Division Rules, Network Formation, and the Evolution of Wealth

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Abstract

In our model, each of $n > 2$ agents is endowed with an exogenous amount of initial wealth. Each individual may establish at most one link with any agent he prefers each period. A surplus will be generated from each link and the agents involved in that link will bargain over its division. The payoffs obtained by an agent at each period will be added to the existing wealth (ideal payoff) level of the agent. Suppose in a society (or in a group of individuals), all agents adhere to a particular division rule.

A variety of long-run wealth distributions can arise, depending on the division rule and initial wealth distribution. These can range from situations where the richest agent remains richest to cases where the poor are continually becoming rich. We examine several division rules in detail: egalitarian, equal sacrifice, proportional, dictatorship of the rich, dictatorship of the poor.

From the analysis of these cases, we find that two factors determine the long-run wealth distribution: the size of the gain from a link, and the incentive to link to rich or poor.

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1 Introduction

In an important recent paper, Piccione and Rubinstein (2004) studied a model of strategic network formation in which being wealthier initially did not necessarily imply being wealthier ex post.[1] They used a simple one-shot setup in which each individual can establish at most one link with a weaker agent, and the stronger agent can acquire the total amount of wealth owned by the weaker agent. Although their main result does not hold when agents are restricted to direct transfers only; it holds when indirect appropriations too are allowed.

Needless to say, the limit (long-run) wealth distributions of such setups too can be as interesting as the short-run fluctuations in the wealth rankings of the agents. For instance, when the stage setup is repeated infinitely many times in Piccione and Rubinstein (2004), one can see that the limit distribution is such that the agent with the highest initial wealth will end up with the entire wealth in the economy.

We study a related framework where agents bargain over the wealth created by a link, the links being recreated in each of infinitely many periods.

In our model, each of $n > 2$ agents is endowed with an exogenous amount of initial wealth. As in Piccione and Rubinstein (2004), each individual may establish at most one link with any agent he prefers each period. A surplus will be generated from each link and the agents involved in that link will bargain over its division. Only direct links generate payoffs. The payoffs obtained by an agent at each period will be added to the existing wealth (ideal payoff) level of the agent. The status-quo point for each pair that have a link is the origin.

The Pareto frontier is linear and connects the ideal payoffs of the parties. Each point on the Pareto frontier represents a particular division of the pie; each such point can be assigned as the outcome by a particular division rule. Among them, the midpoint of the Pareto frontier has a focal importance. It assigns payoffs proportional to the ideal

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[1]Although there were a few other papers on networks in the economics literature before that, the most notable work on strategic network formation can be dated back to that of Jackson and Wolinsky (1996). The introductory chapter of Dutta and Jackson (2003) provides an overview of this literature's progression and the role of the papers collected in their edited volume. Their edited volume includes some of the papers that have been very influential in the literature of the formation of networks.

payoffs of the two agents. Suppose in a society (or in a group of individuals), all agents adhere to a particular division rule.

The division of the surplus from a link depends only on the current wealth of the two agents. If the payoff is increasing in the other agent's wealth, we call the division rule *rich-linking* as agents will have an incentive to link with rich agents.[2] If it is decreasing in the other agent's wealth, it is *poor-linking.* The proportional division rule (which coincides with the outcome of the Kalai/Smorodinsky solution of the cooperative bargaining problem) defines the border between rich-linking and poor-linking solutions. Under proportional division, an agent does not have any particular incentive to link with any particular agent.

Whether the division rule is rich-linking or poor-linking determines the structure of the links. A link request by an agent is costless and cannot be declined by any agent. (If both agents request each other, only one link is established between them.) Label agents by initial wealth with the richest agent as agent 1. Given a rich-linking division rule, all agents will want to link to the richest agent they are not already linked with. It follows that the situation where all agents (except agent 1) link to agent 1 and agent 1 links to agent 2 is an equilibrium. Given a poor-linking division rule, the situation where all agents (except agent n) link to agent n and agent n links to agent $(n-1)$ is an equilibrium. Other equilibria exist, and are discussed in the appropriate sections.

There are several other divisions of the pie that are of prominence. They are the egalitarian division, the equal sacrifice division, dictatorship of the poor, dictatorship of the rich. The first one assigns equal payoffs to both agents, while in the last two one of the agents receives his ideal payoff and the other agent receives nothing. In the equal sacrifice outcome, each agent's payoff is equally far from his ideal payoff.

The evolution of the wealth distribution as well as the limit wealth distribution can differ dramatically according to the division rule, even in the simple equilibria. Two aspects of the division rule are important: the share that goes to each individual and the incentive to form links. We examine several cases in order to understand this better.

The egalitarian solution is rich-linking. Oddly, the long-run distribution is egalitarian for everyone but the richest agent. The richest agent gets half of the wealth. The rest is

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[2]Note that the actual division of the surplus may give more to the poor agent.

split equally among the other agents. Here the number of links the rich agent participates in overwhelms the smallness of the gain from linking.

