Reciprocal Roots, Paired Roots and the Saddlepoint Property*

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Abstract. The usual proof of the saddlepoint property in optimal growth models is based on the reciprocal root property. That proof is incomplete. It assumes there are no multiple roots. This paper repairs both the continuous and discrete time versions of the proof. Under a symmetry condition, the reciprocal root property is usually combined with results from the theory of pencils of quadratic forms to establish the saddlepoint property. I employ a simple spectral mapping argument to characterize the saddlepoint property.

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1. Introduction

Since Poincaré, mathematicians have known that the eigenvalues associated with Hamiltonian dynamical systems come in reciprocal pairs. In many economic problems, this implies the saddlepoint property. Unfortunately, the most commonly cited sources simply assume reciprocity alone implies the roots are paired. In essence, they assume there are no multiple roots. Fortunately, this deficiency can be repaired even though multiple roots are possible. A more careful examination of the characteristic equation shows that roots and their reciprocals actually satisfy the same characteristic polynomial. As a result, the reciprocal is not only a root, but must have the same multiplicity. When roots are counted according to multiplicity, they occur in reciprocal pairs. This paired root property holds in any reduced-form optimal growth model. Further, under a symmetry condition, a simple spectral mapping argument establishes a criterion for the saddlepoint property

The saddlepoint property comes in two versions, depending on whether time is continuous or discrete. In continuous time there are an equal number of positive and negative eigenvalues. In discrete time, there are as many eigenvalues with modulus greater than one as have modulus less than one.

The saddlepoint property plays an important role in capital theory. Saddlepoint instability is characteristic of optimal growth problems (Kurz, 1968). By Becker's (1981) Equivalence Principle, saddlepoint instability is also characteristic of perfect foresight equilibria. Further, it is instrumental in Scheinkman's (1976) construction of the stable manifold. It continues to play a key role in characterizing instability, as in Feinstein and Oren (1983).

The saddlepoint property is intimately related to various other stability and regularity properties of optimal paths. Under a symmetry condition, DasGupta and McKenzie (1984, 1985) show that the saddlepoint property implies local strong dynamic regularity, and is equivalent to both local asymptotic stability and the dominant diagonal block condition used by Araujo and Scheinkman (1977). Combining this with Araujo and Scheinkman's arguments shows the saddlepoint property is equivalent to global asymptotic stability.

Most authors refer to McKenzie (1963), Levhari and Liviatan (1972), or a series of papers by Samuelson (e.g. 1968, 1969, 1970) when they need the saddlepoint property. A close inspection of these papers reveals the same lacuna in each of these proofs, regardless of whether time is continuous or discrete. In the discrete-time version, they attempt to prove the saddlepoint property by showing that whenever λ is an eigenvalue, so is its reciprocal. Things are not quite so simple. This reciprocal root property is not enough. The roots must also have the same multiplicity.

A very simple example illustrates what can go wrong. Consider the 4×4 diagonal matrix with diagonal $(2, 2, 2, \frac{1}{2})$. Both 2 and its reciprocal, $\frac{1}{2}$, are roots of this matrix, but the roots are not paired. Even though none of the roots has modulus one, the saddlepoint property fails. The problem is that 2 has multiplicity 3 and $\frac{1}{2}$ has multiplicity 1.

The potential for multiple roots arises whenever there are two or more capital goods. For this reason, Scheinkman (1976) applied state-of-the-art global analysis (the Hirsch-Pugh Stable Manifold Theorem) to obtain the local stable manifold at a stationary optimal program. If the roots had been known to be distinct, the local stable manifold could have been constructed using elementary methods (Scheinkman, 1976, p.19). If there are multiple roots, these methods are insufficient.

The next problem is to obtain a simple characterization of the saddlepoint property. This is possible under an appropriate symmetry condition. Magill and Scheinkman (1979) and DasGupta and McKenzie (1984) use a result concerning simultaneous diagonalization from the theory of pencils of quadratic forms (Gantmacher, 1960, pp. 310-312). In fact, a simple spectral mapping argument yields a characterization which implies their results.

The results in this paper also apply to other economic problems with a similar structure. In particular, linear rational expectations models with quadratic objective functions used by

Hansen and Sargent (1980, 1981) and Whiteman (1983) fall into the same framework. For example, Kollintzas (1986) found the saddlepoint property useful for the solution of such models.

Section Two contains basic facts about polynomials and their roots. Section Three shows that roots occur in reciprocal pairs with the same multiplicity. Section Four then establishes the saddlepoint property under a symmetry condition by means of a simple spectral mapping argument.

