$$\lim_{(x,y)\to(0,0)} \frac{xy - y^2}{\sqrt{x} - \sqrt{y}} \overset{\frac{0}{0}}{=} \lim_{\substack{(x,y)\to(0,0)\\ x>0,\, y>0}} \frac{y(x-y)}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(x_0,y_0)} \frac{y\left((\sqrt{x})^2 - (\sqrt{y})^2\right)}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{y\cancel{(\sqrt{x}-\sqrt{y})}(\sqrt{x}+\sqrt{y})}{\cancel{\sqrt{x}-\sqrt{y}}} = 0 \cdot 0 = 0$$

(2) claim that $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$ DNE.

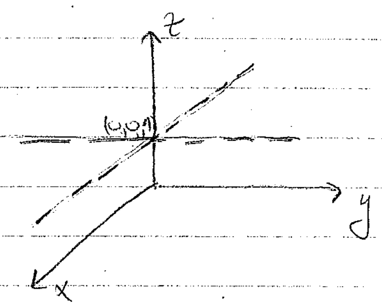
Lecture 02/13/2020

Ex 1 let $f(x,y) = \begin{cases} 1 & \text{if } x\cdot y = 0 \\ 0 & \text{if } xy \neq 0 \end{cases}$ 02/13/2020

a) what is the D of $f(x,y)$? whole R^2

b) what does the graph of $f(x,y)$ look like?

1. $x=0$ or $y=0$ if $xy=0$

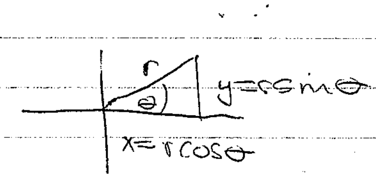


c) $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE $f(0,0)=1$

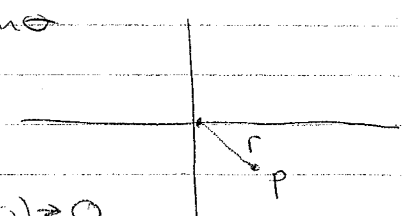
d) At which points is $f(x,y)$ discontinuous? and continuous?

Discontinuous at all points where $x=0$ and $y=0$
Continuous at every other point

Ex 2



a) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$



$(x,y)\to(0,0) \Rightarrow d((x,y),(0,0))\to 0$
$\Rightarrow r\to 0$
$\sqrt{x^2+y^2}$

$\lim_{x\to 0}\frac{\sin x}{x}=1$

$$\frac{x^3+y^3}{x^2+y^2}=\frac{r^3\cos^3\theta+r^3\sin^3\theta}{r^2}=r\left(\cos^3\theta+\sin^3\theta\right)$$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right|=r\left|\cos^3\theta+\sin^3\theta\right|\le r\left(\left|\cos^3\theta\right|+\left|\sin^3\theta\right|\right)\qquad |a+b|\le|a|+|b|$$

$$\le r(1+1)=2r\xrightarrow[r\to 0]{}0$$

limit $=0$

b) $\lim_{(x,y)\to(0,0)}\frac{y^2}{x^2+y^2}=\text{DNE}$ $\qquad\frac{y^2}{x^2+y^2}=\frac{\cancel{r^2}\sin^2\theta}{\cancel{r^2}}=$

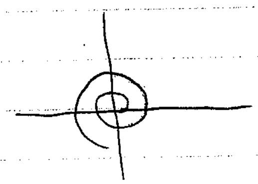
$=\sin^2\theta$ not unique answer

c) $\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{y}=\text{DNE}$ $\qquad\frac{x^2+y^2}{y}=\frac{r^{\cancel{2}}}{\cancel{r}\sin\theta}=\frac{r}{\sin\theta}$

Suppose we choose $\theta=r$ (spiral)

$r\to 0$

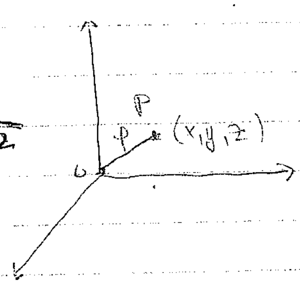
$\theta=2r$



$\lim_{r\to 0}\frac{r}{\sin r}=1$

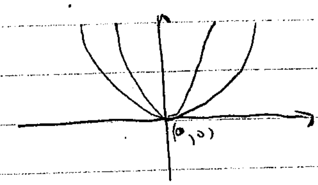
$\lim_{r\to 0}\frac{r}{\sin(2r)}=\frac{1}{2}$ $\Big\}$ not the same limit

d) $\lim_{(x,y,z)\to(0,0,0)}\frac{\sin(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}$ $\qquad\rho=|OP|=\sqrt{x^2+y^2+z^2}$



$$=\lim_{\rho\to 0}\frac{\sin(\rho^2)}{\rho}=\lim_{\rho\to 0}\underbrace{\rho}_{0}\ \underbrace{\boxed{\frac{\sin(\rho^2)}{\rho^2}}}_{1}=0$$

e) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4 - y^2}{x^4 + y^2}$ Approach origin on parabola $y = kx^2$