The equal sacrifice solution is poor-linking. Everyone links with the poorest agent. One might expect the poorest agent to gain wealth and eventually overtake the next poorest while the richest agent (with only one link) grows slowly. Eventually, the poor leapfrog their way up the income distribution. In fact, it is not so simple and for some initial distributions the rich agent will end up growing fastest. Alternatively, other initial distributions lead to the leapfrogging situation described above. The additional links that the poor get can overcome small deficiencies in initial wealth, but when the gap is large, the bigger gain that the richer agent receives is dominant.

The proportional division is link-neutral. As as result, everyone's wealth grows at the same rate. However, the wealth distribution may drift over time, depending on the equilibrium path chosen.

In the case of dictatorship of the rich, the extreme wealth division overcomes the benefits to the poor of poor-linking. Everyone but the poorest person links to someone poorer. All but agent n have a 100% growth rate. Agent n's wealth remains constant.

Finally, the dictatorship of the poor leads to leapfrogging and the income distribution bunches up. Here the extra links the rich get are of no value to them. What counts is that the poor gain more from the links.

Intuitively, it is clear that other long-run behavior is possible. Take a rich-linking division which is arbitrarily close to the proportional division. Eventually, agent 1, with whom everybody links, ends up owning nearly the entire wealth in the society; the evolution of the wealth distribution always preserves the initial wealth ranking, but skews the amounts. Alternatively, a poor-linking division which is arbitrarily close to the proportional division will have all agents other than n link to agent n; this continues until agent n leap-frogs over agent $n - 1$. The same pattern is followed as the poorer agents catch up with richer agents; thus, the wealth distribution does not preserve the initial wealth ranking, and relatively frequent reversals of fortunes are possible. We see this in the egalitarian case.

We set up the model in Section 2. The egalitarian solution is examined in Section 3. Section 4 considers the equal sacrifice case and Section 5 analyzes the proportional

division rule. Finally dictatorships of both rich and poor are considered in Section 6. Some concluding remarks are in Section 7.

2   The Model

The set of agents is $\{1, ..., n\}$, where $n > 2$. Each agent i is initially endowed at time 0 with $w_i > 0$ amount of wealth, such that $w_1 > w_2 > \ldots w_n$. Let w_i^t denote agent i's wealth at period $t = 0, \ldots \infty$. It evolves according to $w_i^{t+1} = w_i^t + b_i^t$ where b_i^t denotes the total bargaining payoffs i will earn from the links he has at period t. It will be convenient to write the payoff as a proportion of current wealth. We set $z_i^t = b_i^t / w_i^t$, which implies $w_i^{t+1} = (1 + z_i^t) w_i^t$.

The bargaining set at time t for agents i and j depends solely on their wealth at time t and is the convex hull of $\{(0,0), (0, w_i^t), (w_j^t, 0)\}$.[3] We denote the set by $S(w_i^t, w_j^t)$ and its Pareto frontier by $\partial(w_i^t, w_j^t)$. Of course, the Pareto frontier is the segment $[(w_i^t, 0), (0, w_j^t)]$. The origin will be taken as the status-quo point.

We will consider consider a variety of bargaining solution concepts. Let F be the bargaining solution. Given a match with wealth levels (w, w'), the payoffs are $F(w, w') = (F_1(w, w'), F_2(w, w'))$. We will require that the solution F be Pareto optimal and that it obey constant returns to scale.[4] We also require that the solution only depend on the wealth levels, not their order. In sum:

Assumption 1. *We assume that* $F : R_+^2 \to R_+^2$ *obeys:*

1. *F exhibits constant returns to scale.*

2. *$(F_1(w, w'), F_2(w, w')) = (F_2(w, w'), F_1(w, w'))$.*

3. *F is on the Pareto frontier $\partial(w, w')$.*

The first two conditions allow us to write $F(w, w') = (wg(w'/w), w'g(w/w'))$ where $g(x) = F_1(1, x)$.

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[3] More general cases could be considered, but at the cost of substantially complicating the analysis.
[4] Recall that the bargaining set itself exhibits constant returns to scale.

Since the bargaining set is the convex hull of $\{(0,0), (w,0), (0,w')\}$, Pareto optimality implies $F_1(w,w')/w + F_2(w,w')/w' = 1$. This can be rewritten $g(w'/w) + g(w/w') = 1$. As this holds for all (w,w'), it is equivalent to $g(x) + g(1/x) = 1$. It follows that any function $g : [0,1] \to [0,1]$ with $g(1) = 1/2$ can be extended to such a function on R_+.

The payoff from the match (w,w') is $(wg(w'/w), w'g(w/w'))$. When $g' > 0$ ($g' < 0$), the payoff to each agent is increasing (decreasing) in the other agent's wealth. Thus both agents will want to link to the richest available agent when $g' > 0$. When $g' < 0$, they will both link to the poorest available agent. Things are more complex if g' changes sign.

