02/20/20

(19) / 14.4

Suppose $f(u, v, w)$ is differentable and

$u = x - y \quad v = y - z \quad w = z - x$

Show that $f_x + f_y + f_z = 0$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x} =$$

$$= f_u \cdot 1 + f_v \cdot 0 + f_w \cdot (-1)$$

$$\underline{f_x} = f_u - f_w$$

$$f_y = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y} =$$

$$\underline{f_y} = f_u \cdot (-1) + f_v \cdot (1) + f_w \cdot 0 = -f_u + f_v$$

$$\underline{f_z} = -f_v + f_w$$

$$f_x + f_y + f_z = \cancel{f_u} - \cancel{f_w} + \cancel{f_v} - \cancel{f_u} - \cancel{f_v} + \cancel{f_w} = 0 \checkmark$$

Ex 2.

$F(x, y)$, but also $y = y(x)$

$g(x) = F(x, y(x))$

$$g'(x) = \frac{\partial F}{\partial x} \cdot \underbrace{\frac{dx}{dx}}_{} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx}$$

$$\underline{g'(x) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx}}$$

$F(x, y)$, $y = x^3$

$$\frac{d}{dx}\left(F(x, x^3)\right) = \frac{\partial F}{\partial x} \cdot \underbrace{1}_{\frac{dx}{dx}} + \frac{\partial F}{\partial y} \cdot \underbrace{3x^2}_{\frac{dy}{dx}}$$

$$= \frac{\partial F}{\partial x}(x, x^3) + \frac{\partial F}{\partial y}(x, x^3) \cdot 3x^2$$

Suppose y is implicitely given as a function of x by
$F(x,y) = c$ $\qquad y = y(x)$

$$\frac{d}{dx}\left(F(x,y)\right) = \frac{d}{dx}(c)$$

$$\frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}}$$ ⬅ valid at all points where $\frac{\partial F}{\partial y} \neq 0$

$$\underbrace{x^2 + 3y^2}_{F(x,y)} = 5 \qquad \frac{\partial F}{\partial x} = 2x \qquad \frac{dy}{dx} = \frac{-2x}{6y} = -\frac{x}{3y}$$

$$\frac{\partial F}{\partial y} = 6y$$ true at all points where $y \neq 0$.

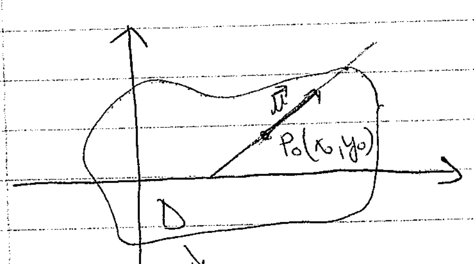
14.5 : Gradient & Directional derivatives $\quad$ 2/20/20

$f(x,y) \leftarrow \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

∇ "nabla" → gradient of f

$f(x,y,z) \quad \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

Directional d. $\qquad \vec{u} = \langle u_1, u_2 \rangle$
$z = f(x,y)$ Suppose $\vec{u}$ is a unit vector (a direction)



domain of $f(x,y)$

$$\begin{cases} x = x_0 + t\cdot u_1 \\ y = y_0 + t\cdot u_2 \end{cases} \iff \vec{r} = \vec{r}_0 + t\cdot\vec{u}$$

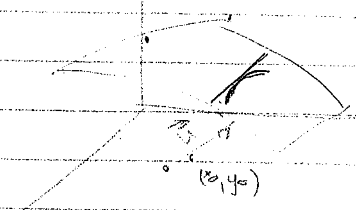
$$|\vec{r}\,'(t)| = |\vec{u}| = 1$$
$$|\vec{r}\,'(s)| = |\vec{u}| = 1$$

* Directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of $\vec{u}$:

$$(D_{\vec{u}}f)(x_0,y_0) = \lim_{s \to 0} \frac{f(x_0+s\cdot u_1,\ y_0+s\cdot u_2) - f(x_0,y_0)}{s} =$$

$$= \frac{d}{ds}\Big(f(x_0+su_1,\ y_0+su_2)\Big)\Big|_{s=0} \underset{\text{chain rule}}{=}$$

$$= \frac{\partial f}{\partial x}\cdot \underbrace{u_1}_{\frac{dx}{ds}} + \frac{\partial f}{\partial y}\cdot \underbrace{u_2}_{\frac{dy}{ds}} = \boxed{\vec{\nabla} f \cdot \vec{u}}$$



$$\vec{\nabla} f = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right>$$
$$\vec{u} = \left< u_1, u_2 \right>$$

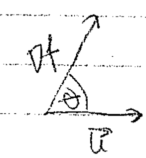
$$\boxed{(D_{\vec{u}}f)(x_0,y_0) = \vec{\nabla} f_{(x_0,y_0)} \cdot \vec{u}}$$

In which direction $\vec{u}$ should move so that the function increases at the highest rate?

