z}^2 = (0, +4/3, -2/3, -2)$, which satisfies asset market clearing.

We have to take into account that we can add or subtract multiples of $(1, -1, -1, 0)$ from both $\boldsymbol{z}^1$ and $\boldsymbol{z}^2$ without the Radner budget sets or the optimal consumption bundles. However, if we replace $\boldsymbol{z}^i$ by $\alpha_i(1, -1, -1, 0) + \boldsymbol{z}^i$, we must set $\alpha_1 = -\alpha_2$ in order to maintain market clearing. So except for renormalization of prices, the equilibria are

$$\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{z}^1 = -\boldsymbol{z}^2 = \begin{pmatrix} 0 \\ -4/3 \\ +2/3 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix},$$

$$\mathbf{x}^1 = \begin{pmatrix} 5/3 \\ 5/3 \\ 5/3 \end{pmatrix}, \text{ and } \mathbf{x}^2 = \begin{pmatrix} 7/3 \\ 7/3 \\ 7/3 \end{pmatrix}.$$

◄

28.4.14 Incomplete Asset Structures

When markets are incomplete, the resulting equilibrium need not be Pareto optimal. It is possible that the equilibrium will be Pareto optimal, but it is not necessary.

Example 28.4.9: Optimal Equilibrium with Incomplete Markets. Consider a model with two consumers, three states, and one good in each state. Consumer preferences are given by $u(\mathbf{x}) = \ln x_1 + \ln x_2 + \ln x_3$. In other words, they are equal-weighted Cobb-Douglas preferences. Endowments are $\boldsymbol{\omega}^1 = (2,1,0)$ and $\boldsymbol{\omega}^2 = (0,1,2)$. The return matrix is

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here rank $\mathbf{R} = 2 < 3 = S$, so we have an incomplete set of assets.

It is easy to see that prices $\hat{\mathbf{p}} = (1,1,1)$ yield an Arrow-Debreu equilibrium with $\hat{\mathbf{x}}^1 = \hat{\mathbf{x}}^2 = (1,1,1)$. The consumers just consume their endowments in state two. There is no trade in state two, and no income need be moved into or out of state two. This is fortunate as the assets do not allow income to be moved in or out of state two. The only changes in income are in states one and three. Consumer one consumes less than endowment income in state one, and more than endowment income in state three. The opposite is true for consumer two.

This is also a Radner equilibrium allocation. It is given by spot prices $\hat{\mathbf{p}}$, allocations $\hat{\mathbf{x}}^i$, asset prices $\hat{\mathbf{q}} = (1,1)$, and portfolios $\hat{\mathbf{z}}^1 = (-1,+1) = -\hat{\mathbf{z}}^2$. The allocation is Pareto optimal even though markets are incomplete. ◄

28.4.15 When is Equilibrium Optimal?

The optimality of the equilibrium in Example 28.4.9 is quite fragile. It can be destroyed by a small change in the endowments that make it necessary to move income in or out of state two. Suppose we change the endowments of the consumers to $\boldsymbol{\omega}^1 = (2, 1+\varepsilon, 0)$ and $\boldsymbol{\omega}^2 = (0, 1-\varepsilon, 2)$ where $-1 < \varepsilon < +1$. The aggregate endowment is unaffected. Now $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{z}}^i$ as in Example 28.4.9 again yield a Radner equilibrium with allocations $\mathbf{x}^1 = (1, 1+\varepsilon, 1)$ and $\mathbf{x}^2 = (1, 1-\varepsilon, 1)$.

These allocations are not Pareto optimal if $\varepsilon \neq 0$. To see that, compare $MRS^1_{12} = 1+\varepsilon$ and $MRS^2_{12} = 1-\varepsilon$. The marginal rates of substitution must be the same for the allocation to be Pareto optimal, and this only happens if $\varepsilon = 0$.