Now consider the Egalitarian, Proportional (which coincides with Kalai-Smorodinsky here), and Equal Sacrifice solutions, as well as the two dictatorial solutions.

The Egalitarian solution, E, is such that its outcome, $E(w,w')$, is on $\partial(w,w')$ where $E_i(w,w') = E_j(w,w')$. The Pareto frontier consists of non-negative (x,y) such that $x/w + y/w' = 1$. Setting $x = y$, we obtain

$$E(w,w') = \left(\frac{ww'}{w+w'}, \frac{ww'}{w+w'} \right).$$

In this case $g(x) = x/(1+x)$.

The Proportional solution, P, is such that its outcome $P(w,w')$ is on $\partial(w,w')$ such that $P_j(w,w')/P_i(w,w') = w'/w$.[5] It is easily calculated as

$$P(w,w') = \left(\frac{w}{2}, \frac{w'}{2} \right).$$

This yields $g(x) = 1/2$.

The Equal Sacrifice solution, ES, is such that its outcome, $ES(w,w')$, is on $\partial(w,w')$ where $w - ES_i(w,w') = w' - ES_j(w,w')$. Solving, we obtain

$$ES(w,w') = \left(\frac{w^2}{w+w'}, \frac{(w')^2}{w+w'} \right).$$

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[5]The outcome of any bargaining solution that satisfies the 'Pareto Optimality', 'Symmetry' and 'Scale Invariance' axioms coincides with the outcome of the Proportional division rule in our setup. Most notably these solutions include the Nash, Kalai/Smorodinsky, Perles/Maschler and Equal Area solutions.

Here, $g(x) = 1/(1 + x)$.

The dictatorial solutions yield a discontinuous g. The Dictatorship of the Rich, DR, is defined as the most preferred point of the rich agent. The Dictatorship of the Poor, DP, is the most preferred point of the poor agent. The dictatorship of the rich yields $g(x) = 1$ for $x < 1$ while the dictatorship of the poor gives $g(x) = 0$ for $x < 1$.

We say that a solution F is *rich-linking* if $F_1(w, w')$ is increasing in w' and *poor-linking* if $F_1(w, w')$ is decreasing in w'. The solution is *neutral* if $F_1(w, w')$ is unaffected by changes in w'. Rich-linking solutions give everyone an incentive to link to the richest available agent. Poor-linking solutions create incentives to link with the poorest.

Theorem 2. *Let g be the function associated with a solution F. If $g' > 0$, F is rich-linking. If $g' < 0$, F is poor-linking. Finally, if g is constant, F is neutral.*

Proof. Since $F_1(w, w') = wg(w'/w)$, $\partial F_1(w, w')/\partial w' = g'(w'/w)$ and the result follows. $\qquad\square$

It follows that the egalitarian solution is rich-linking, the equal sacrifice solution is poor-linking, and the proportional solution is neutral. The two dictatorial solutions yield discontinuous functions g. The dictatorship is the rich is weakly poor-linking, but does not particularly encourage linking to the poorest as the payoff is independent of the poorer agent's wealth. Similarly, the dictatorship of the poor is weakly rich-linking.

Whether g is increasing or decreasing also determines whether the richer or poorer agent gains proportionately more. Assume $w > w'$. The rich agent's payoff is $wg(w'/w)$ while the poor agent gets $w'g(w/w')$. Dividing by their current wealth levels, we find the rich agent gains by the ratio $g(w'/w)$ while the poor agent gains by $g(w/w')$. Since $w/w' > w'/w$, the poor agent gains proportionately more when $g' > 0$ and the rich agent gains proportionately more when $g' < 0$. Both gain in the same proportion when g is constant.

Monotonicity properties of g also inform us about gains when two agents have a link to the same agent (typically either the richest or the poorest). Suppose $w > w'$ for two agents and both link to an agent with w^*. The richer agent gains in proportion $g(w^*/w)$ while the poorer gains $g(w^*/w')$. When g is increasing, the rich agent will

gain proportionately less from linking to a third party than the poor agent. When g is decreasing, the situation is reversed.

3   Egalitarian Solution

The egalitarian solution is rich-linking. In equilibrium, each agent will have links only with the richest other agents. If there are k such links, they will be with the k richest other agents. Since there are no richer agents left unlinked, this is a best response by each player.

There are n possible equilibria. In the first equilibrium (call it equilibrium E_1) each agent links to agent 1. There are $(n-1)$ links in all. There are $(n-1)$ other equilibria. In equilibrium E_i, agent 1 links to agent i. It is the best response for each of the other agents to link to 1, and agent i then links to the best unlinked agent, agent 2. This equilibrium has n links.[6]

The dynamics of equilibrium E_1 are the simplest, and we will first focus our attention on the case where equilibrium E_1 occurs in each period.

