Recursive Utility and the Ramsey Problem[*]

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This paper examines existence, continuity and characterization of optimal paths under "recursive" preferences. Current utility is a fixed (aggregator) function of current consumption and future utility. For suitable aggregators, a useful refinement of the Contraction Mapping Theorem generates the utility function, as in Lucas and Stokey. A broader class of aggregators is handled via a limiting argument analogous to partial summation. The Weierstrass theorem yields the existence of optimal paths. Under somewhat more stringent conditions on the aggregator and technology, optimal paths are continuous in initial capital stocks, and are characterized by generalized Euler equations and a transversality condition. *Journal of Economic Literature* Classification Numbers: 022, 111, 213.

1. Introduction

This paper examines the existence, continuity and characterization of optimal paths under time-stationary recursive preferences specified by an aggregator function. Given a suitable aggregator, a refinement of the Contraction Mapping Theorem generates the utility function. A limiting argument, analogous to partial summation, constructs an upper semicontinuous utility function for an even broader class of aggregators. The classical Weierstrass method then demonstrates the existence of optimal paths. Under somewhat more stringent conditions on the aggregator and technology, optimal paths are continuous in the initial capital stocks, and are characterized through generalized Euler equations and a transversality condition.

Recursive utility involves flexible time preference. In contrast, many dynamic economic models rely on additively separable utility functions. Unfortunately, a fixed rate of impatience can cause strange behavior in rather ordinary circumstances. A consumer facing a fixed interest rate will either try to save without limit, or borrow without limit, except in the knife-edge case where the rate of impatience equals the interest rate. This problem is especially severe when there are heterogeneous households. If all households face the same interest rate, and have different rates of impatience, they cannot all be in an interior steady state.

Recursive utility escapes this dilemma by allowing impatience to depend on income. These "recursive" or "time-stationary" preferences, introduced by Koopmans [33], formalize a

method used by Fisher [28]. Fisher employed a simple division between current consumption and future utility.[1] With recursive preferences, current utility is taken as a fixed function (the aggregator) of current consumption and future utility. Koopmans found axioms on preferences that would allow the utility function to be characterized by such an aggregator.

Lucas and Stokey [37] investigated the converse for bounded aggregators. But what about unbounded aggregators? I provide a converse for these. Given an aggregator, recursive substitution can be used to construct the utility function on an appropriate space of consumption paths. A new concept of impatience, β-myopia, helps build the link between the properties of the aggregator and the utility function. The parameter β combines information about both discounting and the asymptotic growth properties of the aggregator. This result applies to a wide variety of utility functions, including all those with constant elasticity of intertemporal substitution.

Taking the aggregator as fundamental provides detailed information about preferences in a compact form. First, it is a lot easier to specify an aggregator than a recursive utility function. Koopmans, Diamond and Williamson [34] found an aggregator that had a specific property (increasing marginal impatience), but the corresponding utility function has never been explicitly computed. It does not have a closed form expression. Second, the aggregator, with its sharp distinction between current and future consumption, often makes it easier to incorporate hypotheses about intertemporal behavior. It can be quite difficult to translate axioms into usable conditions on the utility function. The normality conditions used by Lucas and Stokey [37], Benhabib, Jafarey and Nishimura [5] and Benhabib, Majumdar and Nishimura [6] to study dynamics are imposed directly on the aggregator. Only recently has Epstein [25] discovered conditions on the utility function that imply a similar normality condition in models with continuous-time recursive utility. Finally, if we impose behavioral conditions as axioms, there is the question of their consistency. With aggregators, this is never a problem. Once the utility function exists, consistency is automatic.

Of course, the use of the aggregator does partially obscure the actual utility function and its properties. Fortunately, the aggregator usually contains all the information required to construct the utility function. I recover it via a contraction mapping, following Lucas and Stokey [37]. Their method only applies to bounded utility functions. I introduce a refinement of the Contraction Mapping Theorem, the Weighted Contraction Theorem, which applies to a much broader class of utility functions that includes many standard examples. Aggregators that allow $-\infty$ as a value require further treatment. The weighted contraction is combined with a "partial sum" technique to construct the utility functions.

Although weighted function techniques have long been utilized in the mathematics literature (e.g., Stein [47]), they are relatively new to dynamic programming (Lippman [35], Wessels [53], Waldman [52], Bhattacharya and Majumdar [10, 11]), and have only now found their way to economics. Instead of considering bounded functions, consider functions obeying a growth condition. This new function space has a natural norm induced by the growth condition. This introduces the problem of verifying that the map is a contraction. Denardo's [23] N-stage contraction argument has been used for this both by Lippman and

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[1] Fisher actually referred to "real income", but he also emphasized that income would ideally be measured in terms of utility.

by Bhattacharya and Majumdar. In this paper, as elsewhere in economics (and in Blackwell [12]), monotonicity properties are used to show that a map is a contraction. This new form of the weighted contraction has already proven useful in other contexts (Epstein and Zin [27]). It can also be used to lift the artificial restriction to bounded utility so common in stochastic dynamic models.[2]

Two additional results follow from the construction of the utility function. First, depending on its asymptotic marginal felicity, the aggregator can often be bounded by certain standard functions, which systematically determine the parameter β. Second, it enjoys useful continuity properties. As in Beals and Koopmans [3], Majumdar [39] and Lucas and Stokey [37], the combination of continuity or upper semicontinuity with compactness of the feasible set immediately yields existence. Further, Majumdar showed that convexity properties can then be used to find support prices via the Hahn-Banach theorem. This is quite different from the routes taken by Gale [30], Brock [14] and Sutherland [50]. In particular, value loss techniques are, at best, difficult to apply to recursive preferences. Fortunately, continuity can still be used. As a bonus, this route to existence is conceptually much simpler than the other approaches.

This method requires the choice of an appropriate commodity space. There are two conflicting tendencies here. When the space is small and the topology is strong, it is relatively easy to obtain a continuous utility function. The concept of β-myopia measures how far we must go. The space must also be large enough to contain the feasible set, and the topology weak enough to make it compact. In particular, the space must contain the path of pure accumulation.[3]

When the utility function is actually continuous, much more can be said. The Maximum Theorem yields a continuous maximizer correspondence. In addition, with a concave utility function and convex technology, optima are characterized by a combination of suitably generalized Euler equations and the transversality condition. The weighted norms constructed to show existence are utilized in a squeezing argument that demonstrates the necessity of the transversality condition. They also implicitly enter through the continuity property required for sufficiency of the transversality condition and the generalized Euler equations.

Of course, these techniques apply to the simplest recursive case, the additively separable utility function. As such, they bring many scattered results under one roof. They can deal with piecewise linear, time-varying (Brock and Gale [15]), strongly productive (Gale and Sutherland [31]), or nonclassical technologies (Dechert and Nishimura [21]) as well as undiscounted (or even upcounted) utility in a unified way.