In a similar model where preferences obey some smoothness conditions, Magill and Quinzii (1996, pg. 102) show that under incomplete markets, the set of endowments where the equilibrium is Pareto optimal is a set of measure zero. In other words, the typical situation is that incomplete markets lead to equilibria that are not Pareto optimal.

To get some intuition about this, return the equilibrium considered above, Example 28.4.9. We can this of the equilibrium as depending on the endowments $(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2) \in \mathbb{R}^6_+$. The endowments near $\big((2,1,0),(0,1,2)\big)$ that lead to a Pareto optimal equilibrium obey the constraints $\omega^1_1 = \omega^1_2 = 1$. As such, they form a 4-dimensional submanifold of a 6-dimensional parameter space. Such sets have measure zero.

Similar results can be obtained by making small changes to preferences rather than endowments as in Geanakoplos and Polemarchakis (1986). We can do this in Example 28.4.9 by replacing the preferences with $u_1(\mathbf{x}) = \ln x_1 + (1+\varepsilon)\ln x_2 + \ln x_3$ and $u_2(\mathbf{x}) = \ln x_1 + (1-\varepsilon)\ln x_2 + \ln x_3$. The equilibrium allocation is unchanged at $\mathbf{x}^1 = \mathbf{x}^2 = (1,1,1)$ and once again $MRS^1_{12} = 1+\varepsilon$ and $MRS^2_{12} = 1-\varepsilon$, which are only equal when $\varepsilon = 0$. When $\varepsilon \neq 0$, the equilibrium is not Pareto optimal.

28.4.16 Non-Optimal Equilibrium I

In general, if assets are to be both traded and non-trivial, we need at least two assets. If there are two states, that completes the market. So incompleteness with asset trading requires at least three states, as in the next example. This example also yields an equilibrium which is not Pareto optimal. However, by completing the asset market we get an equilibrium allocation that is not only Pareto optimal, but is a Pareto improvement over the original equilibrium.

Example 28.4.10: Non-optimal Equilibrium with Incomplete Markets. Since we have three states to deal with, we will strip the model down to essentials, with a single consumption good in each state ($m = 1$). There are two consumers with endowments $\boldsymbol{\omega}^1 = (1, 2, 3)$ and $\boldsymbol{\omega}^2 = (3, 2, 1)$, so there are always 4 units of the consumption good available. Each consumer has utility $u_i(\mathbf{x}) = \ln x_1 + \ln x_2 + \ln x_3$.

There are two assets. The first has return vector $\mathbf{r}^1 = (1, 1, 0)^\mathsf{T}$ and the second has return vector $\mathbf{r}^2 = (1, 0, 0)^\mathsf{T}$. That means the payoff matrix is

$$\mathbf{R} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We normalize spot prices so that $p_{1,s} = 1$ for each state s. The equilibrium price vector is $\hat{\mathbf{p}} = (1, 1, 1)$. With only one good in each state, consumer demand is

$$\mathbf{x}^1 = \begin{pmatrix} 1 + z_1^1 + z_2^1 \\ 2 + z_1^1 \\ 3 \end{pmatrix} \text{ and } \mathbf{x}^2 = \begin{pmatrix} 3 + z_1^2 + z_2^2 \\ 2 + z_1^2 \\ 1 \end{pmatrix}.$$

Notice that consumption in state three is independent of the portfolio chosen.

Given portfolios $\mathbf{z}^i$, we compute indirect utility as $V_1(\mathbf{z}^1) = \ln(1 + z_1^1 + z_2^1) + \ln(2 + z_1^1) + \ln 3$ and $V_2(\mathbf{z}^2) = \ln(3 + z_1^2 + z_2^2) + \ln(2 + z_1^2)$.

We take asset two as numéraire, making the asset prices $\mathbf{q} = (q, 1)$. Then $z_2^i = -qz_1^i$ by the asset budget constraint. To find asset demands we must maximize $V_1(z_1^1) = \ln\left(1 + (1-q)z_1^1\right) + \ln(2 + z_1^1)$ and $V_2(z_2^2) = \ln\left(3 + (1-q)z_1^2\right) + \ln(2 + z_1^2)$.