2. Polynomials and Reciprocity

I will need the following well-known result on the factorization of polynomials to establish that roots come in reciprocal pairs. This proposition is an easy consequence of the Fundamental Theorem of Algebra and the fact that such a factorization must be unique [see Hoffman and Kunze (1971, pp. 136-138)]. The importance of this factorization is that it takes multiplicity into account.

PROPOSITION. *Let $p(\lambda)$ be a polynomial of degree m over the reals. It has a unique factorization of the form*

$$p(\lambda) = \alpha \prod_{i=1}^{j} (\lambda - \lambda_i)^{n_i}$$

where α is a constant, the λ_i are the distinct roots of p, the n_i are their multiplicities, and $\sum_{i=1}^{j} n_i = m$.

The above factorization will be used in the following two lemmas on paired roots. Lemma 1 is used to prove the paired root property in continuous time, and Lemma 2 applies to discrete time.

LEMMA 1. *Suppose $p(\lambda) = p(\rho - \lambda)$. Then both λ_i and $\rho - \lambda_i$ are roots with the same multiplicity n_i.*

PROOF. Use the proposition to factor $p(\lambda) = p(\rho - \lambda)$.

$$\begin{aligned} p(\lambda) = p(\rho - \lambda) &= \alpha \prod_{i=1}^{j} [(\rho - \lambda) - \lambda_i]^{n_i} \\ &= (-1)^m \alpha \prod_{i=1}^{j} [\lambda - (\rho - \lambda_i)]^{n_i}. \end{aligned}$$

By the proposition, this factorization is unique. Hence both λ_i and $\rho - \lambda_i$ are roots with multiplicity n_i. □

LEMMA 2. *Suppose $p(\lambda)$ is a polynomial of degree m that satisfies $p(\lambda) = \gamma\lambda^m p(1/\delta\lambda)$ for some constant γ with $p(0) \neq 0$. Both λ_i and $1/\delta\lambda_i$ are roots of the same multiplicity n_i.*

PROOF. Again use the proposition to factor $p(\lambda)$.

$$\begin{aligned} p(\lambda) = \gamma\lambda^m p(1/\delta\lambda) &= a\lambda^m \alpha \prod_{i=1}^{j} [(\delta\lambda)^{-1} - \lambda_i]^{n_i} \\ &= \gamma\alpha \prod_{i=1}^{j} [\delta^{-1} - \lambda_i\lambda]^{n_i} \\ &= \alpha' \prod_{i=1}^{j} [\lambda - (\delta\lambda_i)^{-1}]^{n_i}. \end{aligned}$$

Once again, unique factorization guarantees that both λ_i and $1/\delta\lambda_i$ are roots of multiplicity n_i. □

3. Reciprocity and Paired Roots

I will consider both the discrete and continuous time cases. Following Levhari and Liviatan (1972), I consider reduced-form utility functions. These have the form $\int_0^\infty e^{-\rho t} v(k, \dot{k}) dt$ in continuous time and $\sum_{t=0}^\infty \delta^t v(k_t, k_{t+1})$ in discrete time. Here v, the reduced-form felicity function, is concave and twice continuously differentiable. The non-negative n-vector of capital stocks at time t is denoted by k. An initial stock k_0 is given. The technology is given by a closed convex set $\mathbf{T}$. In the continuous-time case, $k(t)$ is a twice continuously differentiable function of time t with $(k, \dot{k}) \in \mathbf{T}$ and $k(0) = k_0$. In the discrete-time case, $(k_t, k_{t+1}) \in \mathbf{T}$ for all t.

As usual, interior optimal paths must satisfy the Euler equations. In continuous time the Euler equations are

$$e^{-\rho t}\partial v/\partial k^i = d(e^{-\rho t}\partial v/\partial \dot{k}^i)/dt,$$

and in discrete time they are

$$\delta\partial v(k_t, k_{t+1})/\partial k_t^i + \partial v(k_{t-1}, k_t)/\partial k_t^i = 0$$

for $t = 1, 2, \ldots$.

We linearize these equations to study motion about the steady state. The linearized equation is obtained from the Taylor expansion of the Euler equations about the steady state capital stock $k = k*$. In continuous time this procedure yields following system of equations.

$$\begin{aligned} Adz/dt &= -(B - B' - \rho A)z + (C + \rho B)y \\ dy/dt &= z \end{aligned}$$

where $y = k - k*$, and $A = (\partial^2 v/\partial \dot{k}^i \partial \dot{k}^j)$, $B = (\partial^2 v/\partial \dot{k}^i \partial k^j)$, and $C = (\partial^2 v/\partial k^i \partial k^j)$ are the matrices of partial derivatives, evaluated at the steady state $\dot{k} = 0$, $k = k*$. This system describes the approximate behavior of optimal paths in a neighborhood of the steady state.