Section Two introduces the aggregator. The Weighted Contraction Theorem is proved in Section Three. The utility function is constructed in Section Four, using the Weighted Contraction Theorem and a "partial sum" approach. Optimal paths are studied in the next

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[2] Examples where it may prove useful include Lucas [36], Prescott and Mehra [44] and Danthine and Donaldson [20]. A weakness of these (and other) papers is that explicitly solvable examples frequently involve unbounded utility, while the general theory only applies to bounded utility. Weighted contractions can remedy this.

[3] This choice of the commodity space is similar in motivation to the choice of both commodity and price spaces in Aliprantis, Brown and Burkinshaw [1]. In their terminology, this commodity space is the Riesz ideal generated by the path of pure accumulation.

two sections. Existence and continuity are examined in Section Five. Finally, the Euler equations and transversality condition characterize optimal paths in Section Six. Concluding comments are gathered in Section Seven.

2. The Aggregator Approach

The aggregator approach to recursive utility was initiated by Lucas and Stokey [37]. They showed how a recursive utility function could be constructed from an aggregator function W, under the assumption that W was bounded. The aggregator maps $\mathbf{X} \times \mathbf{Y}$ to $\mathbf{Y}$, where $\mathbf{X}$ is a subset of $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbf{Y}$ is a subset of $\mathbb{R}$.[4] The aggregator W must take values in $\mathbf{Y}$ so that the recursion map T_W can be defined below. The recursive utility function is the unique fixed point of T_W.

For the recursion map, we first define the projection π and shift S by $\pi X = x_1$ and $SX = (x_2, x_3, \dots)$ for $X \in \mathbb{R}^\infty$. The key property that makes a utility function U recursive is that $U(X) = W(\pi X, U(SX))$. Intuitively, we can find U by recursively substituting it in this equation. This substitution is performed by the recursion operator T_W defined by $(T_W U)(X) = W(\pi X, U(SX))$. Thus $(T_W^N 0)(X) = W(x_1, W(x_2, \dots, W(x_N, 0)) \cdots))$.

Without loss of generality, I may assume $0 \in \mathbf{Y}$. In fact, if $\mathbf{Y}$ is bounded below, I may even assume $W(0,0) = 0$. When $W(0,0) \neq 0$, a modest adjustment of the utility scale can remedy the situation. Use the adjusted aggregator, $V(x,y) = W(x, y + U(0)) - U(0)$. Both aggregators generate equivalent utility functions, and $V(0,0) = 0$. When applied to the Uzawa aggregator, this yields $U(0) = W(0, U(0)) = (-1 + U(0))e^{-u(0)}$, so $U(0) = 1/(1-v)$ where $v = e^{u(0)}$. The adjusted aggregator is then $V(x,y) = [y + v/(1-v)]e^{-u(x)} - 1/(1-v)$.

More formally, $W : \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ is an *aggregator* if:

Aggregator.
W1) W is continuous on $\mathbf{X} \times \mathbf{Y}$ and increasing in both x and y.
W2) W obeys a Lipschitz condition of order one, i.e., there exists $\delta > 0$ such that $|W(x,y) - W(x,y')| \leq \delta|y - y'|$ for all x in $\mathbf{X}$ and y, y' in $\mathbf{Y}$.[5]
W3) $(T_W^N y)(X)$ is concave in X for all N and all constants $y \in \mathbf{Y}$.

There are several important points about (W2). First, the Lipschitz bound δ is independent of x and y. Second, $\delta < 1$ is not required. This permits undiscounted or even upcounted models. Third, a variable δ could be handled, but would require much finer control over both the program space and the aggregator to give significant additional information. The uniform bound is chosen to make life simpler, and maintains a balance between power and ease of use.

The sole purpose of condition (W3) is to guarantee concavity of the utility function. It is not required for the existence results. Curiously, the aggregator need not be jointly concave in x and y for the associated utility function to be concave. Although the Uzawa aggregator

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[4] I use the notational convention that lowercase denotes real numbers and uppercase denotes vectors in $\mathbb{R}^\infty$.

[5] This property (uniformly bounded time perspective) is similar to those studied axiomatically by Koopmans [33] and Koopmans, Diamond and Williamson [34].

is not concave, the corresponding utility function $U(X) = -\sum_{t=1}^{\infty} \exp[-\sum_{\tau=1}^{t} u(x_\tau)]$ is concave. Epstein previously [24] gave sufficient conditions for the concavity of generalized Uzawa utility functions. When the utility function is the limit of the functions $T_W^N(0)(X)$, (W3) insures concavity is inherited by U. Thus,

LEMMA 1. *Suppose (W3) holds and $T_W^N(0)(X) \to U(X)$. Then U is concave on its domain. If, in addition, W is strictly concave in x and strictly increasing in y, then U is strictly concave. Conversely, if U is concave, condition (W3) holds for all y in the range of U.*

3. THE WEIGHTED CONTRACTION THEOREM

The Contraction Mapping Theorem is a useful technique for proving existence theorems. The commonly used forms of this theorem require bounded utility functions. This rules out many of the usual utility functions, including those with constant elasticity of marginal utility. Even logarithmic utility is excluded. This is a severe restriction. Fortunately, a modification of the contraction technique can be used even when the utility functions are not bounded. This is the weighted contraction method.

Let $f \in \mathcal{C}(\mathbf{A}; \mathbf{B})$, the space of continuous functions from $\mathbf{A}$ to $\mathbf{B}$. Suppose $\varphi \in \mathcal{C}(\mathbf{A}; \mathbf{B})$ with $\mathbf{B} \subset \mathbb{R}$ and $\varphi > 0$. The function f is φ-*bounded* if the φ-*norm* of f, $\|f\|_\varphi = \sup\{|f(x)|/\varphi(x)\}$ is finite.[6] The φ-norm turns $\mathcal{C}_\varphi(\mathbf{A}; \mathbf{B}) = \{f \in \mathcal{C}(\mathbf{A}; \mathbf{B}) : f$ is φ-bounded$\}$ into a Banach space.[7] In particular, $\mathcal{C}_\varphi(\mathbf{A}; \mathbf{B})$ is a complete metric space. A transformation $T: \mathcal{C}_\varphi \to \mathcal{C}_\varphi$ is a *strict contraction* if $\|Tx - Ty\|_\varphi \leq \theta\|x - y\|_\varphi$ with $\theta < 1$. For such T, we have:

CONTRACTION MAPPING THEOREM. *A strict contraction on a complete metric space has a unique fixed point.*

The proof is well-known, and can be found in various standard references (e.g., Reed and Simon [45], Smart [46]).