28.4.17 Non-Optimal Equilibrium II

The first-order conditions are

$$0 = (1 - q)(2 + z_1^1) + 1 + (1 - q)z_1^1$$
$$0 = (1 - q)(2 + z_1^2) + 3 + (1 - q)z_1^2.$$

Thus $2(1 - q)z_1^1 = -1 - 2(1 - q)$ and $2(1 - q)z_1^2 = -3 - 2(1 - q)$. Market clearing requires $z_1^1 + z_1^2 = 0$, so $0 = -4 - 4(1 - q)$, yielding $q = 2$. The equilibrium asset prices are $\hat{\mathbf{q}} = (2, 1)$.

Substituting $q = 2$ in the first-order conditions yields equilibrium asset demands of $\hat{\boldsymbol{z}}^1 = (-1/2, q/2) = (-1/2, +1)$ and $\hat{\boldsymbol{z}}^2 = (1/2, -q/2) = (+1/2, -1)$. The equilibrium consumption vectors are then $\hat{\mathbf{x}}^1 = (3/2, 3/2, 3)$ and $\hat{\mathbf{x}}^2 = (5/2, 5/2, 1)$.

This equilibrium is not Pareto optimal. Although the marginal rate of substitution between consumption in states one and two is the same for both consumers, it differs between the consumers for the other two pairs of states. Thus $MRS_{13}^1 = 2$ and $MRS_{13}^2 = 2/5$ while $MRS_{23}^1 = 3/2$ and $MRS_{23}^2 = 2/5$.

In both cases, consumer one has the higher marginal rate of substitution, indicating that Pareto improvements can be made by increasing consumer one's consumption in states one and two while decreasing it in state three, and decreasing consumer two's consumption in states one and two while increasing it in state three, with the terms of trade between the respective marginal rates of substitution. For example, $\mathbf{x}^1 = \mathbf{x}^2 = (2, 2, 2)$ is a Pareto improvement (and Pareto optimum). This is the Arrow-Debreu allocation, and would be the equilibrium allocation if we had a complete set of assets.

In fact, there is a whole set of Pareto optima that are Pareto improvements. Since this is a problem with three goods and equal weighted Cobb-Douglas utility, we know from Example 19.2.5 that each consumer gets a share of $(4, 4, 4)$ at every Pareto optimum. To improve on the equilibrium, we need $u_1\big(\alpha(4, 4, 4)\big) = \ln 4^3\alpha^3 \geq u_1(\hat{\mathbf{x}}) = \ln 27/4$ and $u_2\big((1-\alpha)(4, 4, 4)\big) = \ln 4^3(1-\alpha)^3 \geq u_2(\hat{\mathbf{x}}) = \ln 25/4$. This works if $1 - \frac{1}{4}(\frac{25}{4})^{1/3} \geq \alpha \geq \frac{3}{4}(\frac{1}{4})^{1/3}$. The approximate range for α is $[0.472, 0.539]$.

Such Pareto improvements cannot be made with the incomplete return matrix $\mathbf{R}$ because there is no way to trade consumption in state three for consumption in states one or two using the assets available. ◀

28.4.18 Adding Assets may not be Pareto Improving

A natural question is what can be done about the non-optimality of equilibrium? We know that complete markets are Pareto optimal, but are they Pareto improving? For that matter, is it Pareto improving to add assets to an incomplete market, regardless of whether the additional assets complete the market or not?

In general, the answer to the last question is no. Hart (1975) used a multiperiod version of the Radner model to show that adding assets when markets are incomplete may not result in a Pareto improvement. It may make one or more consumers worse off.

