Name: Solution key Panther ID: ____________

Exam 1 Calculus 3 Spring 2020

Important Rules:

A. Any electronic device (cell phone, calculator of any kind, smart-watch, etc.) should be turned off at the beginning of the exam and placed in your bag, NOT in your pocket. Electronic items, notes, texts, or formula sheets should NOT be used at any time during the examination. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.

Violations of any type of this rule will lead to a score of zero on this exam, possibly an automatic grade F for the course and a report for academic misconduct.

B. Unless otherwise mentioned, to receive full credit you must show your work. Answers which are not supported by work might receive no credit. Solutions should be concise and clearly written. Incomprehensible work might not be considered.

1. (15 pts) Given the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, find each of the following:

(a) (5 pts) A vector $\mathbf{w}$ of magnitude 5 in the direction opposite to the direction of $\mathbf{u}$.

$$\vec{w} = -5\frac{\vec{u}}{|\vec{u}|} = -\frac{5}{\sqrt{5}}(\vec{i} - 2\vec{k}) = -\sqrt{5}\,\vec{i} + 2\sqrt{5}\,\vec{k}$$

(b) (5 pts) The angle between $\mathbf{u}$ and $\mathbf{v}$ (answer as inverse trig. function ok).

$$\vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cos\theta$$

so $$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}$$

$$\cos\theta = \frac{3-2}{\sqrt{5}\cdot\sqrt{14}} = \frac{1}{\sqrt{70}}$$

$$\underline{\theta = \arccos\left(\frac{1}{\sqrt{70}}\right)}$$

(c) (5 pts) The vector $\text{proj}_{\mathbf{v}}\mathbf{u}$ (the projection of $\mathbf{u}$ onto $\mathbf{v}$.)

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{1}{14}(3\vec{i} + 2\vec{j} + \vec{k})$$

2. (14 pts) Circle True or False. You do not have to justify your answer for these. Assume that $\mathbf{u}, \mathbf{v}$ are arbitrary vectors in $\mathbf{R}^3$ unless stated otherwise.

(a) For any vectors $\mathbf{u}, \mathbf{v}$ in $\mathbf{R}^3$, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$. **True** **False**

(b) The plane perpendicular to the z-axis at $(0, 0, 4)$ is given by $z = 4$. **True** **False**

(c) If two lines in $\mathbf{R}^3$ do not intersect, then they are parallel. **True** **False**

(d) If two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ do not intersect, then $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}$. **True** **False**

(e) $x^2 + z^2 = 4$ is a circle in 3-space. **True** **False**

(f) If $\mathbf{r}(t)$ is the position vector of a particle at time t, then the speed is given by $|\mathbf{r}'(t)|$. **True** **False**

(g) The graph of $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 5\,\mathbf{k}$ is an ellipse in the plane $z = 5$. **True** **False**

3. (15 pts) (a) (9 pts) Show that the line $x = 2,\ y = 3 + 2t,\ z = -2 - 2t$ intersects the plane $x + 2y + 3z = 4$ and find the point of intersection.

Solving the system $\begin{cases} x = 2 \\ y = 3+2t \\ z = -2-2t \\ x+2y+3z = 4 \end{cases}$, get $2 + 2\cdot(3+2t) + 3(-2-2t) = 4$

$2 + \cancel{6} + 4t - \cancel{6} - 6t = 4$.

so $t = -1$

Thus, the line intersects the plane at the point $P_0(2, 1, 0)$.

(b) (6 pts) Is the line $x = 2,\ y = 3 + 2t,\ z = -2 - 2t$ perpendicular to the plane $x + 2y + 3z = 4$? Briefly justify.

No, because the directional vector $\vec{v} = \langle 0, 2, -2 \rangle$ of the line is not parallel to the normal vector $\vec{n} = \langle 1, 2, 3 \rangle$ of the plane

(they are not scalar multiples)

4. (12 pts) Match the following equations with the appropriate surface:

(i) $x^2 - 2y^2 + 3z^2 = 1$ (b)

(ii) $2y^2 + 3z^2 = 1$ (d)

(iii) $(x+1)^2 + 2(y-1)^2 + 3(z-2)^2 = 10$ (a)

(iv) $x - 2y^2 - 3z^2 = 1$ (e)

(v) $(x+1)^2 - 2(y-1)^2 - 3(z-2)^2 = 10.$ (c)

(vi) $x^2 = 2y^2 + 3z^2$ (f)

(a) ellipsoid (b) hyperboloid with one sheet (c) hyperboloid with two sheets

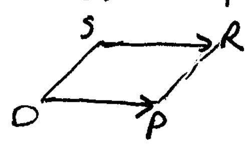
(d) elliptic cylinder (e) elliptic paraboloid (f) elliptic cone

typo

5. (20 pts) ~~A~~ quadrilateral (a 4-sided polygon lying in a plane in $\mathbf{R}^3$) has corners at $O(0,0,0)$, $P(1,5,2)$, $R(4,10,1)$ and $S(3,5,1)$. $S(3,5,-1)$

(a) (5 pts) Show that this shape is a parallelogram. Explain your reasoning briefly.