Note that both agents receive the same absolute gain from a link. We also know that the gains agents receive from linking to agent 1 are decreasing in their own wealth. Thus 2 gains more than 3, who gains more than 4, etc. Moreover, agent 1's absolute gain is equal to the total gains of all of the other agents. There are no changes in the ordering of agents by wealth in this equilibrium. The fifth wealthiest individual remains fifth wealthiest forever.

In this equilibrium, player $i > 1$ has only one link, and wealth evolves according to $w_i^{t+1} = (1 + z_i^t)w_i^t$ where z_i^t is

$$z_i^t = \frac{w_1^t}{w_i^t + w_1^t} = \frac{1}{1 + (w_i^t/w_1^t)}. \tag{1}$$

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[6]As a real-life example consider paper submissions to the Econometric Society Meetings where each author can submit at most one paper. Despite that restriction, the programs of these meetings are replete with instances where several prominent authors ending up having several papers on the program since many other authors deem their chances of being on the program much higher if they submit a joint work with a prominent author.

In the case of player 1, there are $(n-1)$ links, and z_1^t is given by

$$z_1^t = \sum_{i=2}^{n} \frac{w_i^t}{w_i^t + w_1^t} = \sum_{i=2}^{n} \frac{1}{(w_1^t/w_i^t) + 1}. \tag{2}$$

Under these dynamics, there is a steady state wealth distribution of $(w_1, w, \ldots, w)$ where $w_1/(w_1 + w) = (n-1)w/(w_1 + w)$ so $w_1 = (n-1)w$. In that case $z_1 = (n-1)/(w_1 + w) = w_1/(w_1 + w) = z_i$ for $i \neq 1$. As a result, wealth grows by a factor of

$$\left[\frac{w_1}{w_1 + w} + 1 \right] = \left[\frac{(n-1)w}{(n-1)w + w} + 1 \right] = \left[\frac{2n-1}{n} \right].$$

Before analyzing the dynamics, we note that for each t, $w_1^t z_1^t = \sum_{i=2}^{n} w_i^t z_i^t$. In other words, fully half of the social gain from linking is captured by the richest individual.

Let A^t be the total wealth of $i = 2, \ldots, n$ at time t. Suppose $w_1^t > A^t$. Then $w_1^{t+1} = w_1^t + z_1^t$ and $A^{t+1} = A^t + z_1^t$ since the growth is split equally between agent 1 and everyone else. It follows that $w_1^{t+1} > A^{t+1}$. So if $w_1^1 = w_1 > A^1 = \sum_{i=2}^{n} w_i$, $w_1^t > A^t$ for all t and if $w_1 < A^1$ then $w_1^t < A^t$ for all t.

Now we consider the behavior of the shares over time.

Theorem 3. *In equilibrium E_1, $\lim w_1^t/A^t = 1$.*

Proof. Suppose $w_1^t > A^t$. Then

$$\frac{w_1^{t+1}}{A^{t+1}} = \frac{w_1^t + z_1^t}{A^t + z_1^t} < \frac{w_1^t}{A^t}.$$

Similarly, if $w_1^t < A^t$, then w_1^t/A^t rises over time. In the first case the ratio is bounded below by 1 and falls, in the second it is bounded above by 1 and rises. Either way it must converge to a limit r.

Now consider the $w_1^t > A^t$ case and suppose the limit $r < 1$. Then

$$r = \lim \frac{w_1^t}{A^t} = \lim \frac{w_1^{t+1}}{A^{t+1}} = \lim \frac{w_1^t + z_1^t}{A^t + z_1^t} = \lim \frac{w_1^t/A^t + z_1^t/A^t}{1 + z_1^t A^t}.$$

It is easy to show $1/2 < z_1^t/A^t < 1$, so we may take a subsequence where z_1^t/A^t converges. Call the limit α. Then $r = (r + \alpha)/(1 + \alpha)$, so $r = 1$. The case $w_1^t < A^t$ is similar. $\square$

Theorem 4. *In equilibrium E_1, the wealth distribution converges monotonically and without changes in order to the steady state wealth distribution examined earlier.*

Proof. Of course, wealth never decreases. We already know that wealthier agents get larger absolute gains, which insures that the order doesn't change.