Whether additional assets will yield Pareto improvements in equilibrium depends on the both the economy and the assets. Geanakoplos and Polemarchakis (1986) investigated the constraints that incomplete financial markets can impose on equilibria. Generically, these constraints bind. When they bind, adding assets may lead to an equilibrium that is not Pareto improving.

Eulu (1995) showed that in a rich class of economies with incomplete assets, it is possible to make arbitrary perturbations in equilibrium utilities by adding another asset. These perturbations can be either or negative, indicating that there is nothing special about Hart's example. It happens all too often. Cass and Citanna (1998) showed that this sort of thing is generic when $S - K \geq I - 1$.

28.5 Derivatives: Creating New Assets from Old

There is still an unanswered question: Where do assets come from? The requirement $\sum_i z_k^i = 0$ implies that consumers may create the assets out of thin air by offering them for sale. We don't have an endowment of assets. So why can our consumers only offer specific assets, those in the return matrix $\mathbf{R}$? One possible answer is that either the legal system or the rules of an organized asset exchange limit trade to particular assets.

One way the asset structure might arise is from firms. A return vector could derive from the profits of a firm. A share of the firm would entitle the holder to a share of the profits in each state—the assets would be shares of the firm. In this case we would set up the model a little differently, and consumers would start with a share portfolio at time zero. Consumers could trade them, or even sell them short (thus effectively creating more of the asset).[7]

There is a potential problem here. As long as there production takes place solely in period one, once uncertainty is resolved, all the owners will agree on what profit maximization means. However, if production takes place over time, say with inputs in period zero and outputs in period one, complications arise. We have to think in terms of maximizing expected profits. If there are complete asset markets, there are unique market probabilities of each state and all can agree on what expected profit maximization means. With incomplete markets, different consumers may have different opinions on the probabilities and disagree on the meaning of expected profits.

[7] Note that under certainty, trading the firm shares would be pointless since a share of profit worth \$1 will have a price of \$1.

28.5.1 Options

Another possibility is to explore methods of creating new assets from old assets—derivative assets. One way to do this is to introduce options.[8]

Options are assets based on an underlying primary asset. Consider a *call option*, which is a contract to sell a specified amount of the underlying asset at a specified price (the *strike price*) at a specified time or range of times (the *exercise date*). The buyer of the call obtains the right to buy an amount of the underlying asset at the strike price on the exercise date(s). The option part is that the buyer of the call does not have to exercise the option to buy the underlying asset.

Put options are similar, but the writer agrees to buy a specified amount of the underlying asset at the strike price on the exercise date(s). The buyer of a put gets the option of buying the asset at the strike price on the exercise date. Once again, buyer of the put option is the one that has the choice of whether or not to exercise it.

This means there are two prices associated with an option. The price of the option itself and the strike price. Since we have only two periods available, our options will be very simple. At time zero, options are bought and sold. At time one, the purchaser of the option has the choice of exercising it. If they exercise it, they either buy or sell the specified amount of the underlying asset at the strike price. If they opt to not exercise the option, no further trades are made involving the option. In practice, options are often settled in cash rather than by delivery of the underlying asset. In our simple one-good models, the good that the asset pays in is also the numéraire, so there is no real difference between the two types of settlement.

[8] The fundamental papers on option pricing are Black and Scholes (1973) and Merton (1973).

28.5.2 Calls and Puts

European and American options differ in the way they are exercised. American options can be exercised at any time before the option expires. European options can only be exercised at the expiration date. In our two period model there is no time in-between, and so no distinction between the two. We will define these options as if they are true European options.

European call option. Given a underlying asset $\mathbf{r}$, a *call option* with strike price c (cost of exercising the option) gives the buyer of the option the right to buy one unit of the asset at price c at time one, *after the state is revealed*. We denote the call option with strike price c by $\mathbf{r}(c)$.[9]

If we are in a state where $r_s > c$, it will be worthwhile to exercise the option and buy the asset at the previously agreed strike price of c. By exercising the option, we gain the value of the asset (r_s) at the cost of paying the strike price c, for a net gain of $(r_s - c)$. Such an option is said to be "in the money".