In applications, the main problem is to show that T is a strict contraction. An easy way to do this is by using monotonicity properties, as is common in dynamic programming. In the weighted contraction context, this yields the following form of the theorem.

WEIGHTED CONTRACTION MAPPING THEOREM (MONOTONE FORM). *Let $T: \mathcal{C}_\varphi \to \mathcal{C}$ such that*

(1) *T is non-decreasing ($\xi \leq \psi$ implies $T\xi \leq T\psi$).*

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[6] This type of weighting must be distinguished from another type of weighting previously used in economics. The second type of weighting deals with the underlying commodity space rather than functions on it. Examples include Chichilnisky [17, 18] and Magill [38] for continuous-time models and Chichilnisky and Kalman [19] and Dechert and Nishimura [21] for discrete-time models.

[7] Define V by $Vf = \varphi f$. Since φ is continuous, Vf is a continuous function whenever f is continuous. Further, V is an isometric isomorphism from the Banach space of bounded continuous functions to $\mathcal{C}_\varphi(\mathbf{A}; \mathbf{B})$. Hence $\mathcal{C}_\varphi(\mathbf{A}; \mathbf{B})$ is also a Banach space.

(2) $T(0) \in \mathcal{C}_\varphi$.
(3) $T(\xi + A\varphi) \leq T\xi + A\theta\varphi$ for some constant $\theta < 1$ and all $A > 0$.
Then T has a unique fixed point.

Proof. For all $\xi, \psi \in \mathcal{C}_\varphi$, $|\xi - \psi| \leq \|\xi - \psi\|_\varphi \varphi$. So, $\xi \leq \psi + \|\xi - \psi\|_\varphi \varphi$ and $\psi \leq \xi + \|\xi - \psi\|_\varphi \varphi$. Properties (1) and (3) yield $T\xi \leq T\psi + \theta\|\xi - \psi\|_\varphi \varphi$ and $T\psi \leq T\xi + \theta\|\xi - \psi\|_\varphi \varphi$. Thus $\|T\xi - T\psi\|_\varphi \leq \theta\|\xi - \psi\|_\varphi$.

Setting $\psi = 0$, we have $\|T\xi - T(0)\|_\varphi \leq \theta\|\xi\|_\varphi$, and so $\|T\xi\|_\varphi \leq \theta\|\xi\|_\varphi + \|T(0)\|_\varphi < \infty$ by property (2). Hence $T : \mathcal{C}_\varphi \to \mathcal{C}_\varphi$. As $\theta < 1$, T is a strict contraction on $\mathcal{C}_\varphi$. By the contraction mapping theorem, it has a unique fixed point. ∎

This form of the Contraction Mapping Theorem is particularly adapted for use on dynamic programming problems, including economic models with a recursive structure. In particular, it can be used to establish the existence and continuity of recursive utility functions. It can also be used on stochastic programming problems such as those in Lucas' asset pricing model [36]. Other variants are possible. By using the techniques of Bhakta and Mitra [9], the requirement that $\theta < 1$ can be relaxed. Further, a version of the Local Contraction Mapping Theorem can also be obtained.

4. THE EXISTENCE OF RECURSIVE UTILITY

The recursive utility function will generally be defined on a space smaller than $\mathbb{R}^\infty$. Define $\mathbf{X}(\beta) = \{X \in \mathbb{R}_+^\infty : |X|_\beta < \infty\}$ where $|X|_\beta = \sup |x_t/\beta^{t-1}|$ is the β-weighted ℓ^∞ norm. Here $\beta \geq 1$. The utility function will be a function on $\mathbf{X}(\beta)$ that is continuous in the topology generated by the β-norm (β-topology). These norms can be thought of as having the discount factor $1/\beta$ built in. Topologies of this type have been used by Chichilnisky and Kalman [19] and Dechert and Nishimura [21] to study optimal paths.

One notion of impatience is the concept of myopia introduced by Brown and Lewis [16]. Their basic idea was to use the continuity properties of the utility function to measure impatience. This notion was founded on the observation that Mackey continuity presupposes a certain degree of impatience. This idea may be further developed by considering continuity with respect to a variety of different topologies. In particular, we may use the β-topology. We call a utility function $\beta - myopic$ if it is defined and continuous on $\mathbf{X}(\beta)$.

Let $\mathbf{A} \subset \mathbb{R}^\infty$ with $\pi(\cup_{N=0}^\infty S^N \mathbf{A}) \subset \mathbf{X}$. Both the shift S and projection π are continuous in any topology on $\mathbf{A}$ that is stronger than the relative product topology, as are the β-topologies. Given a positive function φ, continuous on $\mathbf{A}$, let $\mathcal{C} = \mathcal{C}(\mathbf{A}; \mathbf{Y})$ and $\mathcal{C}_\varphi = \mathcal{C}_\varphi(\mathbf{A}; \mathbf{Y})$. Since all the functions involved are continuous, $T_W : \mathcal{C}_\varphi \to \mathcal{C}$.

CONTINUOUS EXISTENCE THEOREM. *Suppose the topology on $\mathbf{A}$ is stronger than the relative product topology, $W : \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ obeys (W1) and (W2), φ is continuous, $W(\pi X, 0)$ is φ-bounded, and $\delta\|\varphi \circ S\|_\varphi < 1$. Then there exists a unique $U \in \mathcal{C}_\varphi$ such that $W(\pi X, U(SX)) = U(X)$. Moreover, $(T_W^N 0)(X) \to U(X)$ in $\mathcal{C}_\varphi$.*

Proof. Since W is increasing in y, the recursion operator T_W is increasing. Now $|T_W(0)|/\varphi(X) = |W(x_1, 0)|/\varphi(X) < \infty$ because $W(\pi X, 0)$ is φ-bounded. Finally, $T_W(\xi + A\varphi) = W(x_1, \xi(SX) +$

$A\varphi(SX)) \leq W(x_1, \xi(SX)) + A\delta\varphi(SX) \leq T_W\xi + A\delta\|\varphi \circ S\|_\varphi\varphi(X)$. The Weighted Contraction Theorem, with $\theta = \delta\|\varphi \circ S\|_\varphi < 1$, shows that T_W is a contraction, and has a unique fixed point U.

Now consider $\|U(X) - (T_W^N 0)(X)\|_\varphi \leq \delta^N\|U(S^N X)\|_\varphi \leq \|U\|_\varphi(\delta\|\varphi \circ S\|_\varphi)^N$. As the last term converges to zero, $(T_W^N 0)(X) \rightarrow U(X)$. ∎

In fact, the full force of (W1) was not employed in the proof. The aggregator need not be increasing in x for the theorem to hold.