$$\vec{u} = \frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$$

$$\vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f| \cdot \underbrace{|\vec{u}|}_{1} \cos\theta$$

$-1 \leq \cos\theta \leq 1$



maximised when $\theta = 0$

so $\vec{u}$ should be in direction of gradient

$$\vec{u} = -\frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$$

direction for the highest decrease for $f(x,y)$

uvijek koristiti unit vec

(12) /14.5

a) $f(x,y) = 2x^2 + y^2$, $P_0(-1,1)$, $\vec{v} = 3\vec{i} - 4\vec{j}$

its direction is $\vec{u}$

$\vec{u} = \frac{1}{5}(3i - 4j)$

$$(D_{\vec{u}} f)_{P_0} = (\nabla f)_{P_0} \cdot \vec{u}$$

$$\nabla f = \langle 4x, 2y \rangle \qquad (D_{\vec{u}} f)_{P_0} = \frac{3}{5} \cdot (-4) + \left(-\frac{4}{5} \cdot 2\right) = -4$$

$$(\nabla f)_{P_0} = \langle -4, 2 \rangle$$

b) In which direction does the function increases the

In the direc. of gradient

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{20}} \langle -4, 2 \rangle$$

(22)

$$g(x,y,z) = xe^y + z^2 \quad , P_0\left(1, \ln 2, 1/2\right)$$

Find the dir. of $\vec{u}$ in which g decrease most at P_0 and find the rate of decrease w.r.t. that

$$\nabla g = \langle e^y, xe^y, 2z \rangle \qquad (\nabla g)_{P_0} = \langle e^{\ln 2}, 1 \cdot e^{\ln 2}, 2 \cdot 1/2 \rangle$$

$$(\nabla g)_{P_0} = \langle 2, 2, 1 \rangle$$

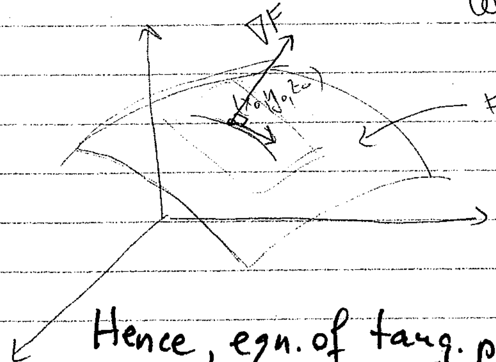
$$\vec{u} = -\frac{\nabla g}{|\nabla g|} = -\frac{1}{3} \langle 2, 2, 1 \rangle$$

$$D_{\vec{u}}\, g = \nabla g \cdot \left(-\frac{\nabla g}{|\nabla g|}\right) = -\frac{|\nabla g|^{\cancel{2}}}{\cancel{|\nabla g|}} = -|(\nabla g)_{P_0}| = -3$$

14.6 **Tangent planes** 2/20/2020

Surface given as $F(x,y,z)=c$

$\downarrow$ level surface of $F(x,y,z)$



theorem: At any point (x_0,y_0,z_0) of the surface $F(x,y,z)=c$
$(\nabla F)(x_0,y_0,z_0)$ is normal to surface
($\nabla F(x_0,y_0,z_0)$ is normal to tangent plane of the surface)

Hence, eqn. of tang. plane is $\boxed{\frac{\partial F}{\partial x}(x_0,y_0,z_0)(x-x_0)+\frac{\partial F}{\partial y}(x_0,y_0,z_0)(y-y_0)+\frac{\partial F}{\partial z}(P_0)(z-z_0)=0}$

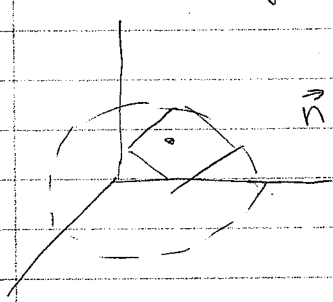
Proof: Suppose $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$ is a curve lying on the ~~plane~~ surface and such that $\vec{r}(t_0)=\langle x_0,y_0,z_0\rangle$

$$F(x(t),y(t),z(t))=c \quad \Big|\frac{d}{dt}$$

$$\left\{\frac{\partial F}{\partial x}\cdot x'(t)+\frac{\partial F}{\partial y}\cdot y'(t)+\frac{\partial F}{\partial z}\cdot z'(t)=0\right\}$$

$\nabla F\cdot\vec{r}\,'=0 \implies (\nabla F)(x_0,y_0,z_0)$ is $\perp$ to any $\vec{r}(t_0)$ and its normal to the surface

Ex. Find the eqn. of the tangent plane to the ellipsoid: $x^2+3y^2+2z^2=6$ at $(1,1,1)$



$$\nabla F=\langle 2x,6y,4z\rangle$$
$$\vec{n}=(\nabla F)_{(1,1,1)}=\langle 2,6,4\rangle$$

$$2(x-1)+6(y-1)+4(z-1)=0$$