If we are in a state where $r_s < c$, it would be foolish to exercise the option. We would lose additional money by doing so. At that point it is worth nothing. Such an option is "out of the money".

Thus $\mathbf{r}(c)$ gives us a return vector of $r_s(c) = \max\{0, r_s - c\}$. If $\mathbf{r} = (4, 3, 2, 1)$, $\mathbf{r}(3.5) = (0.5, 0, 0, 0)$, $\mathbf{r}(2.5) = (1.5, 0.5, 0, 0)$, $\mathbf{r}(1.5) = (2.5, 1.5, 0.5, 0)$ and $\mathbf{r}(0.5) = (3.5, 2.5, 1.5, 0.5)$. Here we have generated a complete set of assets by writing options on a single underlying asset.

In some cases, we cannot complete the market with options. An extreme case is the asset $\mathbf{r} = (1, 1, 1, 1)$. If $c < 1$, $\mathbf{r}(c) = (1 - c)\mathbf{r}$, while if $c \geq 1$, $\mathbf{r}(c) = \mathbf{0}$. Writing options on this asset does not create additional independent assets.

European put option. A *European put option* is an option to *sell* an asset at time one at a specified strike price c. It is the opposite of a call. If an option has return vector $\mathbf{r}$, we denote the return vector of the put at strike price c as $\mathbf{r}(c)$.

Since the asset is being sold at price c, it will be worthwhile to sell it if this price is higher than the payoff from the asset itself, if $r_s < c$. The return if the put is exercised $c - r_s$. Therefore the put pays $\max\{0, c - r_s\}$ in state s.

[9] European and American options are not the only possibilities, and other types of options are also traded in the market. In our simple two-period model, there is no distinction between American and European options as there is only one possible time to exercise the option. With continuous time, the difference is significant and the pricing of European options is described by the Black-Scholes formula (Black and Scholes, 1973).

28.5.3 Pricing Options

We can often create a variety of assets by writing options on existing assets.

When markets are complete, the price of an option will depend on the strike price in a piecewise linear fashion. The following example shows how this works.

Example 28.5.1: Pricing Options. Let's consider a contingent goods exchange economy with one good in three states and two consumers with utility $u(\mathbf{x}) = \ln x_1 + \ln x_2 + \ln x_3$. The endowments are $\boldsymbol{\omega}^1 = (1, 2, 3)$ and $\boldsymbol{\omega}^2 = (3, 2, 1)$. It is easy to find the Arrow-Debreu equilibria. They have price vector $\hat{\mathbf{p}} = (1, 1, 1)$ (or a positive scaler multiple) and allocation $\hat{\mathbf{x}}^1 = \hat{\mathbf{x}}^2 = (2, 2, 2)^{\mathbf{T}}$.

We can convert this to a Arrovian securities equilibrium with goods prices $\hat{\mathbf{p}}$ as before, asset prices $\hat{\mathbf{q}} = \hat{\mathbf{p}} = (1, 1, 1)$, goods allocation $\hat{\mathbf{x}}^1$ and $\hat{\mathbf{x}}^2$ as before. We can then calculate the equilibrium portfolios which are $\hat{\mathbf{z}}^1 = (+1, 0, -1)$ and $\hat{\mathbf{z}}^2 = (-1, 0, +1)$.

Consider the Radner equilibrium where the return matrix is

$$\mathbf{R} = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

The matrix $\mathbf{R}$ is invertible, so asset markets are complete. We can find the Radner asset prices by setting $\bar{\mathbf{q}} = \mathbf{R}^{\mathbf{T}}\hat{\mathbf{q}} = (6, 3, 3)^{\mathbf{T}}$.