4.1. *Examples with Aggregator Bounded Below*

The easiest application of the Continuous Existence Theorem is to a bounded aggregator with $\delta < 1$ and $\mathbf{A} = \mathbf{X}(1)$. Take φ as the constant 1, and use the product topology. As in Lucas and Stokey [37], this yields a recursive utility function that is not only β-myopic for all $\beta \geq 1$, but also continuous in the relative product topology on $\mathbf{X}(1)$. In particular, this applies to the Uzawa aggregator with $u(0) > 0$.

Another application is to W with $0 \leq W(x, 0) \leq A(1 + x^\eta)$ as in the case where $W(x, 0)$ has asymptotic exponent or asymptotic elasticity of marginal felicity (see Brock and Gale [15]) less than $\eta > 0$ with $\delta\beta^\eta < 1$. In this case, take $\mathbf{A} = \mathbf{X}(\beta)$ and $\varphi(X) = 1 + |X|_\beta^\eta$. Then $\|\varphi \circ S\|_\varphi = \beta^\eta$, and the recursive utility function is β-myopic. When $0 \leq W(x, 0) \leq A(1 + \log(1 + x))$, a similar argument shows that U is β-myopic for all $\beta < \infty$.[8] In fact, when W is concave in x, $W(x, 0) \leq W(1, 0) + \alpha(x - 1)$ for some supergradient α. (If differentiable, $\alpha = W_1(1, 0)$.) Thus we may set $\varphi(X) = 1 + |X|_\beta$ for a β-myopic utility function when the aggregator is concave in x with $\delta\beta < 1$.

Relaxing the condition $\beta^\eta\delta < 1$ risks losing existence on $\mathbf{X}(\beta)$. Let $W(x, y) = x^\eta + \delta y$ and take $\beta = \delta^{-1/\eta}$. The utility function cannot be defined when X is given by $x_t = \beta^t$. No utility function can be constructed from the aggregator on $\mathbf{X}(\beta)$. A smaller space must be used.[9]

4.2. *Unbounded Aggregators*

The Continuous Existence Theorem can also be used indirectly to deal with aggregators that are not bounded below, such as $W(x, y) = \log x + \delta y$. These obey:

W1'). *W is increasing in both x and y, upper semicontinuous on $\mathbf{X} \times \mathbf{Y}$, continuous when $x > 0$ and $y > -\infty$ and obeys $W(x, -\infty) = W(0, y) = -\infty$ for all $x \in \mathbf{X}$ and $y \in \mathbf{Y}$.*

For aggregators satisfying (W1'), paths that are near 0 can pose problems. When $W(x, y) = \log x + \delta y$ these problems result in a utility function that is upper semicontinuous but not lower semicontinuous, but are not severe enough to preclude existence of the utility function. To circumvent the problems posed by paths that are too close to zero, I consider a region that excludes them as the set $\mathbf{A}$. More precisely, choose $\gamma \leq \beta < \infty$, and set $_\gamma|X| = \inf |xt/\gamma^{t-1}|$ if $0 < \gamma$ and $_0|X| = \infty$. Then take $\mathbf{A} = \mathbf{X}(\beta, \gamma) = \{X \in \mathbb{R}_+^\infty : 0 <_\gamma |X| \text{ and } |X|_\beta < \infty\}$. This is the set of paths that have a growth rate between γ and β. Thus $\mathbf{X}(\beta, 0)$ is just our old friend $\mathbf{X}(\beta)$.

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[8] Take $c > 0$ such take $\delta(c + \log\beta)/c < 1$ and set $\varphi(x) = c + \log(1 + x)$.

[9] However, Streufert [49] has recently discovered cases where $\beta^\eta\delta > 1$ and U exists. These seem to require $W_{22} < 0$.

These aggregators permit $U(X) = -\infty$ as a solution. However, by restricting our attention to $\mathbf{X}(\beta, \gamma)$, we see that this is not a reasonable solution. Intuitively, we expect to obtain the utility function by recursive substitution, as the limit of $(T_W^N u)(X) = W(x_1, W(x_2, \ldots, W(x_N, u) \ldots))$ with u constant. In fact, under appropriate conditions, the Continuous Existence Theorem applies on $\mathbf{X}(\beta, \gamma)$, yielding a unique φ-bounded utility function Ψ. Of course, the iterates $T_W^N u$ converge to Ψ on $\mathbf{X}(\beta, \gamma)$.

By using a process analogous to partial summation, Ψ can be extended to a utility function on all of $\mathbf{X}(\beta)$. This extension is upper semicontinuous and recursive. Further, it is the *only* recursive upper semicontinuous extension of Ψ to $\mathbf{X}(\beta)$.

UPPER SEMICONTINUOUS EXISTENCE THEOREM. *Suppose $W : \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ obeys (W1'), the Lipschitz condition (W2) holds whenever W is finite, and there are increasing functions g and h with $g(x) \leq W(x, 0) \leq h(x)$. Set $\varphi(X) = \max\{h(|X|_\beta, -g(_\gamma|X|)\}$ and suppose $\varphi > 0$ with $\delta\|\varphi \circ S\|_\varphi < 1$ for some $\beta > \gamma > 0$ with $\beta \geq 1$. Then there exists a unique U that is φ-bounded on $\mathbf{X}(\beta, \gamma)$, obeys $W(\pi X, U(SX)) = U(X)$ and is β-upper semicontinuous on $\mathbf{X}(\beta)$.*

Proof. First, temporarily give $\mathbf{A} = \mathbf{X}(\beta, \gamma)$ the discrete topology. As all functions are continuous there, and $W(x, 0)$ is clearly φ-bounded, the Continuous Existence Theorem applies, yielding a unique φ-bounded recursive utility function $\Psi : \mathbf{X}(\beta, \gamma) \to \mathbb{R}$.

Second, let Z be an arbitrary element of $\mathbf{X}(\beta, \gamma)$ and define the "partial sums" on all of $\mathbf{X}(\beta)$ by

$$\Psi_N(X|Z) = [T_W^N \Psi(S^N Z)](X) = W(x_1, W(x_2, \ldots, W(x_N, \Psi(S^N Z)) \ldots)).$$

Now for $Z, Z' \in \mathbf{X}(\beta)$,

$$\begin{aligned}
|\Psi_N(X|Z) - \Psi_N(X|Z')| &\leq \delta^N |\Psi(S^N Z) - \Psi(S^N Z')| \\
&\leq \delta^N M[\varphi(S^N Z) + \varphi(S^N Z')] \\
&\leq M'(\delta\|\varphi \circ S\|_\varphi)^N
\end{aligned}$$

for some M'. The first step uses the Lipschitz bound (W2). The second uses the φ-boundedness of Ψ on $\mathbf{X}(\beta, \gamma)$, and the third uses the fact that $\varphi(S^N Z) \leq (\|\varphi \circ S\|_\varphi)^N \varphi(Z)$ for any $Z \in \mathbf{X}(\beta, \gamma)$. It follows that if $\lim_{N\to\infty} \Psi_N(X|Z)$ exists, it must be independent of Z. Note that for $X \in \mathbf{X}(\beta, \gamma)$, $\Psi_N(X|X) = \Psi(X)$, so $\lim_{N\to\infty} \Psi_N(X|Z)$ exists on $\mathbf{X}(\beta, \gamma)$ and is equal to Ψ there.