$$\frac{x^4 - y^2}{x^4 + y^2} \underset{\uparrow}{=} \frac{x^4 - (kx^2)^4}{x^4 + (kx^2)^2} = \frac{\cancel{x^4}(1-k^2)}{\cancel{x^4}\underbrace{(1+k^2)}}$$

if $y = kx^2$

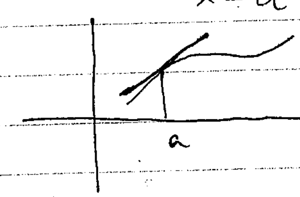
depends on k

DNE

$f(x)$ $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

slope of tangent line to the graph at $x = a$

instantenous rate of change of y w.r.t. x at $x = a$



$z = f(x,y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}(x_0, y_0) \overset{\text{def.}}{=} \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

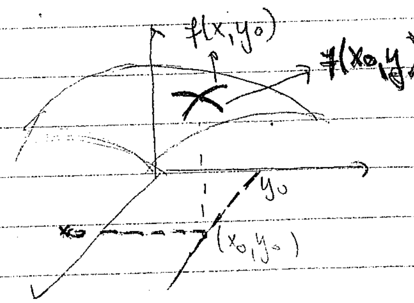
$$\frac{\partial f}{\partial y}(x_0, y_0) \overset{\text{def.}}{=} \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

rate of change of f w.r.t. y at (x_0, y_0)

— " — w.r.t. x at (x_0, y_0)

$\dfrac{\partial f}{\partial x} = f_x$

$\dfrac{\partial f}{\partial y} = f_y$



$f(x,y) = x^2 \ln(x+2y)$

$f_x = \frac{\partial}{\partial x}\left(x^2 \ln(x+2y)\right) = \frac{\partial}{\partial x}(x^2) \cdot \ln(x+2y) + x^2 \frac{\partial}{\partial x}(\ln(x+2y))$

$= 2x\ln(x+2y) + x^2 \frac{1}{x+2y} \cdot \frac{\partial}{\partial x}(x+2y) =$

$= 2x\ln(x+2y) + \frac{x^2}{x+2y}$

$f_y = \frac{\partial}{\partial y}\left(x^2 \ln(x+2y)\right) = x^2 \frac{\partial}{\partial y}(\ln(x+2y)) = x^2 \cdot \frac{1}{x+2y} \cdot 2$

$f_y = \frac{2x^2}{x+2y}$

Higher order partial derivatives:

$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$

$\frac{\partial^2 f}{\partial y \partial y} = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \boxed{f_{xy}}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$

Ex. $f(x) = x e^{(xy)}$

$f_x = e^{xy} + xy e^{xy} = e^{xy}(1+xy)$

$f_y = x^2 e^{xy}$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(e^{xy}(1+xy)\right) =$$
$$= xe^{xy}(1+xy) + xe^{xy} = xe^{xy}(2+xy) \; *$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(x^2 e^{xy}\right) = 2xe^{xy} + x^2 y e^{xy} =$$
$$= xe^{xy}(2+xy) \; *$$

$f_{xy} = f_{yx}$ most of the time

mixed second partials theorem

• If the mixed second partials exist on an open region and are continuous at (x_0, y_0) then $\boxed{\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)}$

$$f(x,y) = \begin{cases} 1 & \text{if } xy = 0 \\ 0 & \text{if } xy \neq 0 \end{cases}$$

z-value $= 1$

$$\frac{\partial f}{\partial x}(0,0) \overset{\text{def}}{=} \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{1-1}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{1-1}{h} = 0$$

In this ex. both partial der. at $(0,0)$ exist, but $f(x,y)$ is not even continuous at $(0,0)$

$$\frac{\partial f}{\partial x}(1,0) = 0$$

0 zato što nije fixno na x ili y osi

$$\frac{\partial f}{\partial y}(1,0) = \lim_{h \to 0} \frac{\overbrace{f(1,h)}^{0} - f(1,0)}{h} = \lim_{h \to 0} \frac{0-1}{h} = \text{DNE}$$

$f(x)$ f is differentable at $x=a \iff \boxed{f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}}$

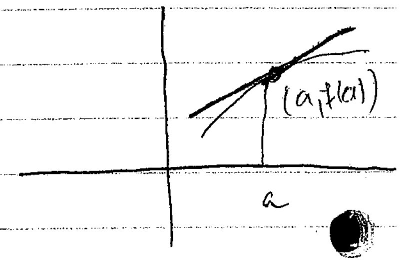
exists and is finite

$$\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a} - f'(a)\right] = 0$$

$$\Rightarrow \lim_{x\to a} \frac{f(x) - (f(a) + f'(a)(x-a))}{(x-a)} = 0.$$

$$y = f(a) + f'(a)(x-a)$$

$$f(x) \approx f(a) + f'(a)(x-a)$$



$f(x,y) = z$

(x_0, y_0)

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

- Def: We say that $f(x,y)$ is differentable at (x_0,y_0) if $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ both exist and are finite and $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - (f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0))}{\|(x,y) - (x_0,y_0)\|} = 0$