We will now consider call options written on asset one. If the strike price c obeys $1 \leq c \leq 5$, $\mathbf{r}^1(c) = (5 - c, 0, 0)^{\mathbf{T}}$. We know that the Arrovian security $(1, 0, 0)$ has price 1. Applying the Arbitrage Pricing Theorem, we find $\mathbf{r}^1(c)$ has price $(5 - c)$ whenever $1 \leq c \leq 5$. When $0 \leq c < 1$, $\mathbf{r}^1(c) = (5 - c, 1 - c, 0)$. We can think of this as a linear combination of the first two Arrovian securities, so its price is $(6 - 2c)$. Of course if the strike price is $c > 5$, the option is never in the money and must have price zero.

We have exploited the fact that we started with a complete set of assets. As a result, the equilibrium goods allocations, spot market prices, and prices of existing assets remain unchanged. The addition of options only affects the possible equilibrium portfolios. ◀

This may not the case if the market is initially incomplete. If the option is not a linear combination of existing assets, it will change the equilibrium allocations. We will examine that further shortly.

28.5.4 Options and Risk I

Our next example of option pricing is a little different. Markets will be complete, with one risk-free asset and one risky asset. We will consider an option on the risky asset, which will carry an adjustable amount of risk.

Example 28.5.2: Options with Varying Risk. Let asset one be risk-free: $\mathbf{r}^1 = (1,1)^{\mathsf{T}}$. Asset two is risky, with the amount of risk depending on a parameter α: $\mathbf{r}^2 = (4+\alpha, 2-\alpha)^{\mathsf{T}}$ for $\alpha \geq 0$. If the two states were equally likely, α would define a mean-preserving spread of $(4,2)^{\mathsf{T}}$. Take asset one as numéraire and let $q_2 = q$.

If the states are equally likely, the expected payoff of $\mathbf{r}^2$ is 3, which would be its price if the market probabilities are $(1/2, 1/2)$. Under any probabilities $(\pi, 1-\pi)$, the mean varies with α. It is $2 + 2\pi + (2\pi - 1)\alpha$. A short calculation shows that the mean is 3 if and only if $\pi = 1/2$, which means it is a mean-preserving spread if and only if $\pi = 1/2$.

However, we don't know the probabilities. In that case set $q_1 = 1$ and $q_2 = q$ with $2 - \alpha < q < 4 + \alpha$ (this ensures $\mathbf{q}$ is arbitrage-free) and consider the option $\mathbf{r}^2(c)$ for $2 < c < 4$. This choice of strike price ensures that the option will be in the money in state one and out of the money in state two. Thus $\mathbf{r}^2(c) = (4 + \alpha - c, 0)^{\mathsf{T}}$.

28.5.5 Options and Risk II

Since any three vectors in $\mathbb{R}^2$ are linearly dependent, and $\mathbf{r}^1$ and $\mathbf{r}^2$ are linearly independent, $\mathbf{r}^2(c)$ is a linear combination of $\mathbf{r}^1$ and $\mathbf{r}^2$. Here

$$\mathbf{R} = \begin{pmatrix} 1 & 4+\alpha \\ 1 & 2-\alpha \end{pmatrix} \text{ and } \mathbf{R}^{-1} = \frac{-1}{2(1+\alpha)} \begin{pmatrix} 2-\alpha & -(4+\alpha) \\ -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{r}^2(c) = \frac{4+\alpha-c}{2(1+\alpha)} \left((\alpha-2)\mathbf{r}^1 + \mathbf{r}^2 \right).$$

The option price can now be calculated using the Arbitrage Pricing Theorem.

$$q_2(c) = \frac{4+\alpha-c}{2(1+\alpha)}(\alpha - 2 + q).$$

Note that q, the price of the underlying asset is positively related to $q_2(c)$, the option price because $4 + \alpha - c > 0$.