The third step is to show $U(X) = \lim_{N\to\infty} \Psi_N(X|Z)$ exists and is β-upper semicontinuous on all of $\mathbf{X}(\beta)$. For κ arbitrary, take $X \in \mathbf{X}(\beta)$ with $|X|_\beta \leq \kappa$ and set $z_t = \kappa\beta^{t-1}$. Since $x_t \leq z_t$, $\Psi_N(X|Z)$ is a decreasing sequence. Its limit $U(X)$, which is also its infimum, must exist. Further, each of the Ψ_N is the composition of non-decreasing β-upper semicontinuous functions, so their infimum $U(X)$ is also β-upper semicontinuous on $\{X : |X|_\beta \leq \kappa\}$. Since upper semicontinuity is a local property, U is β-upper semicontinuous on all of $\mathbf{X}(\beta)$.

The next step is to show that U is recursive. If $\pi X = 0$ or if $U(SX) = -\infty$, (W1') implies $W(\pi X, U(SX)) = -\infty = U(X)$. Otherwise, we have $W(\pi X, U(SX)) = W(\pi X, \lim_{N\to\infty}\Psi_N(SX|SZ)) = \lim_{N\to\infty} W(\pi X, \Psi_N(SX|SZ)) = \lim_{N\to\infty} \Psi_{N+1}(X|Z) = U(X)$. Therefore $W(\pi X, U(SX)) = U(X)$ for all $X \in \mathbf{X}(\beta)$.

The last step is uniqueness. Let Φ be a β-upper semicontinuous recursive utility function that is φ-bounded on $\mathbf{X}(\beta, \gamma)$. Since Ψ is unique, Φ, Ψ and U agree on $\mathbf{X}(\beta, \gamma)$. When $z_t = |X|_\beta \beta^{t-1}$, $X \leq Z$ and so $\Phi(X) \leq \Psi_N(X|Z)$. Thus $\Phi(X) \leq \lim_{N \to \infty} \Psi_N(X|Z) = U(X)$. If $x_t = 0$ for some t, $U(X) = -\infty = \Phi(X)$. If $x_t > 0$ for all t, set $z_t = \gamma^{t-1}$ and consider the sequence $X^n = (x_1, \ldots, x_n, z_{n+1}, z_{n+2}, \ldots)$. By construction, $\Phi(X^n) = \Psi_n(X|Z)$. Since $\gamma < \beta$, $X^n \to X$ in the β-topology. By upper semicontinuity of Φ, $\Phi(X) \geq \lim_{n \to \infty} \Psi_n(X|Z) = U(X)$. It follows that $\Phi(X) = U(X)$, and thus U is the unique such function. $\blacksquare$

Aggregators with $-1 + \min\{0, \log x\} \leq W(x, 0) \leq c + \log(1 + x)$ fall into this framework. Given $\delta < 1$ and $\beta \geq 1$, the constant c may be assumed large enough that $\delta(c + \log \beta)/c < 1$. Take $\gamma = 1$ and let $\varphi(X) = \max\{c + \log(1 + |X|\beta), 1 - \min\{0, \log 1|X|\}\}$. As $\delta\varphi(SX)/\varphi(X) \leq \delta(c + \log \beta)/c < 1$ since $_1|X| \leq_1 |SX|$ and $|SX|_\beta \leq \beta|X|_\beta$, the utility function exists on $\mathbf{X}(\beta, 1)$ for any β. In other cases, upcounting ($\delta > 1$) may be allowed. When $-x^\eta \leq W(x, 0) \leq 0$ with $\eta < 0$, we set $\varphi(X) =_\gamma |X|^\eta$ so $\delta\varphi(SX)/\varphi(X) \leq \delta\beta^\eta \leq \delta\gamma^\eta < 1$. As $\eta < 0$, $\beta^\eta < 1$ and there are γ that permit $\delta > 1$. The Upper Semicontinuous Existence Theorem applies to these examples.

The "partial sum" approach works on a wider range of aggregators than considered in the theorem. For example, if there is a function $v(x)$ with $v(x) = W(x, v(x))$, "partial sums" can be defined on $\{X : |X|_1 \leq \kappa\}$ by $[T_W^N v(\kappa)](X)$. These form a decreasing sequence, so their limit is an upper semicontinuous function $U(X)$. As $W(x_1, U(SX)) = \lim[T_W^{N+1} v(\kappa)](X) = U(X)$, this yields a recursive utility function. This recursive utility function may fail to be lower semicontinuous. One such example is $W(x, y) = -1 + e^{-x} y$ so that $v(x) = -1/(1 - e^{-x})$ and utility is $U(X) = -\sum_{t=1}^\infty \exp(-\sum_{\tau=0}^t x_\tau)$. Consideration of the sequence $X^n = (x_1, \ldots, x_n, 0, \ldots)$, where $x_t = 2\log(t + 1)/t$, shows that this utility function is not lower semicontinuous since $U(X^n) = -\infty$ but $U(X) > -\infty$. Note that $\delta = 1$ in this example.

5. EXISTENCE AND SENSITIVITY OF OPTIMAL PATHS

The existence of optimal paths is just one of the useful facts that follow from continuity of the utility function and compactness of the feasible set. When the aggregator defines a continuous utility function, a modern version of Weierstrass' theorem, the Maximum Theorem (see Berge [7] or Klein and Thompson [32]), can be used to show continuity of optimal paths. For example, when the budget set (and hence the optimal path) depends continuously on a parameter vector ω, the maximizer correspondence $\mathbf{m}(\omega)$ will be continuous.

MAXIMUM THEOREM. *Suppose* $\mathbf{F}(\omega)$ *is* $\beta-$*lower semicontinuous in* ω *and* $\beta-$*compact-valued.*

(1) *If* U *is* β-*upper semicontinuous, there exists a* $C^* \in \mathbf{F}(\omega)$ *with* $U(C^*) = \sup\{U(C) : C \in \mathbf{F}(\omega)\}$.
(2) *If* U *is* β-*continuous, the value function* $J(\omega) = \sup U(\mathbf{F}(\omega))$ *is continuous and the maximizer correspondence* $\mathbf{m}(\omega)$ *is upper semicontinuous. Further, if* U *is strictly concave, then* $\mathbf{m}(\omega)$ *is a continuous function of* ω.