Fix $i \in \{3, \ldots, n\}$ and let $b^t = w_2^t / w_i^t$. Now

$$\frac{w_2^{t+1}}{w_2^t} = 1 + \frac{w_1^t}{w_2^t + w_1^t} < 1 + \frac{w_1^t}{w_i^t + w_1^t} = \frac{w_i^{t+1}}{w_i^t}.$$

Thus $b^t \geq 1$ declines over time. Call the limit b. Then

$$b = \lim \frac{w_2^t}{w_i^t} = \lim \frac{a_2^{t+1}}{w_i^{t+1}} = \lim \frac{w_2^t}{w_i^t} \left[\frac{1 + w_2^t/(w_2^t + w_1^t)}{1 + w_i^t/(w_i^t + w_1^t)} \right]$$

Now the $w_j^t/(w_j^t + w_1^t)$ are bounded, so we may consider a subsequence where they converge to c_2 and c_i for $j = 2, i$, respectively. Then $b = b(1 + c_2)/(1 + c_i)$. Since $b \geq 1$, $c_2 = c_i$. But then, $\lim w_i^t/w_1^t = \lim w_2^t/w_1^t$. Each individual other than 1 will receive the same asymptotic share of wealth. We conclude that $\lim w_1^t/(w_1^t + A^t) = 1/2$ and $\lim w_i^t/(w_1^t + A^t) = 1/2(n-1)$ for $i \neq 1$. $\qquad \square$

We now turn our attention to equilibrium E_2. Again, the ordering of agents 4 though n, each of whom only link with agent 1, doesn't change. Agents 2 and 3 have both links with 1 and with each other. The gain from their mutual link is the same, while agent 2 gains more by linking with 1. Thus agent 2 remains wealthier than agent 3. The only complication arises if agent 2 overtakes agent 1. This doesn't happen. Agent 1 has links with both 2 and 3. Agents 1 and 2 receive equal gains from their mutual link, but agent 1 gains more from the link with 3. This insures that 1 remains wealthier than 2. Once again, the wealth ordering remains unchanged over time.

There is only one possible steady state wealth distribution. Let $(w_1, \ldots, w_n)$ be a steady state income distribution. Setting all growth rates equal shows $w_i = w$ for $i > 3$ and that $w_2 = w_3$. Additional calculations show that $w = 0$ and thus the only possible steady state has wealth distribution $(1/3, 1/3, 1/3, 0, \ldots, 0)$

The same arguments as in Theorem 4 establish that agents 4 and above receive the same asymptotic share of wealth.

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Theorem 5. *In equilibrium E_2, the wealth distribution will converge to the steady state* $(1/3, 1/3, 1/3, 0, \ldots, 0)$.

Proof. We know that all agents remain in the same place in the wealth distribution. Only the magnitude of their share may change. The key is to compare the growth factors of agents 3 and 4. We consider time period t, but drop time subscripts for simplicity. Agent 4 has growth factor $1 + w_1/(w_1 + w_4) < 2$. Agent 3 has growth factor $1 + w_1/(w_1 + w_3) + w_2/(w_2 + w_3)$. We claim agents 3's growth factor is greater than 2. It is enough to show $w_1/(w_1 + w_3) + w_2/(w_2 + w_3) > 1$. Clearing the denominators, this is equivalent to $w_1 w_2 + w_1 w_3 + w_1 w_2 + w_2 w_3 > w_1 w_2 + w_1 w_3 + w_2 w_3 + w_3^2$. Simplifying, this is equivalent to $w_1 w_2 > w_3^2$, which is true as $w_1 > w_2 > w_3$.

Now agent 4's share relative to agent 3 shrinks monotonically. If it converges to a positive number, agent 4's growth factor will be bounded below 2. But then his share relative to 3 (and hence 1 and 2) must converge to zero, which is a contradiction. Therefore the share of agents 4 and above must converge to zero.

Since agents 4 and above have shares that converge to zero, agent 1's growth factor will approach $1 + \lim[w_2^t/(w_1^t + w_2^t) + w_3^t/(w_1^t + w_3^t)]$. This will be no larger than the asymptotic growth rate of agent 2, which itself will be no larger than the asymptotic growth rate of agent 3.

Comparison of the growth rates shows that if the shares of agents 1, 2, and 3 do not converge to 1/3 each, then agent 3's growth factor is bounded above the growth factor of 1 and 2. This implies 3 eventually becomes wealthier than 2, which is impossible. Thus all three shares converge to 1/3, which is the steady state value. $\square$

For the other $(n-2)$ equilibria we have to distinguish two ways of keeping the same equilibrium over time. One maintains linkages based on initial wealth, the other based on current wealth. In the first case, the dynamics are similar to equilibrium E_2, with all of the single-linked individuals finding their shares (but not absolute wealth) converging to zero, and the multiply-linked individuals each having shares of 1/3. In the second case, the wealth ordering is no longer maintained over time. The wealthiest single-linked agent will eventually be overtaken, at which point the link structure changes. As a result, the dynamics become more complex. Of course, cases where the equilibrium selected is different in every period are not easily analyzed.

4 Equal Sacrifice Solution

The equal sacrifice solution is poor-linking. In equilibrium, each agent will have links only with the poorest other agents. If there are k such links, they will be with the k poorest other agents. Since there are no poorer agents left unlinked, this is a best response by each player.

There are n possible equilibria. In equilibrium ES_i, agent n links to agent i. It is the best response for each of the other agents to link to n, and agent i then links to the best unlinked agent, agent $(n-1)$. This equilibrium has n links. In the last equilibrium (call it equilibrium ES_n) each agent links to agent n. There are $(n-1)$ links in all.