We can increase the riskiness of asset two by increasing α. To see the effect of changes in α, let $\hat{\alpha} > \alpha$ and define $\hat{\mathbf{r}}^2$ as $\mathbf{r}^2$ using $\hat{\alpha}$ instead of α. Let

$$\mathbf{r}^3 = \hat{\mathbf{r}}^2(c) - \mathbf{r}^2(c) = (\hat{\alpha} - \alpha, 0)^{\mathsf{T}} > \mathbf{0}.$$

The Arbitrage Pricing Theorem tells us $q_3 = \hat{q}_2(c) - q_2(c)$. Since $\mathbf{r}^3 > \mathbf{0}$, the price $q_3 = \pi(\hat{\alpha} - \alpha) > 0$. It follows that $\hat{q}_2(c) \geq q_2(c)$, that the option $\mathbf{r}^2(c)$ becomes more valuable as underlying asset becomes more risky. This increase in value occurs regardless of whether $\mathbf{r}^2$ is a mean-preserving spread. It makes sense that the price increases with α. The option is not affected by the downside risk. The probability that the option is in or out of the money remains the same. However, when α increases, the payoff when the option is exercised goes up, increasing the value of the option. ◄

28.5.6 Completing Markets with Options

We finish our exploration of asset markets with a model with an incomplete asset structure which can be completed by use of an option.

Example 28.5.3: Incomplete Market Completed by Option. Suppose there is one good in each of two states. There are two consumers with utility $u(\mathbf{x}) = \ln x_1 + \ln x_2$. Endowments are $\boldsymbol{\omega}^1 = (3,1)^{\mathbf{T}}$ and $\boldsymbol{\omega}^2 = (1,3)^{\mathbf{T}}$. We start with return matrix

$$R = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

There is only one asset. This means there is no asset one can sell in order to buy the only asset. It cannot be traded in Radner equilibrium. We normalize the spot prices to $\mathbf{p} = (1,1)$ (any state scalar multiple will do). Both consumers can only afford their endowments. The Radner equilibrium allocation of goods is $\mathbf{x}^1 = (3,1)^{\mathbf{T}}$ and $\mathbf{x}^2 = (1,3)^{\mathbf{T}}$. This is not Pareto optimal.

We complete the market by adding the option $\mathbf{r}^1(1) = (1,0)^{\mathbf{T}}$. With complete markets, we obtain the Arrow-Debreu allocation, which is easily seen to be $\hat{\mathbf{x}}^1 = \hat{\mathbf{x}}^2 = (2,2)^{\mathbf{T}}$ with Arrow-Debreu prices $\hat{\mathbf{p}} = (1,1)$ (or any positive scalar multiple). The equivalent Arrovian securities model has asset prices $\mathbf{q} = \hat{\mathbf{p}} = (1,1)$. Then $\mathbf{r}^1 = 2\mathbf{e}^1 + \mathbf{e}^2$ and $\mathbf{r}^1(1) = \mathbf{e}^1$. By the Arbitrage Pricing Theorem, $q_1 = 3$ and $q_1(1) = 1$. The asset budget constraint is then $3z_1^i + z_2^i = 0$. The consumption budget constraints are then

$$\begin{aligned} x_1^i &= \omega_1^i + 2z_1^i + z_2^i = \omega_1^i - z_1^i \\ x_2^i &= \omega_2^i + z_1^i. \end{aligned}$$

Setting $x_s^i = 2$ (the Arrow-Debreu allocation) we solve for $\bar{\mathbf{z}}^i$, obtaining $\bar{\mathbf{z}}^1 = (1,-3)$ and $\bar{\mathbf{z}}^2 = (-1,+3)$. The Radner equilibrium is then $\hat{\mathbf{p}} = (1,1)$, $\bar{\mathbf{q}} = (3,1)$, $\bar{\mathbf{z}}^1 = -\bar{\mathbf{z}}^2 = (+1,-3)$ and $\hat{\mathbf{x}}^1 = \hat{\mathbf{x}}^2 = (2,2)$.

In this case, the complete markets equilibrium is not only Pareto optimal, but is a Pareto improvement over the incomplete markets equilibrium. ◄

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