In many cases, the β-topology coincides with the product topology. The product topology is generally easier to work with. The following lemma provides the connection.

LEMMA 2. *Let $\alpha < \beta$ and let $\mathbf{F}$ be an α-bounded set. The β-topology and the relative product topology coincide on $\mathbf{F}$.*

Proof. Let k be an α-bound for $\mathbf{F}$. Clearly the β-topology is stronger than the product topology on $\mathbf{F}$. Suppose $X^n \to X$ in the product topology. Given $\epsilon > 0$, choose M such that $k(\alpha/\beta)^M < \epsilon$. As $X^n \to X$, there is an N with $\sup\{|x_t^n - x_t|/\beta^{t-1} : t \leq M\} < \epsilon$ for $n > N$. But then $|X^n - X|_\beta < 2\epsilon$ for $n > N$, and $X^n \to X$ in the β-topology. The two topologies are identical. ∎

It is crucial that $\beta > \alpha$. Majumdar [39] gives an example illustrating why norm-bounded feasible sets are not compact in the norm topology. The same sort of problem would occur here if $\beta = \alpha$. In fact, $\beta = \alpha = 1$ is precisely Majumdar's case.

One application is to a one-sector model of optimal capital accumulation (*Ramsey model*). In the classical Ramsey model, the technology is described by a (gross) *production function*. The production function f is a continuous, non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$. Note that $f(0) \geq 0$. In the time-varying Ramsey model, the technology is described by a sequence, $\{f_t\}$, of such production functions. Given this production technology, the set of feasible paths of accumulation from initial stock k (the *production correspondence*) is $\mathbf{P}(k) = \{X \in \mathbf{X}(\infty) : 0 \leq x_t \leq f_t(x_{t-1}), x_0 = k\}$. The set of feasible paths, $\mathbf{F}(k)$ is $\{C \in \mathbf{X}(\infty) : 0 \leq c_t \leq f_t(x_{t-1}) - x_t$ for some $X \in \mathbf{P}(k)\}$. Define f^t inductively by $f^1 = f_1$ and $f^t = f_t \circ f^{t-1}$. The *path of pure accumulation* is $\{f^t(k)\}_{t=1}^\infty$. Both $\mathbf{F}(k)$ and $\mathbf{P}(k)$ are closed in the product topology and $\mathbf{F}(k) \subset \mathbf{P}(k) \subset \mathbf{X}_{t=1}^\infty [0, f^t(k)]$. As this last set is compact by Tychonoff's Theorem, $\mathbf{F}(k)$ is also compact in the product topology.

When $\lim[f^t(k)/\alpha^t] < \infty$, both $\mathbf{P}(k)$ and $\mathbf{F}(k)$ are α-bounded subsets of $\mathbf{X}(\beta)$. More generally, we call the technology α-bounded if $\mathbf{P}(k)$ is α-bounded. This happens in the case of exogeneous technical progress where $f_t(x) = e^{nt}x^\rho$. The path of pure accumulation grows at asymptotic rate $\exp\{n/(1-\rho)\}$, so the technology is α-bounded for $\alpha > \exp\{n/(1-\rho)\}$. As any concave production function obeys $f(x) \leq f(a) + \xi(x-a)$ whenever ξ is a supergradient at a (e.g. $\xi = f'(a)$), it is α-bounded for any $\alpha > \xi$. Thus, any stationary, concave, production technology is α-bounded for all $\alpha > f'(\infty)$. Provided U is upper semicontinuous on $\mathbf{X}(\beta)$ for some $\beta > \alpha$, Lemma 2 and the Maximum Theorem combine to show existence of at least one optimal path.

To obtain continuity of the value function, it is enough to show that the production correspondence $\mathbf{P}(k)$ is product lower semicontinuous since the set of feasible paths is the continuous image of the production correspondence. For k' near k, $\mathbf{P}(k') \subset \mathbf{P}(k+1)$. Locally, everything takes place in an α-bounded set, and we may use the product topology.

For lower semicontinuity, it is enough to show lower semicontinuity for the subbasic open sets $\mathbf{G}(Y, \epsilon, N) = \{X \in \mathbb{R}_+^\infty : |x_t - y_t| < \epsilon$ for all $t < N\}$. Let $\epsilon, N > 0$ be given. Take $Y \in \mathbf{P}(k)$. By continuity of the f_t, we can choose δ with $|f^t(k') - f^t(k)| < \epsilon$ for all $t \leq N$ when $|k - k'| < \delta$. For any such k', take the path $x_t = \min\{y_t, f^t(k')\}$. Note that $f^t(k') + \epsilon > f^t(k) \geq y_t$ for $t \leq N$, so $y_t \geq x_t > y_t - \epsilon$ for all $t \leq N$. Hence $X \in \mathbf{G}(Y, \epsilon, N)$. Further, $f_{t+1}(x_t) = \min\{f_{t+1}(y_t), f^{t+1}(k')\} \geq x_{t+1}$ and $x_1 \leq f_1(k')$, so $X \in \mathbf{P}(k')$. It follows $\mathbf{P}(k') \cap \mathbf{G}(Y, \epsilon, N) \neq \varnothing$ whenever $|k - k'| < \delta$, establishing lower semicontinuity.

An immediate application is to demonstrate β-continuity of optimal paths as a function of initial capital stock. One consequence is that $c_t(k)$ is continuous in k for each t. In

general, this only holds for $\beta > \alpha$. For $\beta = \alpha$, it can fail even in models with additively separable utility. Amir, Mirman and Perkins [2] and Dechert and Nishimura [22], using a non-convex stationary technology, find that optimal paths converge to zero if the initial capital stock is below some critical value. The critical value is itself a steady state, and the only optimal path from the critical value is to remain there. Above the critical value, optimal paths converge to a steady state that lies above the critical value. They assume a maximum sustainable stock, so $\alpha = 1$ will do. The optimal path is clearly not norm ($\alpha = 1$) continuous.

Variations on this are possible. Stronger forms of the maximum theorem allow the utility function to depend on the parameter ω. If the bounds of Section Four hold uniformly in ω, the optimal paths will be continuous in ω.[10] A simple example is an optimal growth model with additively separable utility $W(x, y) = u(x) + \delta y$. Take $(k, \delta) = \omega \in \Omega = \mathbb{R}_+ \times [0, \delta]$ with a strictly concave, bounded u and $\delta < 1$. With a stationary concave production function f, a unique optimal path $\{c_t(k, \delta)\}$ exists. Further, $\{c_t(k, \delta)\}$ is β-continuous, hence $c_t(k, \delta)$ is a continuous function of (k, δ) for all $(k, \delta) \in \Omega$.