When A is non-singular, the system is non-degenerate and its properties are described by the characteristic equation

$$p(\lambda) = \det[A\lambda^2 + (B - B' - \rho A)\lambda - (C + \rho B)] = 0.$$

The characteristic polynomial $p(\lambda)$ obeys

$$\begin{aligned} p(\rho-\lambda) &= \det[A(\rho-\lambda)^2 + (B - B' - \rho A)(\rho-\lambda) - (C + \rho B)] \\ &= \det[A\lambda^2 + (B' - B - \rho A)\lambda - (C + \rho B')] \\ &= \det[A\lambda^2 + (B - B' - \rho A)\lambda - (C + \rho B)] \end{aligned}$$

since A and C are symmetric. But this last expression is just $p(\lambda)$, so $p(\lambda) = p(\rho - \lambda)$. By Lemma 1, both λ_i and $\rho - \lambda_i$ are roots of the same multiplicity.

In the discrete-time case, we again apply Taylor's theorem to the Euler equations to get the following linearized equations about the steady state $k*$.

$$\begin{aligned} \delta B' z_{t+1} &= -(A + \delta C)z_t - By_t \\ y_{t+1} &= z_t \end{aligned}$$

where $y_t = k_t - k*$, and the matrices of partial derivatives, $A = (\partial^2 v/\partial k_2^i \partial k_2^j)$, $B = [\partial^2 v/\partial k_2^i \partial k_1^j]$, and $C = [\partial^2 v/\partial k_1^i \partial k_1^j]$ are evaluated at $k_1 = k_2 = k*$.

When B is non-singular, this system has the characteristic equation

$$p(\lambda) = \det[\delta B' \lambda^2 + (A + \delta C)\lambda + B] = 0.$$

The characteristic polynomial $p(\lambda)$ then obeys

$$\begin{aligned} p(\lambda) &= \lambda^{2n}\delta^n \det[B' + (A + \delta C)(\delta\lambda)^{-1} + \delta B(\delta\lambda)^{-2}] \\ &= \lambda^{2n}\delta^n \det[\delta B(\delta\lambda)^{-2} + (A + \delta C)(\delta\lambda)^{-1} + B'] \\ &= \lambda^{2n}\delta^n \det[\delta B'(\delta\lambda)^{-2} + (A + \delta C)(\delta\lambda)^{-1} + B] \end{aligned}$$

since A and C are again symmetric. Thus p obeys $p(\lambda) = \lambda^{2n}\delta^n p(1/\delta\lambda)$. Lemma 2 now shows that both λ_i and $1/\delta\lambda_i$ are roots of the same multiplicity.

This establishes the following theorem.

Paired Root Theorem. *Suppose roots are counted according to multiplicity. The characteristic roots of the continuous-time reduced-form optimization problem occur in pairs* $\{\lambda_i, \rho-\lambda_i\}$ *when* $\partial^2 v/\partial \dot{k}^i \partial \dot{k}^j$ *is non-singular. When* $\partial^2 v/\partial k_2^i \partial k_1^j$ *is non-singular, the characteristic roots of the discrete-time reduced-form optimization problem occur in pairs* $\{\lambda_i, 1/\delta\lambda_i\}$.

4. Reciprocity and the Saddlepoint Property

The paired root property is the major step in establishing the saddlepoint property. Only one step remains. We must show that the roots have either the proper sign or modulus. When time is continuous, we say the *saddlepoint property* holds if the paired roots have opposite signs. When time is discrete, we say the *saddlepoint property* holds provided one member of each pair has modulus less than one, while the other has modulus greater than one. This can be established when ρ is near 0, or δ is near 1.

When the matrix of cross partial derivatives, B, is symmetric, the analysis can be simplified. We need only look at an $n \times n$ matrix rather than a $2n \times 2n$ matrix.

Saddlepoint Theorem. *Suppose* B *is symmetric. Let* $\Sigma = \{z : \rho^2 \; Re\, z \leq -(Im\, z)^2\}$ *and* $\mathbb{R} = \{z : (Re\, z)^2/(1+\delta)^2 + (Im\, z)^2/(1-\delta)^2 \leq 1\}$. *The saddlepoint property is equivalent to* $\Sigma \cap \sigma(A^{-1}(B+\rho C)) = \varnothing$ *in continuous time and* $\mathbb{R} \cap \sigma(B^{-1}(A+\delta C)) = \varnothing$ *in discrete time.*

Proof. Let Λ be the set of roots. In the continuous-time case, we must show that λ and $\rho - \lambda$ have opposite signs whenever λ is a root. The saddlepoint property will hold if and only if $\Lambda \cap \mathcal{A} = \varnothing$ where $\mathcal{A}$ is the strip $\{z : 0 \leq \mathrm{Re}\, z \leq \rho\}$ shown at the left in Figure 1.