$\vec{OP} = \langle 1,5,2 \rangle = \vec{SR}$ so OPRS is a parallelogram as $OP \parallel SR$ and $|OP| = |SR|$



($OP \parallel SR$ implies that all 4 points are coplanar)

(b) (7 pts) Assuming it is a parallelogram, with O and R as opposite corners, find its area.

$$\text{Area} = |\vec{OP} \times \vec{OS}|$$

$$\vec{OP} \times \vec{OS} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & 2 \\ 3 & 5 & -1 \end{vmatrix} = -15\vec{i} + 7\vec{j} - 10\vec{k}$$

$$\text{so Area} = |-15\vec{i} + 7\vec{j} - 10\vec{k}| = \sqrt{15^2 + 7^2 + 10^2} = \sqrt{374}$$

(c) (8 pts) Find an equation for the plane that contains the parallelogram.

A normal vector to the plane is

$$\vec{n} = \vec{OP} \times \vec{OS} = -15\vec{i} + 7\vec{j} - 10\vec{k}$$

and using O as a particular point in the plane, the equation of the plane is

$$-15x + 7y - 10z = 0 \quad \text{or} \quad 15x - 7y + 10z = 0.$$

6. (16 pts + 6 bonus) Consider the curve

$$\mathbf{r}(t) = \frac{3}{\sqrt{2}}\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + \frac{3}{\sqrt{2}}\cos t\,\mathbf{k}, \text{ ~~where a is a constant.~~}$$

(a) (8 pts) Find parametric equations for the tangent line to the curve at the point corresponding to $t = \pi/4$.

$$r\left(\frac{\pi}{4}\right) = \left\langle \frac{3}{\sqrt{2}}\cos\left(\frac{\pi}{4}\right), 3\sin\left(\frac{\pi}{4}\right), \frac{3}{\sqrt{2}}\cos\left(\frac{\pi}{4}\right)\right\rangle = \left\langle \frac{3}{2}, \frac{3}{\sqrt{2}}, \frac{3}{2}\right\rangle$$

So, the point $P_0\left(\frac{3}{2}, \frac{3}{\sqrt{2}}, \frac{3}{2}\right)$

$$\vec{r}\,'(t) = \left\langle -\frac{3}{\sqrt{2}}\sin t, 3\cos t, -\frac{3}{\sqrt{2}}\sin t\right\rangle$$

$$\vec{r}\,'\left(\frac{\pi}{4}\right) = \left\langle -\frac{3}{2}, \frac{3}{\sqrt{2}}, -\frac{3}{2}\right\rangle \leftarrow \text{directional vector for the tangent line}$$

One can take the directional vector be $\vec{v} = \frac{2}{3}\vec{r}\,'\left(\frac{\pi}{4}\right) = \langle -1, \sqrt{2}, -1\rangle$

Parametric equations of the tangent line:

$$\boxed{x = \frac{3}{2} - t, \quad y = \frac{3}{\sqrt{2}} + \sqrt{2}\cdot t, \quad z = \frac{3}{2} - t}$$ (equivalent forms OK)

(b) (8 pts) Show that $\mathbf{r}(t)$ represents a circle, by showing that the curve lies on the sphere $x^2 + y^2 + z^2 = 9$ and on a certain plane.

In parametric form $\vec{r}(t)$ is equivalent to $\left(x = \frac{3}{\sqrt{2}}\cos t, y = 3\sin t, z = \frac{3}{\sqrt{2}}\cos t\right)$

For any t $\quad \left(\frac{3}{\sqrt{2}}\cos t\right)^2 + (3\sin t)^2 + \left(\frac{3}{\sqrt{2}}\cos t\right)^2 = \frac{9}{2}\cos^2 t + \frac{9}{2}\cos^2 t + 9\sin^2 t$

$= 9\cos^2 t + 9\sin^2 t = 1$

So the curve is on the sphere $x^2 + y^2 + z^2 = 9$

Notice also that $x = \frac{3}{\sqrt{2}}\cos t = z$, so the curve is also on the plane $x - z = 0$.

Thus, the curve is the intersection of the sphere $x^2 + y^2 + z^2 = 9$ with the plane $x - z = 0$, so it must be a circle

(c)* (6 pts bonus - maybe a bit challenging) Can you find the radius of the circle represented by $\mathbf{r}(t)$?

Since the plane $x - z = 0$ contains the origin, which is the center of the sphere $x^2 + y^2 + z^2 = 9$, the curve of intersection is a "great" circle on the sphere, so it has radius $\underline{3}$. (There should not be a star on this problem)

Very easy bonus!

7. (12 pts) Choose ONE proof. If you do TWO, only the larger score will be considered for this problem, but the second score may give some bonus towards a previous problem where your score is smaller. You can use the back of the page, if needed.

(A) Prove the "geometric formula" for the dot product

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta \,, \text{ where } \theta \text{ is the angle between } \mathbf{u} \text{ and } \mathbf{v}.$$

(B) State and prove the point-normal equation of a plane.

(C) Consider the usual 3d rectangular coordinate system with origin O and let A be an arbitrary point on the x-axis, B be an arbitrary point on the y-axis and C be an arbitrary point on the z-axis. Use vectors to show that

$$(Area(\triangle ABC))^2 = (Area(\triangle AOB))^2 + (Area(\triangle BOC))^2 + (Area(\triangle COA))^2$$

For (A) and (B) see notes or textbook.

I let you to think on your own about (C). You could think of it as a 3D version of Pythagorean Thm.