When the turnpike property holds, β-continuity of optimal paths will imply α-continuity. In fact, if optimal paths starting in some interval of initial stocks converge to the same steady state, α-continuity follows on that interval. Beals and Koopmans [3] have demonstrated how these properties follow from monotonicity of optimal paths, and examined conditions where a convex technology would yield monotonic optimal paths. A necessary and sufficient condition for monotonicity is not known in general. However, in the additively separable case, Dechert and Nishimura [22] carried out an analysis of monotonicity in a reduced form model. In the general aggregator case, the analogous reduced form model has yet to be constructed.

6. Characterization of Optimal Paths

Optimal paths for the Ramsey model are characterized in this section. A useful envelope theorem and the generalized Euler equations are developed first. I then proceed to the main result that the generalized Euler equations, together with the transversality condition, completely characterize optimal paths for a large class of aggregators.

The following assumptions will be maintained throughout this section. The utility function U obeys $U(0) = 0$ and is concave and φ-bounded on $\mathbf{X}(\beta)$ for some φ with $\|\varphi \circ S\|_\varphi < 1/\delta$. In addition, the feasible set $\mathbf{F}$ is generated by an α-bounded technology for some $\alpha < \beta$ given by a sequence continuous, concave, increasing production functions $\{f_t\}$ with $f_t(0) = 0$. As a consequence, the theorems of Sections Five and Six apply. The value function $J(y)$ is defined and continuous in initial income $y = f_1(k)$. When U is differentiable with respect to consumption at time t, denote $\partial U/\partial c_t$ by U_t. Except as noted, assume U is differentiable at each time.

ENVELOPE THEOREM. *The value function J is increasing and concave. If U is differentiable with respect to consumption in period 1, and consumption in period 1 is non-zero,*

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[10] Details may be found in Boyd [13].

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then J is differentiable and obeys $dJ(y)/dy = U_1(C)$ where C is any optimal path from y.

Proof. The first two properties follow from the usual arguments. Differentiability is established as follows.[11] Let $h > 0$, $H = (h, 0, \dots)$, and let C be an optimal path with initial income y so that $J(y) = U(C)$. Clearly, $J(y + h) \geq U(C + H)$ and thus $J(y + h) - J(y) \geq U(C + H) - U(C)$. Dividing by h and taking the limit shows that the right-hand derivative $J'(y+)$ satisfies $J'(y+) \geq U_1(C)$. When c_1 is non-zero, repeating this with $h < 0$, $|h| < c_1$ shows $J'(y-) \leq U_1(C) \leq J'(y+)$. As J is concave, $J'(y+) \leq J'(y-)$, thus $J'(y) = U_1(C)$. ∎

COROLLARY 1. *If U is recursive, and the aggregator is differentiable, then $dJ(y)/dy = W_1(c_1, U(SC))$ where C is any optimal path from y with c_1 non-zero.*

Henceforth, assume that U is differentiable at each time t. Now let C^* be optimal and let K^* be the associated sequence of capital stocks. Set $\mathbf{F}_N = \{K \in \mathbf{F} : k_N \geq k_N^*\}$. Let $V_N(K) = U(f_1(k_0) - k_1, \dots, f_N(k_{N-1}) - k_N, f_{N+1}(k_N) - k_{N+1}^*, f_{N+2}(k_{N+1}^*) - k_{N+2}^*, \dots)$. By the Principle of Optimality, K^* solves the problem of maximizing V_N over $\mathbf{F}_N$. The necessary conditions can be written using the Gâteaux derivative (see Fleming and Rishel [29]) as

$$\sum_{t=1}^{N} (\partial V_N/\partial k_t)(K^*)[k_t - k_t^*] \leq 0$$

for all $K \in \mathbf{F}_N$. Since $\partial V_N/\partial k_t(K^*) = \partial U/\partial k_t(C^*) = U_{t+1}(C^*)f'_{t+1}(k_t^*) - U_t(C^*)$ for $t = 1, \dots, N$, the necessary conditions are

$$\sum_{t=1}^{N} [U_{t+1}(C^*)f'_{t+1}(k_t^*) - U_t(C^*)][k_t - k_t^*] \leq 0 \tag{1}$$

for all $K \in \mathbf{F}_N$. When $0 < k_t^* < f_t(k_{t-1}^*)$ for all t, these are equivalent to the ordinary Euler Equations[12]

$$U_{t+1}(C^*)f'_{t+1}(k_t^*) - U_t(C^*) = 0.$$

Alternatively, the Kuhn-Tucker theorem may be applied to derive first-order conditions. Since f_t is increasing and $k_0 > 0$, $\mathbf{F}_N$ obeys Slater's condition (unless K^* is the path of pure accumulation up to N). The Lagrangian is $\mathcal{L} = V_N + \sum_{t=1}^{N} \lambda_t[f_t(k_{t-1}) - k_t] + \sum_{t=1}^{N-1} \mu_t k_t + \mu_N(k_N - k_N^*)$. The first order conditions are

$$\frac{\partial U}{\partial k_t} = \begin{cases} \lambda_t - f'_{t+1}(k_t^*)\lambda_{t+1} - \mu & \text{for } t = 1, \dots, N-1 \\ \lambda_N - \mu_N & \text{for } t = N. \end{cases} \tag{2}$$

I will refer to these as the *generalized Euler equations*. The multipliers are non-negative and subject to the usual complementary slackness conditions. Of course, the generalized Euler equations are equivalent to eqs. (1).

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[11] This method is adapted from Mirman and Zilcha [41].
[12] When the Inada condition, $W_1(c, U) \to \infty$ as $c \to 0$, holds, consumption will be strictly positive on the optimal path. Since $f_t(0) = 0$, the associated path of capital stocks is also positive and the ordinary Euler Equations are enough.

When U is recursive, the partial derivative of U with respect to consumption at time t is given by

$$U_t(C) = W_2(c_1, U(SC))W_2(c_2, U(S^2C))$$
$$\cdots W_2(c_{t-1}, U(S^{t-1}C))W_1(c_t, U(S^tC)).$$

The *marginal rate of impatience R* is defined by

$$1 + R(C) = W_1(c_1, U(SC))/W_2(c_1, U(SC))W_1(c_2, U(S^2C)).$$

Then $\partial U/\partial k_t(C_{t+1}^{*})=U(C^*)[f'_{t+1}(k_t^*) - 1 - R(S^{t-1}C^*)]$ since $U_t(C)/U_{t+1}(C) = 1 + R(S^{t-1}C)$. The generalized Euler equations can easily be rewritten in terms of the rate of impatience. When the path is interior, they reduce to the ordinary Euler equations

$$f'_{t+1}(k_t^*) = 1 + R(S^{t-1}C^*)$$

I.e., the net marginal product of capital is equal to the marginal rate of impatience.