We know $\lambda \in \Lambda$ is equivalent to $\det[A\lambda^2 + (B - B' - \rho A)\lambda - (C + \rho B)] = 0$. Using the symmetry of B, a little manipulation reveals that $\Lambda = \{\lambda : \det[\lambda^2 - \rho\lambda - A^{-1}(C+\rho B)] = 0\}$. Let $f(\lambda) = \lambda^2 - \rho\lambda$. Since f is at most 2 to 1, and $f(\rho - \lambda) = f(\lambda)$, we find $\Lambda = \{\lambda : f(\lambda) \in \sigma(A^{-1}(B + \rho C))\}$ where σ denotes the spectrum. Now f maps $\mathcal{A}$ to the parabolic region $\mathcal{B} = f(\mathcal{A}) = \{z : \rho^2 \,\mathrm{Re}\, z \leq -(\mathrm{Im}\, z)^2\}$ as in Figure 1. The saddlepoint property is

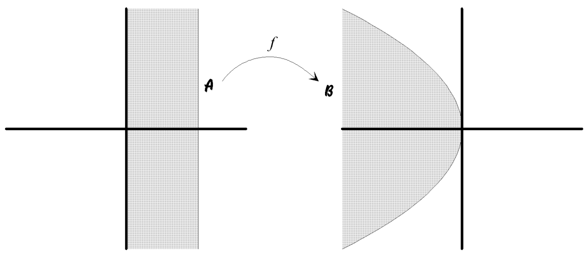

Figure 1

then equivalent to $\mathcal{B} \cap \sigma(A^{-1}(B + \rho C)) = \varnothing$.

In the discrete-time case the forbidden region is the ring $\mathcal{C} = \{z : 1 \le |z| \le 1/\delta\}$ shown at the left in Figure 2. The saddlepoint property is equivalent to $\Lambda \cap \mathcal{C} = \varnothing$. The set of roots is $\Lambda = \{\lambda : \det[\delta B'\lambda^2 + (A + \delta C)\lambda + B] = 0\}$. With B symmetric, $\Lambda = \{\lambda : \det[B^{-1}(A + \delta C) - g(\lambda)] = 0\}$, where $g(\lambda) = -(\delta\lambda^2 + 1)/\lambda$. Now g maps $\mathcal{C}$ to the elliptical region $\mathcal{D} = g(\mathcal{C}) = \{z : (\operatorname{Re} z)^2/(1+\delta)^2 + (\operatorname{Im} z)^2/(1-\delta)^2 \le 1\}$ as shown in Figure 2. It immediately follows that the saddlepoint property is equivalent to $\mathcal{D} \cap \sigma(B^{-1}(A + \delta C)) = \varnothing$. $\square$

The conditions in the Saddlepoint Theorem can sometimes be verified without further calculation. Suppose A and $(C + \rho B)$ are negative definite in the continuous-time case. (This will usually occur when ρ is near zero since v is concave.) Then $-A = |A|$ has a square root $|A|^{1/2}$ and $\sigma(A^{-1}(B + \rho C)) = -\sigma(|A|^{-1/2}(C + \rho B)|A|^{-1/2})$. As $|A|^{-1/2}(C + \rho B)|A|^{-1/2}$ is negative definite, $-\sigma(|A|^{-1/2}(C + \rho B)|A|^{-1/2}) \subset \mathbb{R}_+$. The saddlepoint property follows.

In other cases the Saddlepoint Theorem simplifies matters. Consider the discrete-time

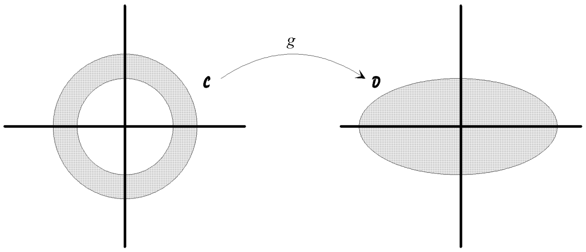

Figure 2

case with $(A + \delta C)$ negative definite. As $-(A + \delta C)$ has a square root D, we obtain $\sigma(B^{-1}(A + \delta C)) = -\sigma(DB^{-1}D)$ as before. Now $-\sigma(DB^{-1}D) \subset \mathbb{R}$ since the spectrum of a symmetric matrix is real. The saddlepoint property is then equivalent to $\sigma(B^{-1}(A+\delta C)) \cap [-(1 + \delta), (1 + \delta)] = \varnothing$, which is the same condition derived by DasGupta and McKenzie (1984).

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