As in the egalitarian case, the dynamics of equilibrium ES_n are the simplest, and we will focus our attention on the case where equilibrium ES_n occurs in each period.

When players with wealth levels i and j are linked at t in the equal sacrifice case, the player with wealth w_i^t gets $(w_i^t)^2/(w_i^t + w_j^t)$ while the player with wealth level w_j^t gets $(w_j^t)^2/(w_i^t + w_j^t)$.

Consider equilibrium ES_n. Wealth evolves according to $w_i^{t+1} = (1 + z_i^t)w_i^t$. In this equilibrium, player $i > n$ has only one link and z_i^t is

$$z_i^t = \frac{w_i^t}{w_i^t + w_n^t} = \frac{1}{1 + (w_n^t/w_i^t)}. \tag{3}$$

In the case of player n, there are $(n-1)$ links, and z_n^t is given by

$$z_n^t = \sum_{i=1}^{n-1} \frac{w_n^t}{w_i^t + w_n^t} = \sum_{i=1}^{n-1} \frac{1}{(w_i^t/w_n^t) + 1}. \tag{4}$$

Theorem 6. *Suppose the players have initial wealth levels $w_1 > w_2 > \cdots > w_n$ and that*

$$z_{n-1} = \frac{w_{n-1}}{w_{n-1} + w_n} > \sum_{i=1}^{n-1} \frac{w_n}{w_i + w_n} = z_n. \tag{5}$$

Consider equilibrium ES_n. If $i > j > n$, then player i's wealth will grow at a faster rate than player j's wealth. Moreover, $\lim z_i^t = 1$ for $i = 1, \ldots, n-1$. Player n's wealth grows at a decreasing rate. Asymptotically, the wealth of $i = 1, \ldots, n-1$ grows at factor 2 while n's wealth is asymptotically constant.

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Proof. We first show that $w_1^t > w_2^t > \cdots > w_n^t$ and $z_{n-1}^t > z_n^t$ for every t. Proceed by induction. First suppose it is true for time t. Then for $i = 1, \ldots, n-2$,

$$z_i^t = \frac{1}{1 + (w_n^t/w_i^t)} \frac{1}{1 + (w_n^t/w_{i+1}^t)} = z_{i+1}^t.$$

Also,

$$z_{n-1}^t = \frac{w_{n-1}^t}{w_{n-1}^t + w_n^t} \sum_{i=1}^{n-1} \frac{w_n^t}{w_i^t + w_n^t} = z_n^t$$

which implies that wealthier agents grow faster. Thus $w_1^{t+1} > w_2^{t+1} > \cdots > w_n^{t+1}$. Moreover,

$$z_{n-1}^{t+1} = \frac{1}{1 + (w_n t + 1/w_{n-1}^{t+1})} \frac{1}{1 + (w_n^t/w_{n-1}^t)} = z_{n-1}^t$$

and

$$z_n^{t+1} = \sum_{i=1}^{n-1} \frac{w_n^{t+1}}{w_i^{t+1} + w_n^{t+1}} < \sum_{i=1}^{n-1} \frac{w_n^t}{w_i^t + w_n^t} = z_n^t.$$

Thus $z_{n-1}^{t+1} > z_{n-1}^t > z_n^t > z_n^{t+1}$. Since the induction hypothesis is true for $t = 1$, it is true for all t.

In fact, we have shown that growth for the richest $n-1$ agents occurs at an increasing rate while the poorest agent's wealth increases at a decreasing rate. It follows that $\lim_{t \to \infty} w_i^t/w_n^t = 0$. Now $\lim z_i^t = \lim 1/(1 + (w_n^t/w_i^t)) = 1$. Thus the growth factor for player i approaches 2.

Player n's growth rate shrinks with $\lim z_n^t = 0$. Player n's wealth at time t is $w_n^t = w_n \prod_{s=1}^{t-1}(1 + z_n^t)$. This will converge if $\sum_{s=1}^{\infty} \log(1 + z_n^t)$ converges. Since log is concave, that series is dominated by $\sum z_n^t$ Because of the exponential growth in w_i^t, $\sum z_n^t$ converges. This implies player n's wealth approaches a constant. $\square$

This result still leaves the question of long-run income distribution open. We know the bottom agent's share of income falls to zero (even though income itself is rising) since it converges to a constant while everyone else experiences exponential growth. The question is: Does the wealthiest agent end up with 100% of the income, or do the other growing agents keep some fraction?

In fact, the following theorem establishes that only the bottom agent's income share will tend to zero.