The generalized Euler equations, and their equivalent form (1), are instrumental in proving the Transversality Theorem.

TRANSVERSALITY THEOREM. *Suppose U is recursive. A path C^* is optimal if and only if the generalized Euler equations hold and either $k_t^* = 0$ for some t, or $c_t^* > 0$ infinitely often and $k_t^*U_t(C^*)^{\to 0}$ as $t \to \infty$ along any subsequence with $c_t^* > 0$ (the Transversality Condition).*

Proof. Suppose C^* is optimal.[13] As above, the optimal path must satisfy the Euler equations. If $k_t^* = 0$ for some t, we're done. Otherwise, let $y_t^* = f_t(k_{t-1}^*)$ denote the income stream associated with the optimal path C^* and J_t denote the value function at time t with $c_t^* > 0$. Since $J_t(0) = 0$, and J_t is concave, $J_t(y) \geq yJ_t'(y)$ for all $y \geq 0$. Setting $y = y_t^*$ yields

$$J_t(y_t^*) \geq y_t^*W_1(c_t^*, U(S^tC^*)) \geq k_t^*W_1(c_t^*, U(S^tC^*)). \tag{3}$$

Now $J_t(y_t) = U(S^{t-1}C^*)$. Multiplying through by δ^{t-1} and using eq. (3) yields

$$\delta^{t-1}U(S^{t-1}C^*) \geq k_t^*U_t(C^*) \geq 0.$$

Combining the φ-boundedness of U with $\delta\|\varphi \circ S\|_\varphi < 1$ shows $k_t^*U_t(C^*) \to 0$ along any subsequence with $c_t > 0$. The sufficiency of the transversality condition is implied by Lemma 3 since U is continuous. ∎

LEMMA 3. *Suppose U is concave and product lower semicontinuous on the feasible set. Then a path K^* is optimal if it satisfies the generalized Euler equations (1) and either $k_t^* = 0$ for some t, or $c_t^* > 0$ infinitely often and the transversality condition is satisfied.*

Proof. Consider an arbitrary feasible path K with associated C. Define an approximate utility function $\mathcal{U}_N$ by $\mathcal{U}_N(C) = U(c_1, \ldots, c_N, c_{N+1}^*, c_{N+2}^*, \ldots)$ where C^* is the consumption

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[13] This portion of the proof is based on the squeezing argument put forth in Mirman and Zilcha [41] and corrected by Becker [4].

path corresponding to K^*. Since U and f are concave, we have

$$\mathcal{U}_N(C) - U(C^*) \leq \sum_{t=1}^{N-1} (\partial U/\partial k_t)[k_t - k_t^*] - U_N(k_N - k_N^*)$$

If $k_N^* = 0$ for N large, $K \in \mathbf{F}_N$ for N large. By eq. (1), the sum is non-positive. Letting $N \to \infty$, $U(C) \leq U(C^*)$ by lower semicontinuity of U.

Otherwise, $k_t^* > 0$ for all t. In this case, all of the μ_t in eq. (2) are zero by complementary slackness. Further, strictly positive consumption occurs infinitely often. We restrict our attention to such N. These N obey $c_N^* = f_N(k_{N-1}^*) - k_N^* > 0$, so $\lambda_N = 0$. By the lower semicontinuity of U and the transversality condition,

$$U(C) - U(C^*) \leq \limsup \sum_{t=1}^{N-1} (\partial U/\partial k_t)[k_t - k_t^*].$$

If $k_{N-1} \geq k_{N-1}^*$, set $T = N - 1$, else choose $T < N$ with $k_T \geq k_T^*$ and $k_t < k_t^*$ for $t = T + 1, \ldots, N - 1$. By eq. (1), the sum of the first T terms is non-positive, so,

$$U(C) - U(C^*) \leq \limsup \sum_{T+1}^{N-1} (\partial U/\partial k_t)[k_t - k_t^*]$$

$$= \limsup \sum_{T+1}^{N-1} [\lambda_t - \lambda_{t+1} f_t'(k_t^*)][k_t - k_t^*]$$

by eq. (2). Regrouping and using $\lambda_N = 0$ and $\lambda_{T+1}(k_{T+1} - k_{T+1}^*) \leq 0$ shows

$$U(C) - U(C^*) \leq \limsup \sum_{T+2}^{N-1} \lambda_t[k_t - k_t^* - f_{t-1}'(k_{t-1}^*)(k_{t-1} - k_{t-1}^*)].$$

If $f(k_{t-1}^*) > k_t^*$, complementary slackness implies $\lambda_t = 0$. Otherwise, $f(k_{t-1}^*) = k_t^*$ and so $k_t - k_t^* - f_{t-1}'(k_{t-1}^*)(k_{t-1} - k_{t-1}^*) \leq f_t(k_{t-1}) - f_t(k_{t-1}^*) - f_{t-1}'(k_{t-1}^*)(k_{t-1} - k_{t-1}^*) \leq 0$ by concavity. It follows that the limit supremum is non-positive. Therefore, $U(C) \leq U(C^*)$. Since K was an arbitrary feasible path, K^* is optimal. ∎

If U is not differentiable, a similar result could be obtained by using supergradients instead of derivatives.

7. CONCLUSION

A number of extensions are possible. Similar techniques will work when there are many commodities available at each time. In fact, a Banach space of commodities at each time can be dealt with in similar fashion.[14] A time-varying aggregator (Streufert [48]) can also

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[14] Just define $|X|_\beta = \sup \|x_t/\beta^t\|$ and $_\gamma|X| = \inf \|x_t/\gamma^t\|$ where $\|\cdot\|$ is the Banach norm.

be used, resulting in a recursive type of variable discount rate model (McKenzie [40], Mitra [42]).

Other possible developments involve the weighted contraction and "partial sum" techniques. Stochastic models with recursive preferences have gotten attention as of late (Epstein [24], Bergman [8], Nairay [43]). The Weighted Contraction Theorem has obvious application to the Lucas [36] asset pricing model. Epstein and Zin [27] have found both it and the "partial sum" technique useful for investigating other types of stochastic models.

The "partial sum" approach also suggests how the concepts of "overtaking" and "catching up" can be extended to recursive preferences. In fact, when a program with $u = -\infty$ is compared with programs with finite u, an implicit overtaking criterion is present. Exactly how overtaking can be applied in a general recursive framework remains to be seen.

Other questions remain open. What is the relation between β-myopia and more traditional notions of impatience based on the marginal rate of substitution for consumption in adjacent time periods? The β-myopia concept also points to a relation between the cardinal properties of the aggregator (δ and η) and properties of the underlying preference order. What is the relation between these? In particular, is there a criterion for the existence of the utility function that works for all equivalent aggregators?

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