Theorem 7. *Suppose the players have wealth levels $w_1 > w_2 > \cdots > w_n$ and that equation (5) holds. Consider equilibrium ES_n. If $i, j \neq n$, then $\lim_{t \to \infty} w_i^t / w_j^t$ exists.*

Proof. Now $w_i^t = w_i \prod_{s=1}^{t-1} (1 + z_i^s)$, so

$$\frac{w_i^t}{w_j^t} = \frac{w_i}{w_j} \prod_{s=1}^{s=t-1} \frac{1 + z_i^s}{1 + z_j^s}.$$

Notice that each term of the product is greater than 1 because wealthier agents will see their wealth grow at a faster rate. Convergence of the product is equivalent to convergence of the sum

$$\sum_{s=1}^{\infty} \log \frac{1 + z_i^s}{1 + z_j^s}.$$

Since the logarithm is concave, $\log x \leq x - 1$. It is then enough to show that

$$\sum_{s=1}^{\infty} \left[\frac{1 + z_i^s}{1 + z_j^s} - 1 \right] = \sum_{s=1}^{\infty} \frac{z_i^s - z_j^s}{1 + z_j^s}$$

converges. Rewriting in terms of wealth levels, we obtain:

$$\sum_{s=1}^{\infty} \left[\frac{w_i^t(w_j^t + w_n^t) - w_j^t(w_i^t + w_n^t)}{(w_i^t + w_n^t)(w_j^t + w_n^t)} \middle/ \left(\frac{2w_j^t + w_n^t}{w_j^t + w_n^t} \right) \right] = \sum_{s=1}^{\infty} \frac{w_n^t(w_i^t - w_j^t)}{(w_i^t + w_n^t)(2w_j^t + w_n^t)}.$$

The terms of the last sum are dominated by $w_n^t/(2w_j^t + w_n^t)$. Since w_n^t converges to a constant and w_j^t grows asymptotically by a factor of 2, the sum converges, proving existence of the desired limit. $\qquad\square$

Thus income shares converge to non-zero constants for everyone except the poorest individual, whose share goes to zero.

If equation (5) fails to hold, it becomes difficult to characterize the long-run behavior of the economy. Consider the case where there are 3 players and the initial wealth distribution is $(6, 5, 4)$. In this case, it is possible to show that no player ever gets more than 40% nor less than 26% of the wealth, and that over time, players continue to shift positions in the income distribution. The income distribution does not converge to a limit.

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5 Proportional Solution

When players with wealth levels w_i^t, w_j^t are linked in the Proportional case, they get a gain of $w_i^t/2$, $w_j^t/2$, respectively. These gains do not depend on whom they link with, but only on the fact that there is a link. This means that there is no incentive to link in any particular direction and that there are many equilibria. We will consider two equilibria: circular links and random links.

In the circular link, player i links to $i+1$ for $i = 1, \ldots, n-1$ and player n links to 1, completing the link circle. Each player has exactly two links. The gain to player i is w_i, so in period 2 player i has $w_i^t = 2^{t-1} w_i$. Each player's wealth doubles in every period and each continues to hold the same share of total wealth.

The random case is somewhat more interesting. In this case, each player independently makes a random choice of which player to link with. There is an equal probability $(1/(n-1))$ of linking with any player.

Theorem 8. *In the random equilibrium, each player's wealth grows at an average rate of $2 - 1/(n-1)$.*

Proof. We will start by examining the gains received by one of the players, i. Let $p = 1/(n-1)$ and $q = 1 - p$. For each of the other players, there is a chance p of linking with i and chance q of not linking. The chance that i will have exactly j incoming links is $C_j^{n-1} p^j q^{n-1-j}$ where $C_j^k = k!/(k-j)!j!$ is the number of combinations of k things taken j at a time.

Now consider the outgoing link. There are two ways we can get j total links. There could be j incoming links and a redundant outgoing link. This happens with probability $j/(n-1)$ when there are j incoming links. Or there could be $j-1$ incoming links and a non-redundant outgoing link. This happens with probability $[(n-1)-(j-1)]/(n-1) = (n-j)/(n-1)$ when there are $j-1$ incoming links.

The probability of exactly j total links is

$$\frac{j}{n-1} C_j^{n-1} p^j q^{n-1} + \frac{n-j}{n-1} C_{j-1}^{n-1} p^{j-1} q^{n-j} = C_{j-1}^{n-2} p^{j-1} q^{n-j-1}$$

Let $\beta_j = C_{j-1}^{n-2} p^{j-1} q^{n-j-1}$ be the probability of exactly j links. Now

$$\sum_{j=0}^{n-2} \beta_{j+1} = \sum_{j=0}^{n-2} C_n^{n-2} p^j q^{n-2-j} = (p+q)^{n-2}.$$

Take the partial derivative with respect to p and set $q = 1 - p$ to obtain

$$\sum_{j=0}^{n-2} j C_n^{n-2} p^{j-1} q^{n-j-2} = n - 2.$$

Then the expected number of links is

$$\sum_{j=1}^{n-1} j\beta_j = \sum_{j=0}^{n-2} (j+1)\beta_{j+1} = \sum \beta_{j+1} + p \sum j\beta_{j+1} = 1 + \frac{n-2}{n-1} = 2 - \frac{1}{n-1}.$$

Slightly fewer links will be formed in the random case than in the circular case.

Now let ψ be the vector distribution of number of links and ψ^t its realization at time t. The wealth of agent i at time $t+1$ is then $w_i^t = w_i(1 + \psi_i^1/2) \cdots (1 + \psi_i^t/2)$. Now consider

$$\log w_i^{t+1} = \log w_i + \sum_{s=1}^{t} \log(1 + \psi_i^s/2).$$

Let $X_i^s = \log(1 + \psi_i^s/2)$. Then X is an *iid* vector random variable. The law of large numbers tells us that $\sum_{s=1}^{t} X_i^s/t$ converges to $2 - 1/(n-1)$. $\qquad\square$

On average, each player's wealth grows at the same rate. This does not, however, guarantee the income distribution remains unchanged over time. A run of bad luck could permanently lower the share going to some agents while a run of good luck could permanently raise it.

6   Dictatorship

The dictatorship of the rich is easiest to analyze. If $w > w'$, $DR(w, w') = (w, 0)$. Each agent that can, links to someone poorer. As far as payoffs are concerned, it doesn't matter who the richer agent links to as long as that agent is poorer. Every such link

leads to 100% growth. Only the poorest individual cannot profitably link. As a result, $w_i^t = 2^t w_i$ for $i = 1, \ldots, n - 1$ and $w_n^t = w_n$.

The dictatorship of the poor is a bit more complex. If $w > w'$, $DP(w, w') = (0, w')$. Each agent that can, links to someone richer. Again, the identity doesn't matter. The richest agent cannot profit from any links and retains the same wealth. The others get 100% growth.

To reduce technical complications, we initially presume that w_i/w_j is never a power of 2. Let t_1 be such that $2^{t_1-1} w_2 < w_1$ and $2^{t_1} w_2 > w_1$. At time t_1, agent 2 becomes the wealthiest individual. But then $w_2^{t_1+1} = w_2^{t_1}$. Since agent one is more than half as wealthy as agent 2 at time t_1, person 1 vaults over agent 2 in the next period.[7] Each time an agent becomes leader, his wealth is unchanged in the next period and someone else (with a wealth greater than $1/2$ the leader's) leapfrogs over him. Thus the group that exchange leadership see their wealth grow by a factor that averages less than 2.[8] Eventually, even the poorest individual will join the leader group as his wealth grows by factor 2 until he becomes leader. We sum this up as follows.

Theorem 9. *Suppose $DP(a, b) = (0, b)$ when $a > b$. If w_i/w_j is never a power of 2, wealth will asymptotically grow by a factor of $2^{(n-1)/n}$. Every agent will be wealthiest infinitely often, with the poorest agent having asymptotic wealth greater than $1/2$ that of the richest agent.*

When some w_i/w_j is a power of two, the possibility of a tie in wealth will slow down overall growth. For example, consider the initial distribution of $(4, 2, 1)$. In period 2, the distribution is $(4, 4, 2)$. Then $(6, 6, 4)$, followed by $(9, 9, 8)$, $(13.5, 13.5, 16)$, $(27, 27, 16)$, etc. The possibility of multiple ties adds additional complexity.

7 Concluding Remarks

To our knowledge, ours is the first paper that studies the evolution of wealth distribution of a society in the context of dynamic network formation. It highlights the role that

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[7]Note that agent 3 may become the wealthiest individual at this point.

[8]In fact, it averages $2^{(l-1)/l}$ where l is the current number in the leader group.

different division rules play in such dynamics. With some division rules, the society converges to a particular limit wealth distribution monotonically without changes in order, regardless of the initial wealth wealth distribution. With certain other division rules, the wealth order of individuals keeps changing constantly without converging to any particular limit distribution. Oddly, the egalitarian division rule results in a very un-egalitarian limit distribution of wealth, while a very un-equal division rule may lead to a much more egalitarian limit distribution of wealth (depending on the initial wealth distribution). Division rules that are arbitrarily close to the Proportional division rule, lead to distinctly different evolution paths of link-preference, evolution of wealth, and limit distributions. Remarkably, all of this complex dynamics occur within the confines of a simple triangular wealth possibilities set.

Many papers in the network formation literature paid close attention to the circumstances under which individual incentives lead to efficient networks. It became clear that the total value in a network does not only depend on who is connected but how these individuals are connected. In this paper too one can observe an important link between certain equilibrium network formations and (in)efficiency. Star network formations that arise in equilibrium in the contexts of both rich-linking and poor-linking division rules, fall short of efficiency (i.e., of reaching the total possible maximum value). Efficiency in those contexts is restored when almost star formations arise in equilibria (where multiple agents have multiple links).

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